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In this work, we present a four-parameter lifetime model that can be used to model reliability issues, fatigue life studies, and survival data called the Type II Topp-Leone Inverse Power Lomax distribution. It has the Type II inverse Lomax, Inverse Power Lomax, and Inverse Lomax distributions as sub-models. Some of its statistical properties, including complete and incomplete moments, generating functions, characteristics functions, mean residual life, mean inactivity time, Renyi entropy, Tsallis entropy, order statistics, stress-strength reliability, and weighted probability moment, have formal formulas that we have developed. The model's parameters are estimated using the maximum likelihood estimation technique. The effectiveness of maximum likelihood estimators is evaluated in terms of absolute bias and simulation study standard error. Two lifetime data sets are used to demonstrate how the new model can be applied. Using the same comparative criteria, the proposed distribution offers a better fit than a few well-known distributions.

Keywords: Absolute Bias, Characteristic function, Maximum Likelihood Estimation, Type II Topp-Leone Inverse Power Lomax distribution, Tsallis Entropy.

1. Introduction

Numerous notable distributions have been developed throughout the past century to be used as models in applied sciences. In terms of usefulness, the so-called generalized beta distribution introduced by Eugene et al. (2002) and Jones (2004) tops the list. The key characteristic of the generalized beta distribution's is that it is extremely rich; to our knowledge, it contains more than forty named distributions. Various mathematical techniques have been used to add one or more parameters to a standard probability distribution in order to improve its fit and make it adaptable to different applications in modeling statistical data. This includes: Marshal-Olkin generated family developed and studied by Marshall and Olkin (1997), Kumaraswamy-G family proposed and studied by Cordeiro and de Castro (2011); Alexander et al. (2012) developed and studied the McDonald-G family, Zografos and Balakrishnan (2009) developed gamma-G (type 1) family of distribution, gamma-G (type 2) was proposed and studied by Ristic and Balakrishnan (2012), gamma-G (type 3) was developed by Torabi and Hedesh (2012); Amini et al. (2012) studied log gamma-G family; Exponentiated generalized-G was developed by Cordeiro et al. (2011), Alzaatreh et al. (2013) proposed and developed the Transformed-Transformer (T-X); Exponentiated (T-X) was studied by Alzaghal et al. (2013), Weibull-G family was developed by Bourguignon et al. (2014). The Exponentiated half logistic generated family was studied by Cordeiro et al. (2014), the Gompertz-G generator was developed by Ghosh et al. (2016). The Cauchy family was studied by Alizadeh eta al. (2015), the type II Topp-Leone- G family was developed by Elgarhy et al. (2018), Reyad et al. (2019) proposed the exponentiated generalized Topp-Leone-G family, the Sine Topp-Leone-G family of distributions was studied and developed by Abdulhakim A. Al-Babtain et al. (2020), Jamal and Chesneau (2020) proposed and studied the Sin Kumaraswamy-G family of distribution), the Topp-Leone Marshall-Olkin-G Family of Distributions was developed by Fastel Chipepa et al.(2020) .This study explores the tractability of the type II Topp-Leone-G family to develop a new generalization of the Inverse Power Lomax distribution called the Type II Topp-Leone Inverse Power Lomax distribution with greater scope/areas of applications.

The cumulative density function (CDF) of Type II Top-Leone-G family of distributions is given by

$$F(x; v, \Psi) = 1 - [1 - J(x; \Psi)^2]^v, \quad x \in \mathbb{R}, \tag{1}$$

The associated PDF corresponding to (1) is given by

$$f(x; v, \Psi) = 2vj(x; \Psi)J(x; \Psi)[1 - J(x; \Psi)^2]^{v-1}, \quad x \in \mathbb{R}, \tag{2}$$

Where v is a positive shape parameter and $J(x; \Psi)$ is the CDF of a baseline continuous distribution which sometimes may depend on a parameter vector Ψ. Elgarhy et al. (2018), described $(TIITL - G)$ family as a simplified version of the Kumaraswamy-G family. The added shape parameter v is to possibly control the tail weight and skewness of the CDF of baseline distribution.

The main motivation of this article is to develop a four-parameter distribution that extends the three-parameter distribution to be useful for modeling lifetime data with a variety of shapes of the hazard function that the three-parameter distribution cannot handle. Also, it provides a good parametric fit to skewed data that cannot be properly fitted by distribution and is a suitable model in several areas such as medicine, insurance, biomedical studies, reliability, and seismography. This distribution is known as the Type-II Topp-Leone Inverse Power Lomax distribution, which is more flexible and tractable in modeling lifetime data than the Inverse Power Lomax distribution.

2. Type II Top-Leone Inverse Power Lomax distribution

The Inverse Power Lomax (IPL) distribution, is one of the Distributions that is particularly appealing, is the subject of study in this paper. Mathematically, it can also be written as distribution of $Y = X^{-1}$, where X represents a random variable following the well-known Inverse Power Lomax distribution (see Amal and Marwa, 2018). Consequently, the IPL distribution's cumulative distribution function (CDF) is given by

$$J(x;\lambda,b,\zeta) = \left(1 + \frac{x^{-b}}{\lambda}\right)^{-\zeta} \tag{3}$$

The PDF that corresponds to (1) is given as

$$j(x;\lambda,b,\zeta) = \frac{b\zeta}{\lambda} x^{-b-1} \left(1 + \frac{x^{-b}}{\lambda}\right)^{-\zeta-1} \tag{4}$$

There are numerous uses of inverted distributions, including econometrics, biological sciences, survey sampling, engineering sciences, and challenges in medical research and life testing.

The CDF of the four-parameter Type II Top-Leone Inverse Power Lomax ($TIITLIPL$) distribution is obtained by plugging (3) in (1) and is given by

$$F(x;\lambda,b,\zeta,v) = 1 - \left[1 - \left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta}\right]^{v}, \quad x \in \mathbb{R}, \tag{5}$$

And the corresponding PDF to (5) is given by

$$f(x;\lambda,b,\zeta,v) = \frac{2b\zeta v}{\lambda} x^{-b-1} \left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta-1} \left[1 - \left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta}\right]^{v-1} \tag{6}$$

Where b, ζ, v are positive shape parameters and λ is a scale parameter. Some important probability distribution are special cases of $TIITLIPL$ distribution such as the Type II Top-Leone Inverse Lomax ($TIITLIL$) distribution $(b = 1)$, inverse Power Lomax (IPL) distribution $(v = 1)$, Inverse Lomax (IL) distribution $(b = v = 1)$. Several authors have used the Type II Top-Leone family of distributions to modify the standard probability distribution to induce flexibility and enhance its areas of application in modeling lifetime time data. Such work includes: the type II Topp-Leone generalized inverse Rayleigh distribution by Yahia and Mohammed (2019); the type II Topp-Leone power Lomax distribution by Al-Marzouki et al. (2020); the type II Topp-Leone inverse exponential distribution by Al-Marzouki (2021); the type II Topp-Leone generalized inverted exponential distribution by Al-Saiary and Al-Jadaani (2022); and the type II Topp-Leone Bur XII distribution by Ogunde et al. (2023), among many others. The graphs of the PDF and the CDF of

$TIITLIPL$ distribution is presented is Figures 1 as drawn below for various values of the parameters of the distribution.

