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Cover Page Footnote

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A Generalized Family of Lifetime Distributions and Survival Models

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In lifetime data, the hazard function is a common technique for describing the characteristics of lifetime distribution. Monotone increasing or decreasing, and unimodal are relatively simple hazard function shapes, which can be modeled by many parametric lifetime distributions. However, fewer distributions are capable of modeling diverse and more complicated shapes such as N-shaped, reflected N-shaped, W-shaped, and M-shaped hazard rate functions. A generalized family of lifetime distributions, the uniform-R{generalized lambda} (U-R{GL}) are introduced and the corresponding survival models are derived, and applied to two lifetime data sets. The survival model is applied to a right censored lifetime data set.

Keywords: Hazard function, T-R{Y} framework, generalized lambda distribution, censored data, regression model

Introduction

Lifetime data are often analyzed using either survival distributions or hazard rate functions. The Kaplan-Meier estimator (Kaplan & Meier, 1958) is a popular technique to generate a nonparametric survival distribution function. The Nelson-Aalen estimator (Nelson, 1972; Aalen, 1978) is common for generating a nonparametric cumulative hazard rate function. When covariates are involved, Cox's (1972) semiparametric proportional hazards model is the most commonly applied survival model. Various fully parametric models have also been developed in the literature (see e.g., Lawless, 2003, and references therein).

The use of fully parametric models is preferred in some settings in which the underlying distribution of a lifetime variable follows certain known probability distributions. The estimation of the parameters in fully parametric models may be obtained through full maximum likelihood. If the parametric models provide a good

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fit to the data, a more precise and efficient estimation of the parameters can be achieved (Klein & Moeschberger, 2003). For instance, as mentioned by Collett (2003), "estimates of quantities such as relative hazards and median survival times will tend to have smaller standard errors than they would in the absence of a distributional assumption," (p. 151). In addition, the estimated coefficients are easily interpreted and clinically meaningful (Hosmer et al., 2008).

Various parametric lifetime distributions are used in modeling and analyzing lifetime data in different areas such as engineering, economics, demography, biomedical sciences and other fields concerned with lifetimes. Examples of such distributions are the exponential, Weibull, lognormal, log-logistic, and gamma distributions. In lifetime studies, the most commonly used measures for describing the underlying distribution of a lifetime variable are the survival function or the hazard function. Since the survival functions are monotonically decreasing in time, the shapes of the hazard function can be used to emphasize the difference among lifetime distributions. For this reason, the shape of the hazard function is an important characteristic of a lifetime distribution, which can help guide model selection (Lawless, 2003). For example, monotonically increasing or decreasing hazard function often suggests the lifetime may follow Weibull distribution. Similarly, situations in which the hazard function increases initially and then decreases (hump-shaped) suggests that the log-logistic and the lognormal distributions may be suitable models for the data.

The three-parameter exponentiated Weibull distribution introduced by Mudholkar and Srivastava (1993) has the ability to model bathtub, upside-down bathtub, and monotone hazard rates. However, there are situations in which the hazard function exhibits some form of more complicated behavior such as N-shape, reflected N-shape, M-shape, W-shape, and others. These shapes are in biomedical science and reliability engineering; see for example, Bebbington et al. (2009). Because most of the well-known distributions do not exhibit these shapes, several generalizations, modifications, or extensions to these distributions have been proposed in the literature, (see for instance, generalized weighted Weibull by Domma et al., 2016; Gumbel-Weibull by Al-Aqtash et al., 2014), and the research in this area continues to be quite active.

Members of the class of T-R{generalized lambda} (T-R{GL}) families of distributions introduced by Aldeni et al. (2017), the U-R{GL} family for modeling lifetime distributions are considered here. The hazard function of U-R{GL} family can be monotonic, bathtub, upside-down bathtub, N-shaped, and bimodal shaped. A generalized regression model with the assumption that the lifetime variable follows the U-R{GL} distribution is derived to model right censored survival data.

The U-R{GL} Family of Lifetime Distributions

The T-R{GL} families of distributions proposed in Aldeni et al. (2017) based on the T-R{Y} framework introduced in Aljarrah et al. (2014) (see also Alzaatreh et al., 2014) are defined as follows:

Let Y be a random variable that follows the four-parameter GL distribution proposed by Ramberg and Schmeiser (1974) with quantile function

$$\mathrm{Q}_Y(u) = \mathrm{Q}_Y(u;\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2}, \quad u \in (0,1), \tag{1}$$

and let $\mathrm{F}_T(x)$ and $\mathrm{F}_R(x)$ be the cumulative distribution functions (CDFs) of the random variables T and R, respectively, with corresponding PDFs (if they exist) $\mathrm{f}_T(x)$ and $\mathrm{f}_R(x)$. The CDF (in general) of the random variable X in T-R{GL} families of distributions is defined as

$$\begin{aligned}\mathrm{F}_X(x) &= \int_a^{\mathrm{Q}_Y(\mathrm{F}_R(x))} \mathrm{f}_T(t)\,dt = \mathrm{F}_T\left\{\mathrm{Q}_Y\left(\mathrm{F}_R(x)\right)\right\} \\ &= \mathrm{F}_T\left(\lambda_1 + \frac{\mathrm{F}_R^{\lambda_3}(x) - \left(1-\mathrm{F}_R(x)\right)^{\lambda_4}}{\lambda_2}\right),\end{aligned} \tag{2}$$

$T,\ Y \in (a, b)$ for $-\infty \le a < b \le \infty$ and, accordingly, the corresponding PDF associated with (2) is

$$\begin{aligned}&\mathrm{f}_X(x) \\ &= \mathrm{f}_R(x)\left(\frac{\lambda_3\,\mathrm{F}_R^{\lambda_3-1}(x) + \lambda_4\left(1-\mathrm{F}_R(x)\right)^{\lambda_4-1}}{\lambda_2}\right)\mathrm{f}_T\left(\lambda_1 + \frac{\mathrm{F}_R^{\lambda_3}(x) - \left(1-\mathrm{F}_R(x)\right)^{\lambda_4}}{\lambda_2}\right)\end{aligned} \tag{3}$$

Aldeni et al. (2017) defined the support regions of the random variable T that correspond to different domains of the GL distribution. In particular, if $\lambda_1 = 1/2$, $\lambda_2 = 2$, and λ_3, $\lambda_4 > 0$, then the random variables T and Y have the support [0, 1]. Note that different choices of the random variables T and R lead to different families of generalized R-distributions. In this paper we consider the random variable T to be the standard uniform distribution and define the U-R{GL} family as follows:

Let Y be a random variable that follows the GL distribution with quantile function in (1), and let T be a random variable that follows the standard uniform distribution. For any given random variable R with CDF $F_R(x)$ and PDF $f_R(x)$, then by (2) and (3) the CDF and PDF of the U-R{GL} family of distributions are respectively given by

$$F_X(x) = \frac{1}{2}\left[1 + F_R^{\lambda_3}(x) - S_R^{\lambda_4}(x)\right],\ \lambda_3, \lambda_4 > 0, \tag{4}$$

$$f_X(x) = \frac{1}{2} f_R(x)\left[\lambda_3 F_R^{\lambda_3 - 1}(x) + \lambda_4 S_R^{\lambda_4 - 1}(x)\right], \tag{5}$$

where $1 - F_R(x) = S_R(x)$ is the survival function of the random variable R.

The corresponding hazard function of U-R{GL} family is given by

$$h_X(x) = f_R(x)\left[\frac{\lambda_3 F_R^{\lambda_3 - 1}(x) + \lambda_4 S_R^{\lambda_4 - 1}(x)}{1 - F_R^{\lambda_3}(x) + S_R^{\lambda_4}(x)}\right].$$

When $\lambda_3 = \lambda_4 = 1$ or $\lambda_3 = \lambda_4 = 2$, $F_X(x) = F_R(x)$ and $h_X(x) = h_R(x)$.

Examples of the U-R{GL} Family of Lifetime Distributions

As equation (4) shows, any member of U-R{GL} family is a generalization of the random variable R mapping from CDF of R to the quantile function of the generalized lambda distribution. Although any continuous distribution of the random variable R might be used in the U-R{GL} family, we consider the case when R is a nonnegative random variable representing a lifetime. Because the Weibull and log-logistic are very attractive distributions in lifetime analysis, we propose generalizations to these distributions, namely, the U-W{GL} and U-LL{GL} distributions.

The U-W{GL} Distribution

If a random variable R follows the Weibull distribution with survival function $S_R(x) = e^{-(x/\gamma)^c}$, $x \geq 0$, c, $\gamma > 0$, and PDF $f_R(x) = c\gamma^{-1}(x/\gamma)^{c-1} e^{-(x/\gamma)^c}$, then by (4) and (5) the CDF and PDF of the U-W{GL} distribution are given by, respectively,

$$\mathrm{F}_X\left(x\right)=\frac{1}{2}\left[1+\left(1-\mathrm{e}^{-\left(x/\gamma\right)^c}\right)^{\lambda_3}-\left(\mathrm{e}^{-\left(x/\gamma\right)^c}\right)^{\lambda_4}\right],\quad x\geq 0, c,\gamma,\lambda_3,\lambda_4>0\,, \tag{6}$$

$$\mathrm{f}_X\left(x\right)=\frac{1}{2}c\gamma^{-1}\left(x/\gamma\right)^{c-1}\mathrm{e}^{-\left(x/\gamma\right)^c}\left[\lambda_3\left(1-\mathrm{e}^{-\left(x/\gamma\right)^c}\right)^{\lambda_3-1}+\lambda_4\left(\mathrm{e}^{-\left(x/\gamma\right)^c}\right)^{\lambda_4-1}\right]. \tag{7}$$

The parameters c, λ_3, and λ_4 are shape parameters, which determine the skewness, kurtosis, and bimodality of the U-W{GL} distribution, whereas γ is a scale parameter. When $\lambda_3 = \lambda_4 = 1$ or $\lambda_3 = \lambda_4 = 2$, the U-W{GL} distribution reduces to the Weibull distribution, which has exponential and Rayleigh distributions as special cases.

Figure 1 illustrates different shapes of the U-W{GL} density function when $\gamma = 1$ and for various values of c, λ_3, and λ_4. The graphs in Figure 1 indicate that the U-W{GL} distribution can be left skewed, right skewed, monotonically decreasing (reversed J-shape), unimodal or bimodal. Figure 1 also indicates that the U-W{GL} distributions could have four different modal points, namely, one mode at zero, one positive mode, two modes: one at zero and the other is positive, or two different positive modes.

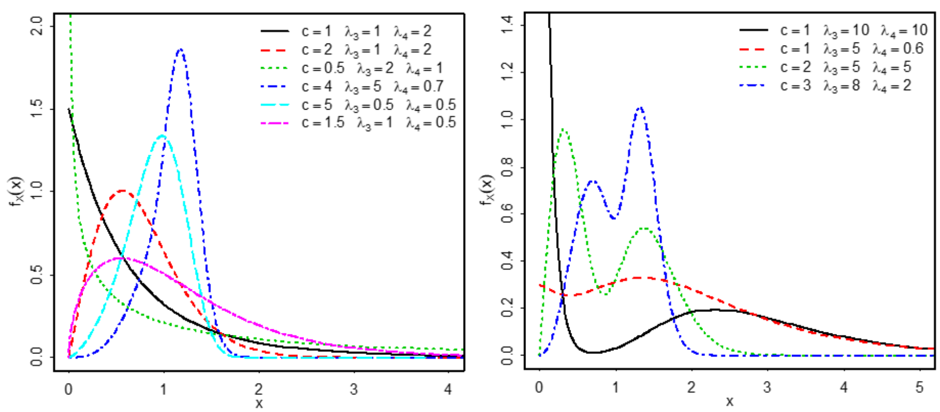


Figure 1. Plots of U-W{GL} distribution when $\gamma = 1$ and for various values of c, λ_3, and λ_4

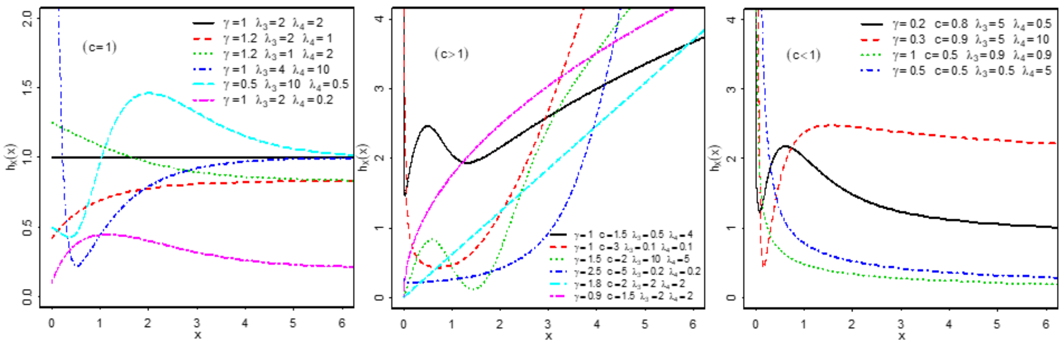


Figure 2. Plots of U-W{GL} hazard function for various values of c, λ_3, λ_4, and γ