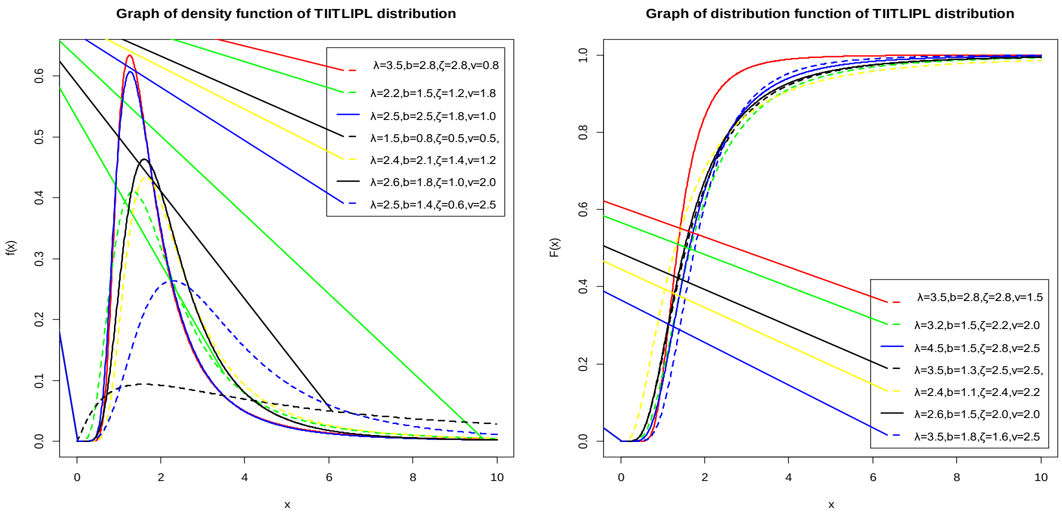

Figure 1. The graph of the density and distribution function of the TIITLIPL distribution

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An expression for the survival function $(S(x))$, hazard function $(h(x))$, cumulative hazard function $£((x))$ and reversed hazard function $(r(x))$ are respectively, given as

$$S(x;\lambda,b,\zeta,v) = 1 - F(x) = \left[1 - \left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta}\right]^{v}, \tag{7}$$

$$h(x;\lambda,b,\zeta,v) = \frac{\frac{2b\zeta v}{\lambda} x^{-b-1}\left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta-1}\left[1 - \left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta}\right]^{v-1}}{\left[1 - \left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta}\right]^{v}}, \tag{8}$$

$$£(x;\lambda,b,\zeta,v) = log\left(1 - \left[1 - \left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta}\right]^{v}\right) \quad , \tag{9}$$

and

$$r(x;\lambda,b,\zeta,v) = \frac{\frac{2b\zeta v}{\lambda} x^{-b-1}\left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta-1}\left[1 - \left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta}\right]^{v-1}}{\left[1 - \left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta}\right]^{v}}, \quad x > 0, \tag{10}$$

The graphs of the hazard function of $TIITLIPL$ distribution is presented is Figures 2, as drawn below for various values of the parameters of the distribution.

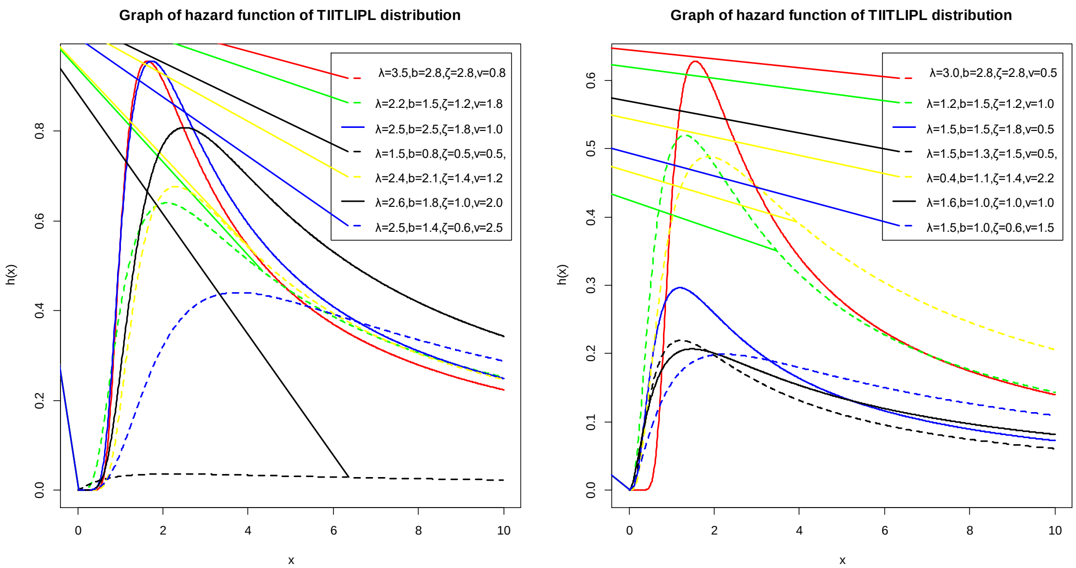

Figure 2. The graph of hazard function of the $TIITLIPL$ distribution

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2.1 Quantiles of the $TIITLIPL$ distribution

Some important features and characteristics of a distribution can be examined through its moments and quantiles, such as skewness, dispersion, and kurtosis. Also, the quantiles of a distribution can be used to generate data from a distribution. The u^{th} quantile (x_u) of the $TIITLIPL$ distribution is determined by solving this quantity:

$$F(x_u) = \mathrm{u},$$

Hence solving equation (5) we obtain

$$x_u = \left[\lambda\left\{\left[1-(1-u)^{1/v}\right]\right\}^{-1/2\zeta} - 1\right]^{-1/b}. \tag{11}$$

In particular, the u^{th} quantile for $u \in (0,1)$ for $u = 0.5, 0.75$, we obtain the middle quartile (median) and the upper quartile of the $TIITLIPL$ distribution given respectively, by

$$x_{0.5} = \left[\lambda\left\{\left[1-(0.5)^{1/v}\right]\right\}^{-1/2\zeta} - 1\right]^{-1/b}, \tag{12}$$

and

$$x_{0.75} = \left[\lambda\left\{\left[1-(0.25)^{1/v}\right]\right\}^{-1/2\zeta} - 1\right]^{-1/b}. \tag{13}$$

An expression for the mode of $TIITLIPL$ distribution can be derived by solving the following equation $\frac{\delta}{\delta x}(log[F(x;\lambda,b,\zeta,v)]) = 0.$

$$\left(\frac{b-1}{x}\right) = \frac{b(2\zeta+1)x^{-b-1}}{\lambda\left(1+\frac{x^{-b}}{\lambda}\right)} + \frac{2\zeta b x^{-b-1}(v-1)}{\lambda\left(1+\frac{x^{-b}}{\lambda}\right)\left[1-\left(1+\frac{x^{-b}}{\lambda}\right)^{\zeta}\right]} = 0$$

2.2 Skewness and Kurtosis Based on Quantile Function for $TIITLIPL$ distribution

The moments of distribution offer an empirical approach to evaluating the skewness and kurtosis of a distribution. However, some cases arise when the moments of distribution do not exist. This is a situation with a heavy tail distribution, such as the distribution. In particular, to measure the skewness of the distribution, we consider Bowley measures of the skewness coefficient defined by

$$\rho = \frac{q(0.75;\lambda,b,\zeta,v) + q(0.25;\lambda,b,\zeta,v) - 2q(0.5;\lambda,b,\zeta,v)}{q(0.75;\lambda,b,\zeta,v) - q(0.25;\lambda,b,\zeta,v)}$$

To estimate the value of kurtosis, we use the Moor's kurtosis (K) coefficient defined by

$$K = \frac{q(0.875;\lambda,b,\zeta,v) - q(0.625;\lambda,b,\zeta,v) + q(0.325;\lambda,b,\zeta,v) - q(0.125;\lambda,b,\zeta,v)}{q(0.75;\lambda,b,\zeta,v) - q(0.25;\lambda,b,\zeta,v)}$$

Here, it could be observed that ρ is a measure of symmetry of the $TIITLIPL$ distribution and K measures the heaviness of the tail distribution of $TIITLIPL$ model, whether it is heavy-tailed or light-tailed

Table 1 drawn below presents the numerical values of the quartiles, Bowley Skewness (ρ) coefficient, and Moor's coefficient (K) of kurtosis of $TIITIPL$ for hypothetical values of the parameters of the distribution taken the value of $v = 1.3$.