The hazard function of U-W{GL} distribution is given by

$$\mathrm{h}_X(x)=\frac{\mathrm{f}_X(x)}{\left(1-\mathrm{F}_X(x)\right)}=\frac{c}{\gamma}\left(\frac{x}{\gamma}\right)^{c-1}\mathrm{e}^{-(x/\gamma)^c}\left(\frac{\lambda_3\left(1-\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_3-1}+\lambda_4\left(\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_4-1}}{1-\left(1-\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_3}+\left(\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_4}}\right). \quad (8)$$

In Figure 2, various shapes of U-W{GL} hazard function for different values of c, λ_3, λ_4, and γ are provided. When $c = 1$, Figure 2 displays six different shapes of the hazard rate function of U-W{GL} including constant, decreasing-constant, increasing-constant, decreasing-increasing-constant, increasing-decreasing-constant, or decreasing-increasing-decreasing-constant. The plots in Figure 2 show that when $c < 1$ the hazard function of U-W{GL} is either monotonically decreasing or reflected N-shape, and when $c > 1$ there are five different shapes including increasing, J-shape, N-shape, U-shape, and W-shape. As stated before, the Weibull distribution has only constant, increasing, or decreasing hazard rates. On the other hand, the U-W{GL} exhibits more than ten different shapes of hazard rates.

The U-LL{GL} Distribution

Let R be a log-logistic random variable with survival function

$$\mathrm{S}_R(x)=\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}, \quad x\geq 0, \alpha, \beta>0$$

and PDF

$$\mathrm{f}_R\left(x\right)=\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-2},$$

then by (4) and (5) the CDF and PDF of the U-LL{GL} distribution are given by, respectively,

$$\mathrm{G}_X\left(x\right)=\frac{1}{2}\left[1+\left(\frac{\left(\frac{x}{\alpha}\right)^{\beta}}{1+\left(\frac{x}{\alpha}\right)^{\beta}}\right)^{\lambda_3}-\left(1-\frac{\left(\frac{x}{\alpha}\right)^{\beta}}{1+\left(\frac{x}{\alpha}\right)^{\beta}}\right)^{\lambda_4}\right],\quad x\geq 0,\alpha,\beta,\lambda_3,\lambda_4>0,\quad (9)$$

$$\mathrm{g}_X\left(x\right)=\frac{1}{2}\left(\frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{2}}\right)\left[\lambda_3\left(\frac{\left(\frac{x}{\alpha}\right)^{\beta}}{1+\left(\frac{x}{\alpha}\right)^{\beta}}\right)^{\lambda_3-1}+\lambda_4\left(1-\frac{\left(\frac{x}{\alpha}\right)^{\beta}}{1+\left(\frac{x}{\alpha}\right)^{\beta}}\right)^{\lambda_4-1}\right]\quad (10)$$

The U-LL{GL} distribution has β, λ_3, and λ_4 as shape parameters and α as a scale parameter. It reduces to the log-logistic distribution when $\lambda_3 = \lambda_4 = 1$ or $\lambda_3 = \lambda_4 = 2$. Some possible shapes of the U-LL{GL} distribution are provided in Figure 3. These graphs indicate that the U-LL{GL} distribution can be monotonically decreasing (reversed J-shape) with one mode at x = 0, right skewed (unimodal) with one mode at $x > 0$, or bimodal with two different modes at $x_1, x_2 > 0$.

The hazard function of U-LL{GL} distribution, is given by

$$
\begin{aligned}
h_X(x) &= \frac{f_X(x)}{\left(1-F_X(x)\right)} \\
&= \left(\frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^2}\right)\left(\frac{\lambda_3\left(\frac{\left(\frac{x}{\alpha}\right)^{\beta}}{1+\left(\frac{x}{\alpha}\right)^{\beta}}\right)^{\lambda_3-1}+\lambda_4\left(1-\frac{\left(\frac{x}{\alpha}\right)^{\beta}}{1+\left(\frac{x}{\alpha}\right)^{\beta}}\right)^{\lambda_4-1}}{1-\left(\frac{\left(\frac{x}{\alpha}\right)^{\beta}}{1+\left(\frac{x}{\alpha}\right)^{\beta}}\right)^{\lambda_3}+\left(1-\frac{\left(\frac{x}{\alpha}\right)^{\beta}}{1+\left(\frac{x}{\alpha}\right)^{\beta}}\right)^{\lambda_4}}\right)
\end{aligned} \tag{11}
$$

Plots of U-LL{GL} hazard function are also given in Figure 3. When $\beta \le 1$, the hazard function is monotone decreasing, and when $\beta > 1$, the hazard function can be either unimodal or bimodal (M-shaped).

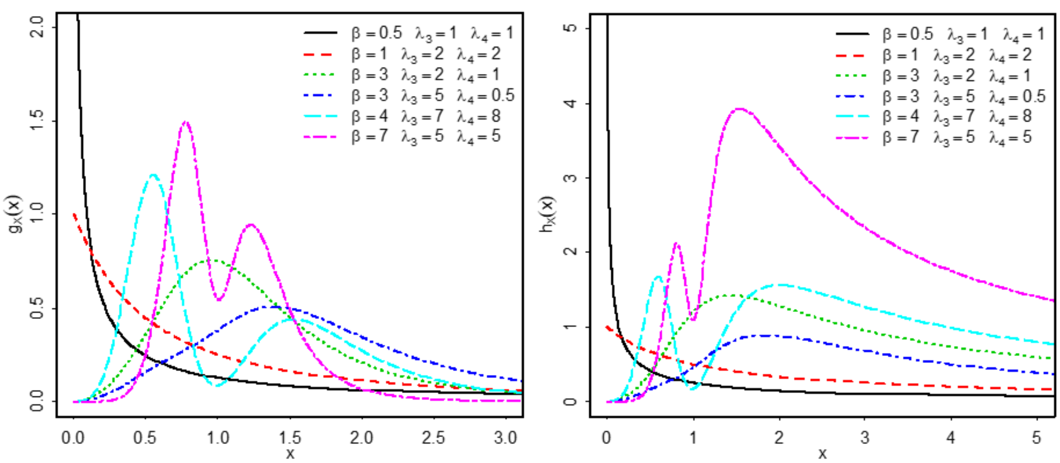


Figure 3. Plots of U-LL{GL} distribution (left) and hazard function (right) when $\alpha = 1$ and for various values of β, λ_3, and λ_4

Some Properties of the U-W{GL} and U-LL{GL} Distributions

Limiting Behavior

Lemmas 1 and 2 address the limiting behaviors of the PDF and hazard function of the U-W{GL} and U-LL{GL} distributions.