Table 1. Quartiles, Skewness and Kurtosis of $TIITLIPL$ distribution

	ζ, b	$q_{0.25}$	$q_{0.5}$	$q_{0.625}$	$q_{0.75}$	$q_{0.875}$	ρ	K
$\lambda = 5.0$	1.2,1.5	0.3511	0.5866	0.7580	1.0268	1.6027	0.3030	1.5645
	1.8,2.5	0.6592	0.8763	1.0143	1.2093	1.5707	0.2107	1.4070
	3.5,3.5	0.9279	1.1199	1.2370	1.3965	1.6768	0.1805	1.3602
	5.5,5.0	1.0478	1.1905	1.2746	1.3860	1.5737	0.1561	1.3285
$\lambda = 10$	1.2,1.5	0.2211	0.3696	0.4775	0.6468	1.0096	0.3023	1.5645
	1.8,2.5	0.4996	0.6641	0.7687	0.9165	1.1904	0.2108	1.4068
	3.5,3.5	0.7612	0.9187	1.0147	1.1456	1.3756	0.1805	1.3605
	5.5,5.0	0.9122	1.0364	1.1096	1.2066	1.3699	0.1563	1.3281
$\lambda = 15$	1.2,1.5	0.2270	0.3482	0.4312	0.4936	0.7705	0.0908	1.7033
	1.8,2.5	0.4248	0.5647	0.6536	0.7793	1.0122	0.2107	1.4071
	3.5,3.5	0.6780	0.8182	0.9037	1.0203	1.2251	0.1805	1.3608
	5.5,5.0	0.8411	0.9556	1.0231	1.1126	1.2633	0.1565	1.3300

2.3 Expansion for the density function

Here, we provide an expression for the expansion of density function of $TIITLIPL$. Consider the binomial series expansion given by

$$(1-z)^{t-1} = \sum_{i=0}^{\infty} (-1)^i \binom{t-1}{i} z^i. \tag{14}$$

for |z| < 1 and t > 0 is a real non-integer. Using the binomial series in equation (14), we obtain an expression for the pdf of $TIITLIPL$ model in equation (6) as

$$f(x; b, \zeta, v, \lambda) = \frac{2b\zeta v}{\lambda} \sum_{i=0}^{v-1} (-1)^i \binom{v-1}{i} x^{-b-1} \left(1 + \frac{x^{-b}}{\lambda}\right)^{-[2\zeta(i+1)+1]}, \tag{15}$$

$$= \sum_{i=0}^{\infty} €_i g_{\lambda,b,\zeta(i+1)}(x),$$

Where,

$$g_{\lambda,b,\zeta(i+1)}(x) = \frac{2b\zeta}{\lambda}(i+1)x^{-b-1}\left(1+\frac{x^{-b}}{\lambda}\right)^{-[2\zeta(i+1)+1]},$$

Is the IPL density with parameters λ, b and $\zeta(i+1)$.

$$€_i = \frac{v}{(i+1)}\sum_{i=0}^{v-1}(-1)^i\binom{v-1}{i}.$$

upon integration, we derive the same mixture representation for the cumulative distribution function of $TIITLIPL$ as

$$F(x;b,\zeta,v,\lambda) = \sum_{i=0}^{\infty} €_i G_{\lambda,b,\zeta(i+1)}(x).$$

Where $G_{\lambda,b,\zeta(i+1)}(x)$ is the CDF of the IPL density with parameters λ, b and $\zeta(i+1)$. The new model is more flexible and tractable and very useful in modeling data exhibiting different shapes of the hazard rate function.

Section 3 provides some precise statistical expressions for the ordinary and the incomplete moments and mean residual life, mean inactivity time, moment generating function, weighted probability moment, strength stress reliability, and order statistics of the $TIITLIPL$ distribution. In the Section 4 we derived an expression for the Renyi and Tsallis entropy of the $TIITLIPL$ distribution. For the $TIITLIPL$ distribution, we carried out simulation study, maximum likelihood estimations of the parameters of the $TIITLIPL$ model and real-life data applications in section 5. In Section 6, we conclude.

3. Statistical Properties of the $TIITLIPL$ distribution

3.1 The ordinary and incomplete moments of $TIITIPL$ distribution

Moments are the anticipated outcome of a particular function of a random variable. Due to the mathematical tractability of the various kinds of moments of $TIITLIPL$ distribution, the moments can be calculated directly. In a similar vein, Bonferroni and Lorenz curves, the mean waiting time, and the mean residual life can be computed using the first incomplete moment. Thus, the r^{th} ordinary moment of a distribution is given by

$$\mu_r' = E(x)^r$$

Thus the s^{th} moment of $TIITLIPL$ distribution is given by

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x;b,\zeta,v,\lambda)dx, \tag{16}$$

Putting equation (15) in (16), we have

$$\mu_r' = \frac{2b\zeta v}{\lambda}\sum_{i=0}^{v-1}(-1)^i\binom{v-1}{i}\int_{-\infty}^{\infty} x^{r-b-1}\left(1+\frac{x^{-b}}{\lambda}\right)^{-[2\zeta(i+1)+1]} dx, \tag{17}$$

by applying change of variables, $y = \frac{x^{-b}}{\lambda}, x^{-b} = x^{-b}y, dx = -{}^{1}/_{b}\,\lambda^{-1/b}y^{-1/b-1}dy$ then plogging it in (17), we have

$$\mu_r' = \frac{2\zeta v}{\lambda}\sum_{i=0}^{v-1}(-1)^i\binom{v-1}{i}\lambda^{b-r/b}\int_{0}^{\infty} y^{-r/b}(1+y)^{-[2\zeta(i+1)+1]}dy, \tag{18}$$

Consequently, taken $y = j(1-j)^{-1}$, $dy = (1-j)^{-2}dj$, then from equation (18), we obtain

$$\mu_r' = 2\zeta v\sum_{i=0}^{v-1}(-1)^i\binom{v-1}{i}\lambda^{-r/b}\int_{0}^{1} j^{-r/b}(1-j)^{r/b-2\zeta(i+1)-3}dj\ , \tag{19}$$

an expression for the r^{th} moments of $TIITLIPL$ distribution is given by

$$\mu_r' = 2\zeta v\sum_{i=0}^{v-1}(-1)^i\binom{v-1}{i}\lambda^{-r/b}B\left[\left\{1-{}^{r}/_{b}\right\},\left\{{}^{r}/_{b}-2\zeta(i+1)-1\right\}\right]. \tag{20}$$

Where $B(.,.)$ is an incomplete beta function. It should be noted that the mean (μ_1') and the variance ($\sigma^2 = \mu_2' - \{\mu_1'\}^2$) of $TIITLIPL$ can respectively, be obtained by taking $r = 1,2$.

Also, the r^{th} central moment of $TIITLIPL$ distribution can also be obtained using the relation

$$\mu_r = \int_{-\infty}^{\infty}(x-\mu)^r f(x;b,\zeta,v,\lambda)dx. \tag{21}$$

Using the binomial expression given in (14) to summarize (21), we have

$$\mu_r = \sum_{p=0}^{r}\binom{r}{p}(-1)^p\mu^p\mu_{r-p.}' \tag{22}$$

It can be observed that $\sigma^2 = \mu_2$. Further, many important quantities can be obtained, namely, the r^{th} cumulant of X given by

$$\mathrm{K}_r = \mu_r - \sum_{p=1}^{r-1}\binom{r-1}{p-1}\mathrm{K}_p\mu_{r-p.}' \tag{23}$$

Taken $\mathrm{K}_1 = \mu_1'$, the Pearson measures of skewness and kurtosis of X is respectively represented by $\gamma_1 = {}^{\mu_3}/_{\mu_2^{3/2}}$ and $£ = {}^{\mu_4}/_{\mu_2^2}$. Table 2 drawn below presents the first six moments, variance ($\boldsymbol{\sigma^2}$), skewness (Υ_1) and kurtosis (£) of type II Top-Leone Inverse power Lomax distribution for various hypothetical values parameters b, and $\boldsymbol{\lambda}$ also for fixed value of parameters $\zeta = 6.5$, and $v = 6.5$.

Table 2. First six moments, σ^2, Υ_1, and £ of $TIITLIPL$ distribution

Moments	$\boldsymbol{\lambda = 1.0}$, $\boldsymbol{b = 6.0}$	$\boldsymbol{\lambda = 2.3}$, $\boldsymbol{b = 8.5}$	$\boldsymbol{\lambda = 3.8}$, $\boldsymbol{b = 10.5}$	$\boldsymbol{\lambda = 4.8}$, $\boldsymbol{b = 14.5}$	$\boldsymbol{\lambda = 6.5}$, $\boldsymbol{b = 14.5}$	$\boldsymbol{\lambda = 8.5}$, $\boldsymbol{b = 16.0}$
$\boldsymbol{\mu'_1}$	0.7404	0.8472	0.9902	1.0132	1.0355	1.0516
$\boldsymbol{\mu'_2}$	0.5093	0.8427	0.9818	1.0285	1.0741	1.1073
$\boldsymbol{\mu'_3}$	0.3213	0.7666	0.9754	1.0462	1.1160	1.1670
$\boldsymbol{\mu'_4}$	0.1851	0.6939	0.9709	1.0664	1.1613	1.2311
$\boldsymbol{\mu'_5}$	0.0987	0.6259	0.9689	1.0892	1.2101	1.2993
$\boldsymbol{\mu'_6}$	0.0509	0.5638	0.9695	1.1148	1.2624	1.3717
$\boldsymbol{\sigma^2}$	NAN	1.1249	0.0013	0.0019	0.0018	0.0014
$\boldsymbol{\gamma_1}$	NAN	-3.6010	13.5766	2.6203	-0.5154	-8.4388
£	4.7333	11.0556	−397.3983	-57.6332	-32.4179	273.3821

It could be observed from Table 2 that the $TIITLIPL$ distribution can be used to model any kind of data, positively or negatively skewed, platykurtic, mesokurtic or leptokurtic.