Lemma 1. The limits of the U-W{GL} and U-LL{GL} density functions as $x \to \infty$ are 0, and the limits as $x \to 0$ are given by

$$\lim_{x\to 0} \mathrm{f}_X(x) = \begin{cases} \infty, & c<1 \text{ or } \lambda_3 c<1 \\ \lambda_4/2\gamma, & c=1 \text{ and } \lambda_3>1 \\ 1/2\gamma, & c>1 \text{ and } \lambda_3 c=1, \\ (1+\lambda_4)/2\gamma, & c=\lambda_3=1 \\ 0, & c>1 \text{ and } \lambda_3 c>1 \end{cases}$$

$$\lim_{x\to 0} \mathrm{g}_X(x) = \begin{cases} \infty, & \beta<1 \text{ or } \lambda_3 \beta<1 \\ \lambda_4/2\alpha, & \beta=1 \text{ and } \lambda_3>1 \\ 1/2\alpha, & \beta>1 \text{ and } \lambda_3 \beta=1 \\ (1+\lambda_4)/2\alpha, & \beta=\lambda_3=1 \\ 0, & \beta>1 \text{ and } \lambda_3 \beta>1 \end{cases}$$

Proof. The proof can be found in Appendix A.

Lemma 2. The limits of U-W{GL} and U-LL{GL} hazard functions as $x \to 0$ are the same as the limits of their densities as $x \to 0$. The limit of U-LL{GL} hazard function as $x \to \infty$ is 0, whereas the limit of U-W{GL} hazard function as $x \to \infty$ is given by

$$\lim_{x\to \infty} \mathrm{h}_X(x) = \begin{cases} 0, & c<1 \\ 1/\gamma, & c=1 \text{ and } \lambda_4 \ge 1 \\ \lambda_4/\gamma, & c=1 \text{ and } \lambda_4<1 \\ \infty, & c>1 \end{cases}$$

Proof. It is straightforward to show the limits as $x \to 0$ from the relation $h_X(x) = f_X(x) / (1 - F_X(x))$. When $x \to \infty$, the result follows by using L'Hôspital's rule.

Transformation

Lemma 3. Let T be a random variable that follows the standard uniform distribution, then

(i) The random variable $X = \gamma(-\log(1 - F_Y(T))^{1/c}$ follows the U-W{GL} distribution, whereas the random variable $X = \alpha(F_Y(T) / 1 - F_Y(T))^{1/\beta}$ follows the U-LL{GL} distribution.

(ii) The quantile function of U-W{GL} distribution is $Q_X(u) = \gamma(-\log(1 - F_Y(u))^{1/c}$, $u \in (0, 1)$ whereas the quantile function of U-LL{GL} distribution is $Q_X(u) = \alpha(F_Y(u) / 1 - F_Y(u))^{1/\beta}$.

Proof. The result in (i) follows directly from the transformation $X = Q_R(F_Y(T))$, where $Q_R(u) = \gamma(-\log(1 - u))^{1/c}$ and $Q_R(u) = \alpha(u / 1 - u)^{1/\beta}$ are the quantile functions of the Weibull and log-logistic distributions, respectively. The result in (ii) follows by using the relation $F_X(x) = F_T(Q_Y(F_R(x)))$, and then solving $F_X(Q_X(u)) = u$ for $Q_X(u)$.

Lemma 3 can be used to simulate a random sample $x_1, x_2, \ldots, x_n$ of size n from a U-W{GL} distribution or a U-LL{GL} distribution by first generating a random sample $t_1, t_2, \ldots, t_n$ from standard uniform distribution and then transforming it to U-W{GL} or U-LL{GL} using the relationships $X = \gamma(-\log(1 - F_Y(T))^{1/c}$ and $X = \alpha(F_Y(T) / 1 - F_Y(T))^{1/\beta}$, respectively, where $F_Y(T)$ is evaluated numerically for different parameter combinations of λ_3 and λ_4. Note that the median M can be also obtained by setting $u = 0.5$ in the quantile functions in Lemma 3 (ii).

Moments

Theorem 1 shows the existence of the r^{th} non-central moments of the U-W{GL} and U-LL{GL} distributions.

Theorem 1. The r^{th} non-central moments of the (i) U-W{GL} and (ii) U-LL{GL} distributions are given by

(i) $$\mathrm{E}\left(X^r\right)=\frac{\gamma^r}{2}\Gamma\left(\frac{r}{c}+1\right)\left[\lambda_4^{-r/c}+\lambda_3\Gamma\left(\lambda_3\right)\sum_{k=0}^{\infty}\frac{\left(-1\right)^k\left(k+1\right)^{-\left(\left(r/c\right)+1\right)}}{k!\ \Gamma\left(\lambda_3-k\right)}\right] \tag{12}$$

When $\lambda_3 > 1$ is an integer, then equation (12) becomes

$$\mathrm{E}\left(X^r\right)=\frac{\gamma^r}{2}\Gamma\left(\frac{r}{c}+1\right)\left[\lambda_4^{-r/c}+\lambda_3\sum_{k=0}^{\infty}\left(-1\right)^k\binom{\lambda_3-1}{k}\left(k+1\right)^{-\left(\left(r/c\right)+1\right)}\right],$$

where

$$\Gamma\left(a\right)=\int_0^{\infty}t^{a-1}\mathrm{e}^{-t}dt$$

is the complete gamma function and

$$\binom{j}{k}=\frac{j\left(j-1\right)\cdots\left(j-k+1\right)}{k!}$$

is the binomial coefficient.

$$\mathrm{E}\left(X^r\right)=\frac{\alpha^r}{2}\left[\lambda_3\,\mathrm{B}\left(\lambda_3+\frac{r}{\beta},1-\frac{r}{\beta}\right)+\lambda_4\,\mathrm{B}\left(1+\frac{r}{\beta},\lambda_4-\frac{r}{\beta}\right)\right], \tag{13}$$

such that $\mathrm{E}(X^r)$ exists when $r < \beta$ and $r < \lambda_4\beta$, where

$$\mathrm{B}\left(a,b\right)=\int_0^1 t^{a-1}\left(1-t\right)^{b-1}dt$$

is the beta function.

Proof. The proof can be found in Appendix B.

Expressions of the statistical measures such as the mean, variance, skewness, and kurtosis can be derived from Theorem 1.

Parameter Estimation and Simulation for U-W{GL} Distribution

Consider the parameter estimation of the U-W{GL} distribution using the method of maximum likelihood. Let x_1, x_2,…, x_n be a random sample of size n from a U-W{GL} distribution defined in equation (7) with the vector of parameters $(\lambda_3, \lambda_4, c, \gamma)^T = \boldsymbol{\theta}$; then the log-likelihood function is given by

$$\begin{aligned}\ell(\boldsymbol{\theta}) &= n\left[\log c - \log 2 - c\log\gamma\right] + (c-1)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\left(x_i/\gamma\right)^c \\ &\quad + \sum_{i=1}^{n}\log\left[\lambda_3\left(1-\mathrm{e}^{-(x_i/\gamma)^c}\right)^{\lambda_3-1} + \lambda_4\left(\mathrm{e}^{-(x_i/\gamma)^c}\right)^{\lambda_4-1}\right]\end{aligned} \tag{14}$$

On setting $z_i = 1-\mathrm{e}^{-(x_i/\gamma)^c}$ and taking the first partial derivatives of equation (14) with respect to λ_3, λ_4, c, and γ, we get

$$\frac{\partial\ell}{\partial\lambda_3} = \sum_{i=1}^{n}\frac{z_i^{\lambda_3-1} + \lambda_3 z_i^{\lambda_3-1}\log(z_i)}{\lambda_3 z_i^{\lambda_3-1} + \lambda_4(1-z_i)^{\lambda_4-1}} \tag{15}$$

$$\frac{\partial\ell}{\partial\lambda_4} = \sum_{i=1}^{n}\frac{(1-z_i)^{\lambda_4-1} + \lambda_4(1-z_i)^{\lambda_4-1}\log(1-z_i)}{\lambda_3 z_i^{\lambda_3-1} + \lambda_4(1-z_i)^{\lambda_4-1}} \tag{16}$$

$$\begin{aligned}\frac{\partial\ell}{\partial c} &= \frac{n}{c} + \frac{1}{c}\sum_{i=1}^{n}\log\left[-\log(1-z_i)\right] \\ &\quad + \frac{1}{c}\sum_{i=1}^{n}\log(1-z_i)\log\left[-\log(1-z_i)\right]\left(1 + \frac{(\lambda_4-1)\lambda_4(1-z_i)^{\lambda_4-1}}{\lambda_3 z_i^{\lambda_3-1} + \lambda_4(1-z_i)^{\lambda_4-1}}\right. \\ &\quad \left. - \frac{(\lambda_3-1)\lambda_3(1-z_i)z_i^{\lambda_3-2}}{\lambda_3 z_i^{\lambda_3-1} + \lambda_4(1-z_i)^{\lambda_4-1}}\right)\end{aligned} \tag{17}$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{-nc}{\gamma} + \frac{c}{\gamma} \sum_{i=1}^{n} \log\left(1 - z_i\right) \left(\frac{\left(\lambda_3 - 1\right) \lambda_3 \left(1 - z_i\right) z_i^{\lambda_3 - 2}}{\lambda_3 z_i^{\lambda_3 - 1} + \lambda_4 \left(1 - z_i\right)^{\lambda_4 - 1}} - \frac{\left(\lambda_4 - 1\right) \lambda_4 \left(1 - z_i\right)^{\lambda_4 - 1}}{\lambda_3 z_i^{\lambda_3 - 1} + \lambda_4 \left(1 - z_i\right)^{\lambda_4 - 1}} - 1 \right) \tag{18}$$

The maximum likelihood estimate (MLE) $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is obtained by setting equations (15) through (18) to zero and solving them numerically. The NLMIXED procedure in SAS is applied in this article. The U-W{GL} reduces to the Weibull distribution when $\lambda_3 = \lambda_4 = 1$ or $\lambda_3 = \lambda_4 = 2$. To find the MLE, we take the initial estimates of parameters λ_3 and λ_4 to be 2 and the initial estimates of parameters c and γ to be the moment estimates of c and γ by assuming the simulated data x_1, $x_2, \ldots, x_n$ is from the Weibull distribution. Using the transformation $Y = \log X$, where Y follows Type 1 extreme-value distribution, the moment estimates for c and γ are

$$\tilde{c} = \frac{\pi}{\sqrt{6} s_y} \quad \text{and} \quad \tilde{\gamma} = \exp\left(\bar{y} + \frac{\delta}{\tilde{c}} \right),$$

where $\bar{y}$ and s_y are the mean and the standard deviation of the sample $y_1, y_2, \ldots, y_n$, and $\delta = \Gamma'(1) \approx 0.57722$ is Euler's constant (Johnson et al., 1994).

A simulation study is conducted to evaluate the MLEs in terms of the average bias and standard deviation of the parameter estimates for different parameter combinations and sample sizes. Following Lemma 3, a random sample of size n from a U-W{GL} distribution can be simulated. Five sample sizes are considered (n = 50, 100, 250, 500, 1000) in this simulation. The simulation study is conducted for a total of five parameter combinations $(\lambda_3, c, \lambda_4)$ = {(3, 0.8, 2), (5, 3, 0.5), (2.5, 2, 3), (5, 1, 5), (4, 4, 3)} when $\gamma = 1$. These combinations are considered to cover different shapes of the distribution, including monotonically decreasing (reversed J-shape), right skewed, left skewed, and bimodal. The MLE of $\boldsymbol{\theta}$ is computed and the process is repeated 500 times for each sample size and each parameter combination. The average bias and standard deviation of the MLEs are reported in Table 1.

In general, the results in Table 1 show that the standard deviations of the MLEs decrease as the sample size increases. In addition, the average biases and standard deviations of the MLEs are somewhat small and seem to be reasonable. It is also noticed that the MLE of the parameter c tend to be overestimated.