3.2 Incomplete moments of $\boldsymbol{TIITLIPL}$ distribution

Incomplete moments of the income distribution form an important characteristic for measuring inequality. For example, the Lorenz and Bonferroni curves can be determined using the incomplete moments of the income distribution. The incomplete moments of $TIITLIPL$ distribution is given by

$$\gamma'_r(t) = \int_{-\infty}^{t} x^r f(x; b, \zeta, v, \lambda) dx. \tag{24}$$

Putting equation (15) in (24), we have

$$\gamma'_r(t) = \frac{2b\zeta v}{\lambda} \sum_{i=0}^{v-1} (-1)^i \binom{v-1}{i} \int_{-\infty}^{t} x^{r-b-1} \left(1 + \frac{x^{-b}}{\lambda}\right)^{-[2\zeta(i+1)+1]} dx, \tag{25}$$

by applying change of variables, $y = \frac{x^{-b}}{\lambda}, x^{-b} = x^{-b}y, dx = -{}^{1}\!/_{b}\, \lambda^{-1/b} y^{-1/b-1} dy$ then plogging it in (25), we have

$$\gamma_r'(t) = \frac{2\zeta v}{\lambda} \sum_{i=0}^{v-1} (-1)^i \binom{v-1}{i} \lambda^{b-r/b} \int_0^t y^{-r/b} (1+y)^{-[2\zeta(i+1)+1]} dy, \tag{26}$$

Consequently, taken $y = j(1-j)^{-1}$, $dy = (1-j)^{-2} dj$, then from equation (26), we obtain

$$\gamma_r'(t) = 2\zeta v \sum_{i=0}^{v-1} (-1)^i \binom{v-1}{i} \lambda^{-r/b} \int_0^1 j^{-r/b} (1-j)^{r/b - 2\zeta(i+1)-3} dj, \tag{27}$$

an expression for the r^{th} incomplete moments of $TIITLIPL$ distribution is given by

$$\gamma_r'(t) = 2\zeta v \sum_{i=0}^{v-1} (-1)^i \binom{v-1}{i} \lambda^{-r/b} B\left[\left\{1 - {}^{r}\!/_{b}\right\}, \left\{{}^{r}\!/_{b} - 2\zeta(i+1) - 1\right\}; \frac{t^{-b}}{\lambda}\right]. \tag{28}$$

By determining the first incomplete moments, we can obtain an expression for the mean residual life and the mean activity time for $TIITLIPL$ distribution as follows:

The mean residual life $[\Gamma_{MRL}(t)]$ of the $TIITLIPL$ distribution is

$$\Gamma_{MRL}(t) = \frac{1 - \gamma_1'(t)}{S(t) - t},$$

$$= \frac{1 - 2\zeta v \sum_{i=0}^{v-1} (-1)^i \binom{v-1}{i} \lambda^{-1/b} B\left[\left\{1 - {}^{1}\!/_{b}\right\}, \left\{{}^{1}\!/_{b} - 2\zeta(i+1) - 1\right\}; \frac{t^{-b}}{\lambda}\right]}{\left[1 - \left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta}\right]^{v} - t},$$

The mean inactivity time ($[\Gamma_{MIT}(t)]$ of the $TIITLIPL$ distribution is

$$\Gamma_{MIT}(t) = t - \frac{\gamma_r'(t)}{F(t)},$$

$$= t - \frac{2\zeta v \sum_{i=0}^{v-1} (-1)^i \binom{v-1}{i} \lambda^{-1/b} B\left[\left\{1 - {}^{1}\!/_{b}\right\}, \left\{{}^{1}\!/_{b} - 2\zeta(i+1) - 1\right\}; \frac{t^{-b}}{\lambda}\right]}{1 - \left[1 - \left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta}\right]^{v}}.$$

3.3 Moment generating function of TIITLIPL distribution

Many important features of a distribution can be derived using its moment generating function and moments. Let X denote a random variable with the probability density function (6). By definition, the moment generating function of $TIITLIPL$ distribution can be obtained by using the relation

$$M_x(t) = E(e^{tx}) = \sum_{r=1}^{\infty} \frac{t^r}{r!} E(X^r) \tag{29}$$

$$= \sum_{r=1}^{\infty} \sum_{i=0}^{v-1} \frac{t^r}{r!} 2\zeta v(-1)^i \binom{v-1}{i} \lambda^{-r/b} B\left[\left\{1 - {r}/{b}\right\}, \left\{{r}/{b} - 2\zeta(i+1) - 1\right\}\right]$$

The characteristic function is obtained by replacing t with it in (29). Thus, the characteristic moments for $TIITLIPL$ distribution is given as

$$\boldsymbol{\varphi(t)} = \sum_{r=1}^{\infty} \frac{(it)^r}{r!} E(X^r), \tag{30}$$

$$= \sum_{r=1}^{\infty} \sum_{i=0}^{v-1} \frac{(it)^r}{r!} 2\zeta v(-1)^i \binom{v-1}{i} \lambda^{-r/b} B\left[\left\{1 - {r}/{b}\right\}, \left\{{r}/{b} - 2\zeta(i+1) - 1\right\}\right].$$

3.4 Weighted probability moment of $TIITLIPL$ distribution

Probability weighted moments (PWMs) can be described as the expectations of certain functions of a random variable, which can only be obtained when the ordinary moments of the random variable exist. The PWMs method can be used to estimate the parameters of a distribution whose inverse form cannot be expressed explicitly. Estimates based on PWMs are often taken to be better than standard moment-based estimates. Sometimes, they are used when maximum likelihood estimates cannot be obtained or are difficult to compute. Suppose r and q are two positive integers, then, the r^{th}, q^{th} weighted probability moments for X, i.e., $\gamma'_{r,q} = E(X^r F(X; b, v, \zeta, \lambda)^q)$is given by

$$\gamma'_{r,q} = \int_{-\infty}^{\infty} x^r f(x; b, \zeta, v, \lambda) F(x; b, \zeta, v, \lambda)^q dx. \tag{31}$$

Plugging (5) and (6) in (31), then followed by using binomial expression in (14), we have

$$\gamma'_{r,q} = \frac{2b\zeta v}{\lambda} \sum_{i=0}^{\infty} \sum_{j=0}^{v(i+i)-1} (-1)^{i+j} \binom{q}{i} \binom{v(i+1)-1}{j} \int_{-\infty}^{\infty} x^r \left(1 + \frac{x^{-b}}{\lambda}\right)^{-\{2\zeta(j+1)+1\}} dx, \tag{32}$$

Also, by applying change of variables, $y = \frac{x^{-b}}{\lambda}, x^{-b} = \lambda y, dx = -{1}/{b}\, \lambda^{-1/b} y^{-1/b-1} dy$ then plogging it in (32), we have

$$\gamma'_{r,q} = \frac{2\zeta v}{\lambda^{\left(\frac{r+b+1}{b}\right)}} \sum_{i=0}^{\infty} \sum_{j=0}^{v(i+i)-1} (-1)^{i+j} \binom{q}{i} \binom{v(i+1)-1}{j} \int_{-\infty}^{\infty} y^{-\left(\frac{r+b+1}{b}\right)} (1 + y)^{-\{2\zeta(j+1)+1\}} dx, \tag{33}$$

Consequently, taken $y = j(1-j)^{-1}$, $dy = (1-j)^{-2}dj$, and substitute it in (33), finally we have an expression for the probability weighted moments of $TIITLIPL$ distribution as

$$\gamma'_{r,q} = \frac{2\zeta v}{\lambda^{\frac{r+b+1}{b}}} \frac{2b\zeta v}{\lambda} \sum_{i=0}^{\infty} \sum_{j=0}^{v(i+i)-1} (-1)^{i+j} \binom{q}{i} \binom{v(i+1)-1}{j} B\left[\frac{-(r+1)}{b}, 2\zeta(1+j) + 1 + \frac{r+1}{b}\right].$$