Table 1. Average bias (standard deviation) for the MLEs when $\gamma = 1$

Actual values				Average bias				
λ_3	c	λ_4	n	$\hat{\lambda}_3$	$\hat{c}$	$\hat{\lambda}_4$	$\hat{\gamma}$	**Mode(s)**
3.0	0.8	2.0	50	-0.073(0.399)	-0.017(0.078)	-0.026(0.281)	0.009(0.131)	Reversed J-shape; one mode at $x = 0$
			100	-0.078(0.410)	-0.010(0.062)	-0.029(0.280)	0.010(0.129)	
			250	-0.060(0.381)	-0.008(0.049)	-0.021(0.275)	-0.002(0.109)	
			500	-0.039(0.372)	-0.004(0.039)	-0.017(0.267)	-0.004(0.103)	
			1000	-0.018(0.353)	-0.002(0.035)	-0.005(0.251)	-0.001(0.096)	
5.0	3.0	0.5	50	-0.013(0.728)	-0.082(0.331)	-0.002(0.066)	-0.005(0.058)	Left-skewed; one mode at $x > 0$
			100	-0.049(0.719)	-0.065(0.262)	-0.003(0.065)	-0.004(0.043)	
			250	-0.012(0.686)	-0.038(0.185)	-0.004(0.060)	-0.004(0.035)	
			500	0.039(0.701)	-0.031(0.127)	-0.006(0.054)	-0.006(0.030)	
			1000	0.044(0.640)	-0.021(0.101)	-0.004(0.049)	-0.004(0.027)	
2.5	2.0	3.0	50	-0.014(0.348)	-0.046(0.205)	-0.022(0.417)	-0.001(0.088)	Right-skewed; one mode at $x > 0$
			100	-0.016(0.342)	-0.020(0.163)	-0.006(0.441)	0.001(0.067)	
			250	-0.017(0.330)	-0.010(0.116)	-0.003(0.412)	-0.001(0.049)	
			500	-0.022(0.304)	-0.004(0.098)	-0.012(0.410)	-0.000(0.042)	
			1000	-0.002(0.280)	-0.004(0.078)	-0.014(0.348)	-0.001(0.036)	
5.0	1.0	5.0	50	0.056(0.717)	-0.004(0.088)	0.068(0.718)	0.002(0.115)	Bimodal; two modes at $x = 0$, $x > 0$
			100	0.043(0.656)	-0.004(0.069)	0.004(0.685)	-0.003(0.098)	
			250	-0.067(0.663)	-0.002(0.047)	-0.019(0.661)	0.001(0.083)	
			500	-0.061(0.640)	-0.001(0.038)	-0.030(0.645)	-0.000(0.073)	
			1000	-0.058(0.615)	-0.002(0.031)	-0.038(0.605)	-0.001(0.066)	
4.0	4.0	3.0	50	0.078(0.539)	-0.023(0.410)	0.071(0.412)	0.001(0.045)	Bimodal; two different modes at $x > 0$
			100	-0.023(0.558)	-0.011(0.333)	0.032(0.399)	0.002(0.031)	
			250	-0.006(0.551)	-0.009(0.219)	0.006(0.400)	-0.000(0.024)	
			500	-0.019(0.492)	-0.007(0.180)	-0.022(0.383)	-0.001(0.020)	
			1000	-0.009(0.470)	-0.011(0.143)	-0.026(0.342)	-0.001(0.017)	

A simulation study is also carried out to examine the performance of the MLEs for the U-LL{GL} distribution. The study showed that the ML method is appropriate for estimating the U-LL{GL} parameters. To save space, the table of values is not reported.

Generalized Regression Models for Survival Data

In this section, we develop a generalized regression model for lifetime data with covariates. The relationship between the explanatory variables or covariates and the lifetime is of interest in the analysis of most censored or uncensored survival data. This relationship might be represented as a linear relationship between the log of the lifetime and the covariate values, which can be described as follows:

Consider the lifetime X_i of the i^{th} individual in the sample ($i = 1, 2,\ldots, n$), and let $\mathbf{v}_i = (1, v_{i1},\ldots, v_{ik})^{\mathrm{T}}$ be a vector of explanatory variables (the 1 is for the intercept). The dependent variable $Y_i = \log(X_i)$ is related to this set of covariates through a regression model. This model can be written as

$$Y_i = \log\ X_i = \mathbf{v}_i^{\mathrm{T}}\boldsymbol{\gamma} + \sigma Z_i\,, \tag{19}$$

where $\boldsymbol{\gamma} = (\gamma_0, \gamma_1,\ldots, \gamma_k)^{\mathrm{T}}$ are the unknown regression coefficients of the values of k explanatory variables, which have an interpretation similar to those in general linear model used in regression analysis, and σ is an unknown scale parameter. The quantity Z_i is the error variable. If Z follows logistic distribution, the model is the log-logistic regression. If Z follows the extreme value distribution, the model is the Weibull regression.

The U-W{GL} and U-LL{GL} Regression Models

Let X be a lifetime variable that follows the U-W{GL} distribution in (7); then the survival function for the U-W{GL} distribution is given by

$$\mathrm{S}_X\left(x\right) = \frac{1}{2}\left[1-\left(1-\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_3} + \left(\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_4}\right], \quad x \ge 0, c, \gamma, \lambda_3, \lambda_4 > 0\,. \tag{20}$$

If we take the log transform of X, and redefine the parameters as $c = 1 / \sigma$ and $\gamma = \mathrm{e}^u$, then Y follows the form of a log linear model such that $Y = \log(X) = \mu + \sigma Z$, where Z is the standardized log(U-W{GL}) distribution with PDF

$$\pi_Z\left(z\right) = \frac{1}{2}\mathrm{e}^z\mathrm{e}^{-\mathrm{e}^z}\left[\lambda_3\left(1-\mathrm{e}^{-\mathrm{e}^z}\right)^{\lambda_3-1} + \lambda_4\left(\mathrm{e}^{-\mathrm{e}^z}\right)^{\lambda_4-1}\right]. \tag{21}$$

Thus, the underlying PDF and survival function, respectively, for Y, are

$$\mathrm{f}_Y\left(y\right) = \frac{1}{2\sigma}\mathrm{e}^{((y-\mu)/\sigma)}\mathrm{e}^{-\mathrm{e}^{((y-\mu)/\sigma)}}\left[\lambda_3\left(1-\mathrm{e}^{-\mathrm{e}^{((y-\mu)/\sigma)}}\right)^{\lambda_3-1} + \lambda_4\left(\mathrm{e}^{-\mathrm{e}^{((y-\mu)/\sigma)}}\right)^{\lambda_4-1}\right], \tag{22}$$

where $y, u \in \mathbb{R}$ and $\sigma, \lambda_3, \lambda_4 > 0$; and

$$S_Y(y) = \frac{1}{2}\left[1 - \left(1 - e^{-e^{((y-\mu)/\sigma)}}\right)^{\lambda_3} + \left(e^{-e^{((y-\mu)/\sigma)}}\right)^{\lambda_4}\right]. \tag{23}$$

It is clear that the log(U-W{GL}) distribution in (22) reduces to log-Weibull (or extreme-value) when $\lambda_3 = \lambda_4 = 1$ or $\lambda_3 = \lambda_4 = 2$.