3.5 Stress strength Reliability Parameter for $TIITLIP$L distribution

Here, we derive the reliability $R = Pr(Y > Z)$ when Y and Z are independent random variable that follows the $TIITLIP$L distribution having sets of the following parameters $(\lambda, b, \zeta_1, v_1)$ and $(\lambda, b, \zeta_2, v_2)$, respectively. Then, the associated stress-strength reliability parameter is given by $R = P(Z < Y)$.

$$R = P(Y < Z) = \int_{-\infty}^{\infty} f_1(b, \zeta_1, v_1, \lambda,) F_2(b, \zeta_2, v_2, \lambda,)\, dx, \tag{34}$$

Consequently, we can write

$$R = F_1(\lambda, b, \zeta_1, v_1) - \frac{2b\zeta_1 v_1}{\lambda} \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{v_1 - 1}{i} \binom{v_2}{j} \int_0^{\infty} x^{-b-1} \left(1 + \frac{x^{-b}}{\lambda}\right)^{-2\zeta_1 i - 2\zeta_2 j} dx, \tag{35}$$

$$= F_1(\lambda, b, \zeta_1, v_1) - 2\zeta_1 v_1 \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{v_1 - 1}{i} \binom{v_2}{j} B[1, 2(\zeta_1 i + \zeta_2 j) - 1].$$

3.6 Order statistics of $TIITLIPL$ distribution

In statistical theory and analytical techniques, order statistics is mostly applied and practiced. Suppose we let $X_{[1]}, X_{[2]}, \dots, X_{[n]}$ be a random variable with corresponding CDF, $F(x)$. let $x_{1:n}, x_{2:n}, \dots, x_{n:n}$ be the corresponding ordered r. Then, the density of r^{th} statistic is represented by

$$f_{r:n}(x) = \text{Ɲ}^* \sum_{l=0}^{n-r} (-1)^l \binom{v_2}{j} f(x) F(x)^{l+r-1}. \tag{36}$$

Where, $\text{Ɲ}^* = \frac{1}{B(r, n-r+1)}$ and $B.,.$ is the beta function. The PDF of the r^{th} statistic of $TIITLIPL$ distribution is obtained by plugging (5) and (6) in (36), correspondingly, we have

$$f_{r:n}(x) = \text{Ɲ}^{*}\,\frac{2b\zeta v}{\lambda}\sum_{l=0}^{n-r}(-1)^{l}\binom{v_2}{j}\left(x^{-b-1}\left(1+\frac{x^{-b}}{\lambda}\right)^{-2\zeta-1}\left[1-\left(1+\frac{x^{-b}}{\lambda}\right)^{-2\zeta}\right]^{v-1}\right)\times\left(1-\left[1-\left(1+\frac{x^{-b}}{\lambda}\right)^{-2\zeta}\right]^{v}\right)^{l+r-1}, \quad (37)$$

By using the binomial expansion given in (14), twice to summarize (37), we have

$$f_{r:n}(x) = \text{Ɲ}^{*}\,\frac{2b\zeta v}{\lambda}\sum_{l=0}^{n-r}\sum_{p=0}^{l+r-1}\sum_{q=0}^{\infty}(-1)^{l+P+q}\binom{n}{l}\binom{l+r-1}{p}\binom{v(p+1)}{q}\times x^{-b-1}\left(1+\frac{x^{-b}}{\lambda}\right)^{-2\zeta q-1}, \quad (38)$$

Finally, we have,

$$f_{r:n}(x) = \text{Ɲ}^{*}\,\frac{2b\zeta v}{\lambda}\sum_{l=0}^{n-r}W^{*}{}_{p,q}\,x^{-b-1}\left(1+\frac{x^{-b}}{\lambda}\right)^{-2\zeta q-1}.$$

Where,

$$W^{*}{}_{p,q} = \sum_{p=0}^{l+r-1}\sum_{q=0}^{\infty}(-1)^{l+P+q}\binom{n}{l}\binom{l+r-1}{p}\binom{v(p+1)}{q}.$$

4. Information measures

4.1 Renyl Entropy of $TIITLIPL$ distribution

Renyl Entropy was introduced and developed by Renyl (1961). It measures the uncertainty embedded in a system. The Renyl entropy of a system can be obtained by using

$$I_{\text{Ʊ}}(X) = \frac{1}{1-\text{Ʊ}}\,log\int_{-\infty}^{\infty} f(x;b,\zeta,v,\lambda)^{\text{Ʊ}}dx, \text{Ʊ} > 0 \ and \ \text{Ʊ} \neq 0. \quad (39)$$

Using binomial expansion (14) in (39) then $f(x)^{\text{Ʊ}}$ can be written as

$$f(x)^{\text{Ʊ}} = \left(\frac{2b\zeta v}{\lambda}\right)^{\text{Ʊ}}\sum_{i=0}^{\infty}(-1)^{p}\binom{\text{Ʊ}(v-1)}{i}x^{-\text{Ʊ}(b+1)}\left(1+\frac{x^{-b}}{\lambda}\right)^{-\text{Ʊ}(4\zeta+1)}, \quad (40)$$

Therefore, the Renyl entropy of $TIITLIPL$ is as follows

$$\int_{-\infty}^{\infty} f(x)^{\text{Ʊ}} = \left(\frac{2b\zeta v}{\lambda}\right)^{\text{Ʊ}}\sum_{i=0}^{\infty}(-1)^{p}\binom{\text{Ʊ}(v-1)}{i}\int_{0}^{\infty} x^{-\text{Ʊ}(b+1)}\left(1+\frac{x^{-b}}{\lambda}\right)^{-\text{Ʊ}(4\zeta+1)}dx.$$

$$= G^* \sum_{i=0}^{\infty} (-1)^p \binom{\mho(v-1)}{i} B\left[\frac{\mho(b+1)+b-1}{b}, b(4\zeta\mho - 2) - \mho + 2\right],$$

Where $G^* = \left(\frac{2b\zeta}{\lambda}\right)^{\mho} \lambda^{\mho\left(\frac{b+\mho-1}{b}\right)}$

Finally, we have an expression for the Renyl entropy of $TIITLIPL$ distribution as

$$I_{\mho}(X) = \frac{1}{1-\mho} log\left\{G^* \sum_{i=0}^{\infty} (-1)^p \binom{\mho(v-1)}{i} B\left[\frac{\mho(b+1)+b-1}{b}, b(4\zeta\mho - 2) - \mho + 2\right]\right\}, \tag{41}$$

4.2 Tsallis Entropy of $TIITLIPL$ distribution

The Tsallis entropy was initially introduced by Havrada and Charvat (1967) and later developed by Tsallis (1988). The Tsallis entropy of the $TIITLIPL$ distribution can be defined as

$$I_T^{(\mho)} = \frac{1}{\mho - 1}\left[1 - \int_{-\infty}^{\infty} f_{TIITLIPL}(x; b, v, \zeta, \lambda)^{\mho}\right], \quad \mho > 0, \mho \neq 1. \tag{42}$$

Since,

$$\int_{-\infty}^{\infty} f(x)^{\mho} = \left(\frac{2b\zeta}{\lambda}\right)^{\mho} \lambda^{\mho\left(\frac{b+\mho-1}{b}\right)} \sum_{i=0}^{\infty} (-1)^p \binom{\mho(v-1)}{i} \times B\left[\frac{\mho(b+1)+b-1}{b}, b(4\zeta\mho - 2) - \mho + 2\right],$$

An expression for the Tsallis entropy of $TIITLIPL$ distribution is given as

$$I_T^{(\mho)} = \frac{1}{\mho - 1}\left\{1 - G^* \sum_{i=0}^{\infty} (-1)^p \binom{\mho(v-1)}{i} B\left[\frac{\mho(b+1)+b-1}{b}, b(4\zeta\mho - 2) - \mho + 2\right]\right\}. \tag{43}$$

5. Simulation study for $TIITLIPL$ model

Here, we carried out a simulation study to examine the behaviour of the MLEs of the $TIITLIPL$ model. We generated $N = 1000$ random samples of size $n = 50, 100, 200, 300, 400$ and 500 from X. Two sets of parameters were used. The MLEs, absolute bias (AB), standard error (SE), and the mean square error (MSE) of b, v, ζ, and λ were obtained. The numerical results were presented in Table 3 and 4 for ($b = 0.3, v = 0.4, \zeta = 0.5$, and $\lambda = 0.6$) and ($b = 0.5, v = 1.4, \zeta = 1.5$, and $\lambda =$

1.6), respectively. From the results obtained, it could be observed that as the number of samples (n) increases the mean square error approaches zero. This indicates that MLEs are consistent in estimating the values of the parameters of the TIITLIPL model.

Table 3. $MLEs$, AB, SE, and MSE of $TIITLIPL$ model

$\boldsymbol{Parameter}$	$\boldsymbol{n}$	$\boldsymbol{ML}$	$\boldsymbol{AB}$	$\boldsymbol{SE}$	$\boldsymbol{MSE}$
$\lambda = 0.3$	50	0.8780	0.5780	0.3575	0.4619
	100	0.6408	0.3408	0.3538	0.4228
	200	0.5385	0.2385	0.1751	0.2320
	300	0.4656	0.1656	0.1816	0.0604
	400	0.3969	0.0969	0.1461	0.0307
	500	0.3718	0.0718	0.1129	0.0179
$b = 0.5$	50	0.7115	0.2115	0.5159	0.3109
	100	0.4038	0.0962	0.3385	0.1238
	200	0.5275	0.0275	0.2547	0.0656
	300	0.4094	0.0906	0.1871	0.0432
	400	0.4473	0.0527	0.1409	0.0226
	500	0.4809	0.0191	0.1183	0.0144
$\zeta = 0.4$	50	0.2288	0.1712	0.6486	0.4499
	100	0.4879	0.0879	0.6239	0.3969
	200	0.2886	0.1114	0.4997	0.2621
	300	0.3351	0.0649	0.3858	0.1531
	400	0.3141	0.0859	0.1662	0.0358
	500	0.3666	0.0334	0.1127	0.0138
$v = 0.6$	50	0.3945	0.2055	0.6523	0.4677
	100	0.6834	0.0834	0.6428	0.4205
	200	0.5783	0.0247	0.3638	0.3644

	300	0.6128	0.0128	0.3471	0.1206
	400	0.7318	0.1318	0.3115	0.1144
	500	0.6487	0.0487	0.2161	0.0491

Table 4. $MLEs$, AB, SE, and MSE of $TIITLIPL$ model

$\boldsymbol{Parameter}$	$\boldsymbol{n}$	$\boldsymbol{ML}$	$\boldsymbol{AB}$	$\boldsymbol{SE}$	$\boldsymbol{MSE}$
$\lambda = 0.5$	50	2.5658	2.0658	4.3356	20.8632
	100	3.2955	2.7955	1.7214	10.7780
	200	1.5855	1.0855	0.6935	1.6593
	300	1.4164	0.9164	0.5317	1.1225
	400	1.1780	0.6780	0.5037	0.7134
	500	1.2128	0.7128	0.4225	0.6866
$b = 1.5$	50	2.0143	0.5143	2.2208	5.1965
	100	0.5269	2.1998	1.8726	8.3458
	200	1.3125	0.7850	1.7084	2.3246
	300	0.9892	0.4471	1.1821	1.5973
	400	1.1640	0.6783	0.6219	0.8469
	500	1.4275	0.8698	0.2593	0.8238
$\zeta = 1.4$	50	0.4216	0.9784	10.3213	107.4865
	100	3.5998	2.1798	7.5113	61.1712
	200	0.6150	0.5101	6.1251	2.3246
	300	0.9529	0.5251	4.7949	1.5973
	400	0.7217	0.4219	2.8494	0.8469
	500	0.5302	0.2388	1.1805	0.8238
	50	0.9886	0.6114	10.6138	113.0265

$v = 1.6$	100	2.7982	1.1982	8.5113	73.8779
	200	2.1101	0.5101	6.1251	37.7771
	300	2.1251	0.5251	4.7949	23.2668
	400	2.0219	0.4219	2.8494	8.2971
	500	1.8388	0.2388	1.1805	1.4506

5.2 Maximum likelihood estimation of $TIITLIPL$ distribution

Suppose $x_1, x_2, \dots, x_n$ be a positive real number drawn from X. Then, the log-likelihood function for $\xi = (b, v, \zeta, \lambda)$ is given by

$$l(\xi) = log\left(\frac{2b\zeta v}{\lambda}\right) - (b+1)\sum_{i=1}^{n} log(x_i) - (2\zeta+1)\sum_{i=1}^{n}\left(1+\frac{{x_i}^{-b}}{\lambda}\right) + (\boldsymbol{v}-\mathbf{1})\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{log}\left[1-\left(1+\frac{\boldsymbol{x_i}^{-b}}{\lambda}\right)^{-2\zeta}\right]. \quad (44)$$

The element of the score vector of ξ, say $\hat{\xi} = \left(\hat{b}, \hat{v}, \hat{\zeta}, \hat{\lambda}\right)$, associated to (42) which exemplify the log-likelihood of the $TIITLIPL$ model, $\hat{b}, \hat{v}, \hat{\zeta},$ and $\hat{\lambda}$ can be obtained as the solution to the simultaneous equations: $\frac{\partial l(\xi)}{\partial \zeta} = 0, \frac{\partial l(\xi)}{\partial \lambda}, \frac{\partial l(\xi)}{\partial b}$ and $\frac{\partial l(\xi)}{\partial b}$, in relation to the parameters. Here, the partial derivatives can be determined as follows:

$$\frac{\partial l(\xi)}{\partial \zeta} = \frac{n}{\zeta} - 2\sum_{i=1}^{n}\left(1+\frac{\boldsymbol{x_i}^{-b}}{\lambda}\right) - 2(v-1)\sum_{i=1}^{n}\frac{\left(1+\frac{\boldsymbol{x_i}^{-b}}{\lambda}\right)^{-2\zeta} log\left(1+\frac{\boldsymbol{x_i}^{-b}}{\lambda}\right)}{\left[1-\left(1+\frac{\boldsymbol{x_i}^{-b}}{\lambda}\right)^{-2\zeta}\right]}, \quad (45)$$

$$\frac{\partial l(\xi)}{\partial \lambda} = -\frac{n}{\lambda} + (2\zeta+1)\sum_{i=1}^{n}\frac{\boldsymbol{x_i}^{-b}}{\lambda^2\left(1+\frac{x_i^{-b}}{\lambda}\right)} + (\boldsymbol{v}-\mathbf{1})\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\frac{\boldsymbol{x_i}^{-b}\left(1+\frac{x_i^{-b}}{\lambda}\right)^{-2\zeta}}{\lambda^2\left(1+\frac{x_i^{-b}}{\lambda}\right)\left[1-\left(1+\frac{x_i^{-b}}{\lambda}\right)^{-2\zeta}\right]}, \quad (46)$$

$$\frac{\partial l(\xi)}{\partial b} = \frac{n}{b} - \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\boldsymbol{log}(\boldsymbol{x_i}) - (2\zeta+1)\sum_{i=1}^{n}\frac{\boldsymbol{x_i}^{-b}\log(\boldsymbol{x_i})}{\left(1+\frac{x_i^{-b}}{\lambda}\right)} + 2\zeta(v-1)\sum_{i=1}^{n}\frac{\boldsymbol{x_i}^{-b}\log(\boldsymbol{x_i})}{\left[1-\left(1+\frac{x_i^{-b}}{\lambda}\right)^{-2\zeta}\right]}, \quad (47)$$

and

$$\frac{\partial l(\xi)}{\partial v} = \frac{n}{v} + \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{log}\left[1 - \left(1 + \frac{\boldsymbol{x_i}^{-b}}{\lambda}\right)^{-2\zeta}\right]. \tag{48}$$

It could be observed from (48) that the following relation exist between the MLEs:

$$\hat{v} = -\frac{1}{n}\sum_{i=1}^{n} log\left[1 - \left(1 + \frac{x_i^{-\hat{b}}}{\hat{\lambda}}\right)^{-2\hat{\zeta}}\right].$$