To incorporate covariates into the U-W{GL} model, we use the log-linear model (19) for the lifetime X_i, where Z_i has the standardized log(U-W{GL}) distribution (21) such that $\boldsymbol{\mu}_i = \mathbf{v}_i^{\mathrm{T}}\boldsymbol{\gamma}$ and σ, λ_3, $\lambda_4 > 0$ are unknown parameters, or equivalently, $X_i = \exp\left(\mathbf{v}_i^{\mathrm{T}}\boldsymbol{\gamma} + \sigma Z_i\right)$, which follows the U-W{GL} distribution in (7).

Similarly, if X follows the U-LL{GL} density function in (10), then $Y = \log(X)$ has the log(U-LL{GL}) distribution. The density function of Y, parameterized in terms of $\beta = 1 / \sigma$ and $\alpha = e^u$, is given by

$$g_Y(y) = \frac{1}{2\sigma}\left(\frac{e^{(y-\mu)/\sigma}}{\left(1+e^{(y-\mu)/\sigma}\right)^2}\right)\left[\lambda_3\left(\frac{e^{(y-\mu)/\sigma}}{1+e^{(y-\mu)/\sigma}}\right)^{\lambda_3-1} + \lambda_4\left(\frac{1}{1+e^{(y-\mu)/\sigma}}\right)^{\lambda_4-1}\right], \tag{24}$$

where $y, u \in \mathbb{R}$ and σ, λ_3, $\lambda_4 > 0$. The survival function of Y is given by

$$S_Y(y) = \frac{1}{2}\left[1 - \left(\frac{e^{(y-\mu)/\sigma}}{1+e^{(y-\mu)/\sigma}}\right)^{\lambda_3} + \left(\frac{1}{1+e^{(y-\mu)/\sigma}}\right)^{\lambda_4}\right]. \tag{25}$$

Maximum Likelihood Estimation

Let the random variables X_i and C_i denote the lifetime and censoring time of i^{th} individual, and the response Y_i represents a log-lifetime or a log-censoring time for i^{th} individual, i.e. $Y_i = \min\{\log(X_i), \log(C_i)\}$ for $i = 1, 2, \ldots, n$. Consider a sample of n independent observations. If all the observations are uncensored, then the log likelihood for the model parameters $\boldsymbol{\theta} = (\lambda_3, \lambda_4, \sigma, \boldsymbol{\gamma}^{\mathrm{T}})^{\mathrm{T}}$ based on the log(U-W{GL}) distribution in (22) can be written as

$$
\begin{aligned}
\mathrm{l}(\boldsymbol{\theta}) &= \sum_{i=1}^{n} \log\left(\mathrm{f}_Y\left(y_i\right)\right) \\
&= -n\log\left(2\sigma\right) + \sum\nolimits_{i=1}^{n} z_i - \sum\nolimits_{i=1}^{n} \mathrm{e}^{z_i} \\
&\qquad + \sum\nolimits_{i=1}^{n} \log\left[\lambda_3\left(1-\mathrm{e}^{-\mathrm{e}^{z_i}}\right)^{\lambda_3-1} + \lambda_4\left(\mathrm{e}^{-\mathrm{e}^{z_i}}\right)^{\lambda_4-1}\right]
\end{aligned} \tag{26}
$$

where $z_i = \left(y_i - \mathbf{v}_i^{\mathrm{T}}\boldsymbol{\gamma}\right)/\sigma$. If some of the observations are right censored, then let C and F be the sets of censored and uncensored observations, respectively. If we assume non-informative censoring such that the observed lifetimes and censoring times are independent, then the log-likelihood function for the vector of parameters $(\lambda_3, \lambda_4, \sigma, \boldsymbol{\gamma}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{\theta}$ is given by:

$$
\begin{aligned}
\mathrm{l}(\boldsymbol{\theta}) &= \sum_{i\in F} \log\left(\mathrm{f}_Y\left(y_i\right)\right) + \sum_{i\in C} \log\left(\mathrm{S}_Y\left(y_i\right)\right) \\
&= -r\log\left(2\sigma\right) + \sum_{i\in F} z_i - \sum_{i\in F} \mathrm{e}^{z_i} \\
&\qquad + \sum_{i\in F} \log\left[\lambda_3\left(1-\mathrm{e}^{-\mathrm{e}^{z_i}}\right)^{\lambda_3-1} + \lambda_4\left(\mathrm{e}^{-\mathrm{e}^{z_i}}\right)^{\lambda_4-1}\right] \\
&\qquad + \sum_{i\in C} \log\left(\frac{1}{2}\left[1-\left(1-\mathrm{e}^{-\mathrm{e}^{z_i}}\right)^{\lambda_3} + \left(\mathrm{e}^{-\mathrm{e}^{z_i}}\right)^{\lambda_4}\right]\right)
\end{aligned} \tag{27}
$$

where r is the number of uncensored observations. The MLE $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ can be obtained by maximizing the log-likelihood function in (26) or (27). We use NLMIXED procedure in SAS to obtain the estimate $\hat{\boldsymbol{\theta}}$. Initial values for $\boldsymbol{\gamma}$ and σ are taken from the fit of the Weibull regression model with $\lambda_3 = \lambda_4 = 1$.

Following the same construction, the log-likelihood function of the parameters $(\lambda_3, \lambda_4, \sigma, \boldsymbol{\gamma}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{\tau}$ based on the log(LL-W{GL}) distribution in (24), is given by

$$
\begin{aligned}
\mathrm{l}(\boldsymbol{\tau}) &= -n\log\left(2\sigma\right) + \sum_{i=1}^{n} z_i - 2\sum_{i=1}^{n} \log\left(1+\mathrm{e}^{z_i}\right) \\
&\qquad + \sum_{i=1}^{n} \log\left[\lambda_3\left(\mathrm{e}^{z_i}/\left(1+\mathrm{e}^{z_i}\right)\right)^{\lambda_3-1} + \lambda_4\left(1/\left(1+\mathrm{e}^{z_i}\right)\right)^{\lambda_4-1}\right]
\end{aligned}
$$

if all the observations are uncensored and

$$
\begin{aligned}
1(\boldsymbol{\tau}) = -r\log(2\sigma) + \sum_{i\in F} z_i - 2\sum_{i\in F}\log\left(1+\mathrm{e}^{z_i}\right) \\
+ \sum_{i\in F}\log\left[\lambda_3\left(\mathrm{e}^{z_i}/\left(1+\mathrm{e}^{z_i}\right)\right)^{\lambda_3-1} + \lambda_4\left(1/\left(1+\mathrm{e}^{z_i}\right)\right)^{\lambda_4-1}\right] \\
+ \sum_{i\in C}\log\left(\frac{1}{2}\left[1-\left(\mathrm{e}^{z_i}/\left(1+\mathrm{e}^{z_i}\right)\right)^{\lambda_3} + \left(1/\left(1+\mathrm{e}^{z_i}\right)\right)^{\lambda_4}\right]\right)
\end{aligned}
$$

if some of the observations are censored.