Analytical expressions for the MLEs cannot be found, of course. To get an approximate numerical solution, we can apply nonlinear optimization techniques like the Newton-Raphson algorithm. Based on some regularity assumptions, for large, the sub-adjacent distribution of can be approximated by a multivariate normal distribution $\mathrm{N}_4\left(0_4, J(\hat{\xi}\,)^{-1}\right)$, where $J(\hat{\xi}\,) = \langle -\frac{\partial^2 l(\xi)}{\partial p \partial q}\rangle_{(p,q)\varepsilon\langle b,v,\zeta\lambda\rangle^2}\Big|_{\xi=\hat{\xi}}$. The 4×4 information matrix is given by

$$J(\hat{\xi}\,) = \begin{vmatrix} \frac{\partial^2 l(\xi)}{\partial b \partial b} & \frac{\partial^2 l(\xi)}{\partial b \partial v} & \frac{\partial^2 l(\xi)}{\partial b \partial \zeta} & \frac{\partial^2 l(\xi)}{\partial b \partial \lambda} \\ \frac{\partial^2 l(\xi)}{\partial v \partial b} & \frac{\partial^2 l(\xi)}{\partial v \partial v} & \frac{\partial^2 l(\xi)}{\partial v \partial \zeta} & \frac{\partial^2 l(\xi)}{\partial v \partial \lambda} \\ \frac{\partial^2 l(\xi)}{\partial \zeta \partial b} & \frac{\partial^2 l(\xi)}{\partial \zeta \partial v} & \frac{\partial^2 l(\xi)}{\partial \zeta \partial \zeta} & \frac{\partial^2 l(\xi)}{\partial \zeta \partial \lambda} \\ \frac{\partial^2 l(\xi)}{\partial \lambda \partial b} & \frac{\partial^2 l(\xi)}{\partial \lambda \partial v} & \frac{\partial^2 l(\xi)}{\partial \lambda \partial b \zeta} & \frac{\partial^2 l(\xi)}{\partial \lambda \partial b \lambda} \end{vmatrix},$$

We can obtain the maximum values of the unrestricted and restricted log-likelihoods to construct the LR statistics for testing some sub-models of the $TIITLIPL$ distribution. For example, we may use the LR statistic to determine if the fit using the $TIITLIPL$ distribution is statistically "better" than a fit using the Type II Topp-Leone Inverse Power Lomax ($TIITLIL$) and IPL distributions for a given data set. In any case, hypothesis testing of the type $H_0: \xi = \xi_0 \; versus \; H_1: \xi \neq \xi_0$ can be carried out by using any of the three asymptotically applicable statistics. For example, the test of $H_0: v = 1 \; versus \; H_1: v \neq 1 \; is \; not \; true$ is equivalent to compare the $TIITLIPL$ with the IPL distribution and the likelihood ratio statistics reduce to

$$w = 2\{l(\hat{b}, \hat{v}, \hat{\zeta}, \hat{\lambda}) - l(\breve{b}, \breve{1}, \breve{\zeta}, \breve{\lambda})\}.$$

Where, $\hat{b}, \hat{v}, \hat{\zeta}$and $\hat{\lambda}$ are the MLEs under H_1 and $\breve{b}, \breve{\zeta}$ and $\breve{\lambda}$ are the MLEs under H_0.

The approximate $100(1-\Psi)\%$ two-sided confidence intervals for $b, v, \zeta \; and \; \lambda$ are given by:

$$\hat{b} \pm Z_{\frac{\Psi}{2}}\sqrt{\mathrm{I}_{bb}^{-1}(\hat{\xi})}, \qquad \hat{v} \pm Z_{\frac{\Psi}{2}}\sqrt{\mathrm{I}_{vv}^{-1}(\hat{\xi})}, \quad \hat{\gamma} \pm Z_{\frac{\Psi}{2}}\sqrt{\mathrm{I}_{\zeta\zeta}^{-1}(\hat{\xi})}, \quad and \quad \widehat{\omega} \pm Z_{\frac{\Psi}{2}}\sqrt{\mathrm{I}_{\lambda\lambda}^{-1}(\hat{\xi})},$$

respectively. Where $I_{bb}^{-1}(\hat{\xi}), I_{vv}^{-1}(\hat{\xi}),\ I_{\zeta\zeta}^{-1}(\hat{\xi})$ and $I_{\lambda\lambda}^{-1}(\hat{\xi})$ are diagonal elements of $I_n^{-1}(\hat{\xi})$, and $Z_{\frac{\zeta}{2}}$ is the upper $\frac{\psi^{th}}{2}$ percentile of the distribution of the standard normal.

5.3 Applications of $TIITLIPL$ model to lifetime data

The section presents practical applications of the $TIITLIPL$ distribution using two lifetime datasets. The comparison of the of Type II Top-Leone Inverse Power Lomax distribution is made with Type II Top-Leone Inverse Lomax ($TIITLIPL$), Inverse Power Lomax (IPL) and Inverse Lomax (IL) distribution. Many accuracy measures including Akaike Information Criterion (AKAIC), Consistent Akaike Information criterion (CAKAIC), Hanan-Quin Information Criterion (HAQIC), Anderson-Darling test (AD), Cramer-von Mises test (CV), and Kolmogorov Smirnoff (KS) statistic are being estimated.

Data set I comprises the failure time of 36 appliances subjected to automatic life test. The data can be found in Lawless (1982).

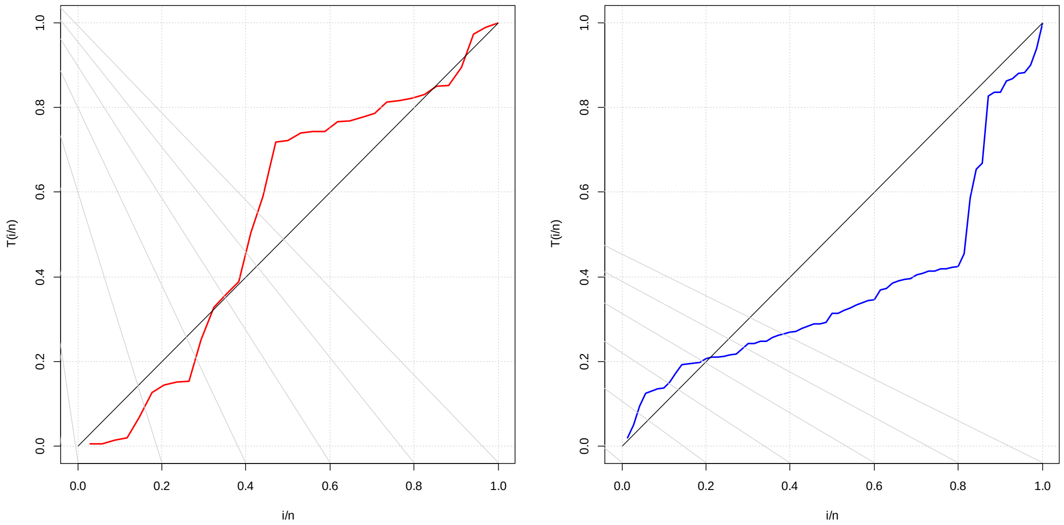
Data set II is available on https://www.ecan.govt.nz/data/riverflow/.

The exploratory data analysis of the data sets is given in Table 5. This shows that data I is positively skewed and over-dispersed with excess kurtosis of 1.0359 (leptokurtic). Also, data II is positively skewed and over-dispersed with excess kurtosis of 12.04 (Leptokurtic). Table 6 and 8 gives the MLEs of the models considered and Table 7 and 9 gives different measures of fit. Figure 3 presents the Total time on Test (TTT) plot for data set 1 and II which shows that the two data sets exhibit non-monotone failure rate. Boxplots for data set I and II are given in Figure 4. The fitted densities for the data sets are given in Figure 5. It can be deduced from these figures that the $KGIL$ distribution fits these three data sets better than other competitive models considered.

The likelihood ratio (LR) test was carried out for the sub-models $TIITLIL, IPL$ and IL of the $KGIL$ distribution for $b = 1, v = 1$, and $b = v = 1$, respectively. Table 10 provides the reports on the LR tests for the two datasets and it is evident from the results obtained that the $TIITLIL$ model have a better fit than its sub-models.

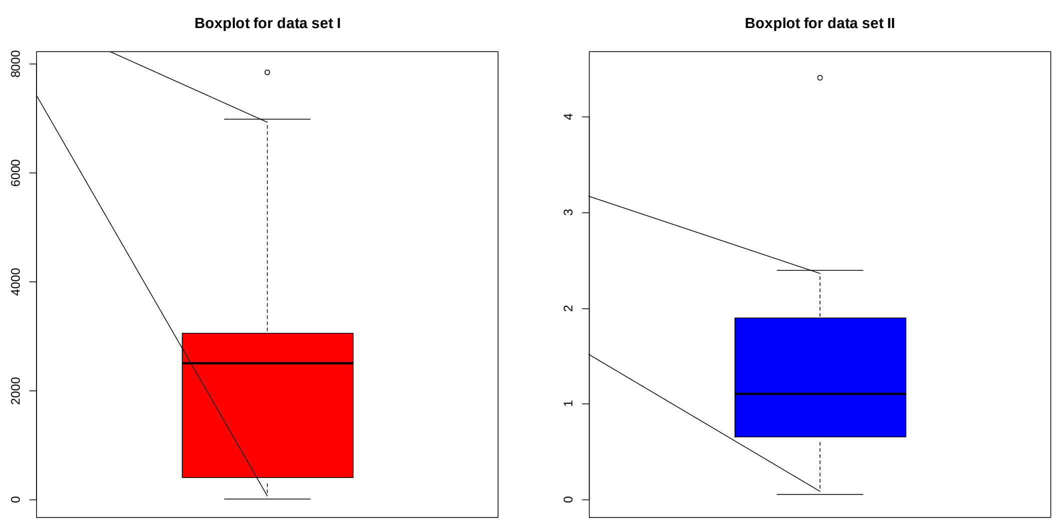
Table 5. Summary statistics of data sets

Data	n	Range	Lower quartile	Median	Upper quartile	mean	Var.	Skew.	Kurt.
$Data\ I$	34	7835	479.2	2511.0	3052.8	2285.8	3865446	1.0708	4.0359
$Data\ II$	70	27.07	0.66	1.11	1.89	2.82	20.68	3.16	15.04



(a) TTT plot to data set I (b) TTT plot to data set II

Figure 3. The graphs of total test time (TTT) plots for the two data sets



(a) Box plot to data set I (b) Box plot to data set II

Figure 4. The graphs of Box plots for the two data sets.

Table 6. MLEs, standard error (in braces), and confidence interval (curly bracket) of the parameters of the models for dataset I.

$Model$	ζ	b	λ	v
$TIITLIPL$	0.0051(0.0022) {−0.0057,0.0560}	0.5694(0.0697) {0.4997,0.0639}	0.8826(0.1709) {0.5476,1.2176}	8.5847(5.4883) {−2.1724,19.3418}
$TIITLIL$	−0.6008 (0.6965) {−1.9659,0.7643}	−(−) {−}	−15.0991(1.8302) {−18.6863, −11.5119}	0.3059(0.0600) {0.1883,0.4235}
IPL	0.0029(0.0010) {0.0094,0.0049}	40.8525(0.0574) {40.7400,40.965}	1.2719(0.3241) {0.6367,1.9071}	−(−) {−}
IL	0.0020(0.0001) {0.0018,0.0022}	−(−) {−}	1.8056(0.3211) {1.1762,2.4350}	−(−) {−}

Table 7. Model selection criteria for data set I.

$Model$	$-l$	$AKAIC$	$CAKIC$	$HAQIC$	CV	AD	KS
$TIITLIPL$	299.40	606.79	608.18	608.88	0.3097	1.7089	0.2302
$TIITPIL$	319.59	645.19	645.99	646.76	0.8552	4.6301	0.3031
IPL	304.57	615.14	615.94	616.70	0.4237	2.3909	0.18901
IL	307.25	618.45	618.84	619.49	0.4804	2.7170	0.2746

Table 8. MLEs and standard error (in braces), confidence interval (curly bracket) of the parameters of the models for dataset II.

$Model$	ζ	b	λ	v
$TIITLIPL$	12.2641(0.3766) {11.5260,13.0022}	6.8277(0.2990) {6.2417,7.4137}	0.1051(0.0331) {0.0402,0.1699}	0.1445(0.0186) {0.1080,0.1810}
$TIITLIL$	2.7374(1.9884) {−1.1599,6.6347}	−(−) {−}	1.9170(0.9477) {0.0595,3.7745}	1.5217(0.3464) {0.84282.2006}
IPL	1.9373(1.1016) {−0.2218,4.0964}	1.3799(0.1867) {1.0140,1.7458}	2.0704(0.8982) {0.3099,3.8309}	−(−) {−}
IL	6.2165(4.5826) {−2.7654,15.1984}	−(−) {−}	5.7548(3.6207) {−1.3418,12.8514}	−(−) {−}

Table 9. Model selection criteria for data set II.

$Model$	$-l$	$AKAIC$	$CAKIC$	$HAQIC$	CV	AD	KS
$TIITLIPL$	122.35	252.69	253.31	256.26	0.0532	0.4555	0.0848
$TIITLIL$	126.74	259.48	259.84	262.16	0.2040	1.3062	0.1267
IPL	126.04	258.08	258.44	260.76	0.1782	1.1714	0.1177
IL	128.59	261.19	261.37	262.97	0.1778	1.1796	0.1637

Table 10. LR test for the two data sets

$model$	$Hypothesis$	LR	$P-value$
Data set I			
$TIITLIPL\ vs\ TIITLIL$	$H_0: b = 1\ vs. H_1: H_0\ is\ false$	40.38	< 0.001
$TIITLIPL\ vs\ IPL$	$H_0: v = 1\ vs. H_1: H_0\ is\ false$	10.34	< 0.00
$TIITLIPL\ vs\ IL$	$H_0: b = v = 1\ vs. H_1: H_0\ is\ false$	15.70	< 0.001
$Data\ set\ II$			
$TIITLIPL\ vs\ TIITLIL$	$H_0: b = 1\ vs. H_1: H_0\ is\ false$	8.78	< 0.001
$TIITLIPL\ vs\ IPL$	$H_0: v = 1\ vs. H_1: H_0\ is\ false$	7.38	< 0.001
$TIITLIPL\ vs\ IL$	$H_0: b = v = 1\ vs. H_1: H_0\ is\ false$	12.48	< 0.001

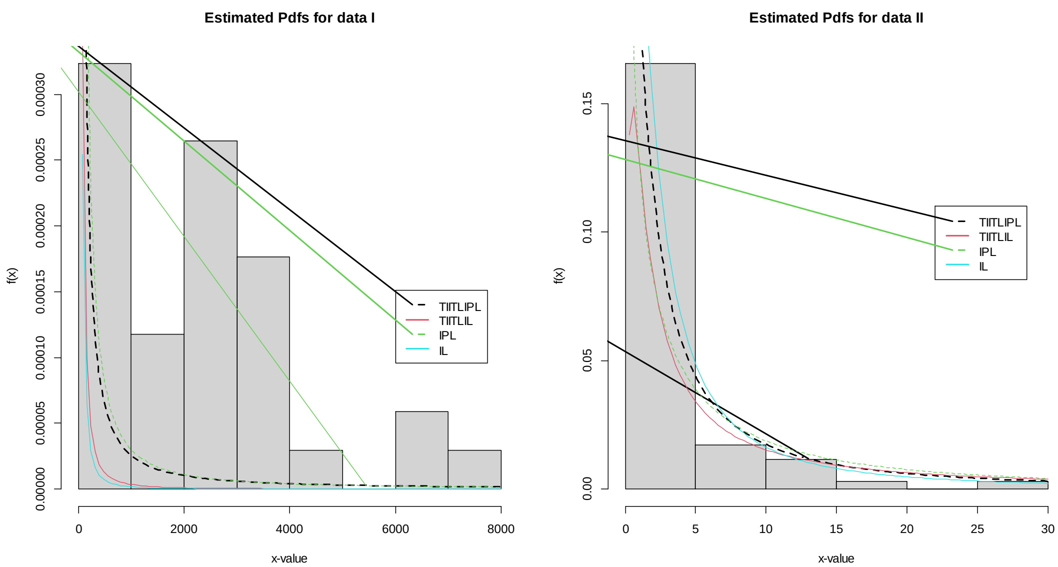


Figure 5. Plots of estimated PDF for the two data sets

5.4 Model Selection

We carried out model selection by using goodness of fit measures including maximized log-likelihood (l), $AKAIC$, $CAKAIC$, $HAQIC$, AD, CM, and KS statistic. Using this goodness of fit criteria, findings from Tables 5 and 7 show that the proposed model gives a better fit than other models considered because it possesses the smallest value of l, $AKAIC$, $CAKAIC$, $HAQIC$, AD, CM, and KS statistic.

6. Conclusions

In this article, we developed a new four-parameter lifetime distribution called the type II Topp-Leone Inverse Power Lomax distribution. Applications of the type II Topp-Leone Inverse Power Lomax model on two lifetime data sets showed that the distribution provided better fits than other competitor models, validating its modeling potential in terms of applicability. We also carried out simulation studies to validate the method of estimation used. Also, we provided some structural statistical properties of the new distribution: quantile function, Bowley skewness, and Moors kurtosis; mixture representations for the probability density; functions; ordinary moments; incomplete moments; mean residual life; mean inactivity time; moment generating function; characteristic function; probability weighted moments; stress-strength reliability; order statistics; and Renyl and Tsallis entropy.

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Declaration of Competing Interest

The authors declare no conflict of interest among the authors.

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