Applications

Fitting Distributions Without Covariates

Two examples are provided to illustrate the flexibility of the U-W{GL} and U-LL{GL} distributions in fitting real data. We recall that the distributions exhibit different hazard shapes. In the first application, we illustrate the applicability of modelling a data set with reflected N-shaped hazard rate function using the U-W{GL}distribution. In the second application, an example of bimodal data is provided comparing the fits of U-W{GL} and U-LL{GL} distributions with other models. The NLMIXED procedure in SAS is used to estimate the parameters.

Kevlar 49/Epoxy Strands Failure Times Data (Pressure at 90%)

In Table 2, the data set (n = 101), which is obtained from Andrews and Herzberg (1985), represents the stress-rupture life in hours of Kevlar 49/epoxy strands when subjected to a constant sustained pressure at the 90% stress level until failure. Al-Aqtash et al. (2014) applied the Gumbel-Weibull (GW) distribution to fit the data,

Table 2. Failure times (in hours) of 101 Kevlar 49/epoxy strands at 90% stress level

0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08,
0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24,
0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60,
0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80,
0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10,
1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51,
1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02,
2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89

and recently, the data was fitted to the exponentiated power generalized Weibull (EPGW) distribution by Peña-Ramírez et al. (2018), and the results were compared to some competitors.

The empirical behavior of the hazard rate function for sample data can be visualized by using the graph of total time on test (TTT). The TTT-plot is obtained through the plot of [*i* / *n*, T(*i* / *n*)] where

$$\mathrm{T}\left(i/n \right) = \frac{\left[\left(\sum_{j=1}^{i} T_{j:n} \right) + (n-i) T_{i:n} \right]}{\sum_{j=1}^{n} T_{j:n}}, \quad i = 1, \ldots, n,$$

and the $T_{j:n}$ are the order statistics of the sample, $j = 1, 2, \ldots, n$. If the observations are generated from a life distribution with constant failure rate; the TTT-plot is approximately straight diagonal line. It is concave (convex) lying above (below) the diagonal line for life distributions with increasing (decreasing) hazard rates. The bathtub-shaped hazard rate corresponds to the TTT-plot first being convex then concave (s-shape). For details about the TTT-plot, we refer the reader to Aarset (1987).

The TTT-plot in Figure 4 indicates decreasing-increasing-decreasing (reflected N-shaped) hazard rate. This suggests that U-W{GL} distribution can fit the failure rates properly; while Weibull distribution cannot.

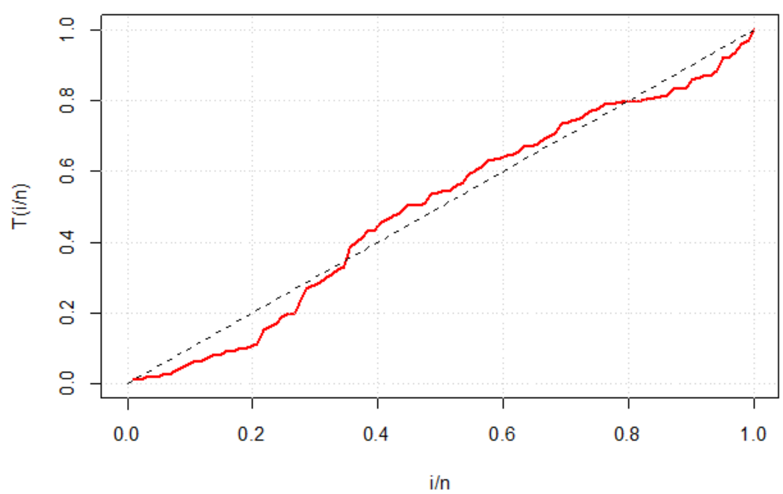
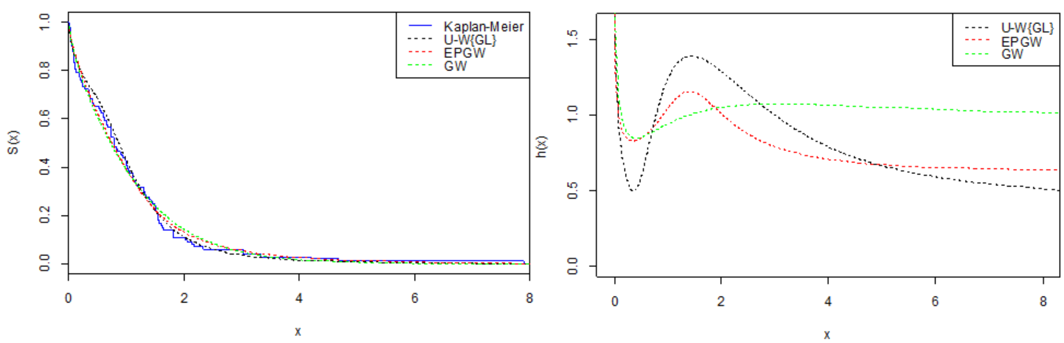


Figure 4. TT plots of Kevlar 49/epoxy strands failure data

Table 3. Parameter estimates and fit statistics for Kevlar 49/epoxy strands failure times data (standard errors in parentheses)

Distribution	U-W{GL}	*GW	EPGW
Parameter estimates	$\hat{\gamma} = 0.1701(0.0851)$	$\hat{\beta} = 1.8064(0.5037)$	$\hat{\alpha} = 0.1666(0.0332)$
	$\hat{c} = 0.6734(0.0716)$	$\hat{\sigma} = 3.2713(0.6459)$	$\hat{\beta} = 0.1201(0.0145)$
	$\hat{\lambda}_3 = 26.3132(22.3672)$	$\hat{\alpha} = 0.9200(0.1594)$	$\hat{\lambda} = 0.1000(0.0738)$
	$\hat{\lambda}_4 = 0.4468(0.1504)$	$\hat{\lambda} = 0.2071(0.1072)$	$\hat{\gamma} = 5.8580(0.1589)$
Log-likelihood	– 98.55	– 100.23	– 99.55
AIC	205.1	208.5	207.1
K-S statistic	0.0576	0.0687	0.0623
p-value	0.8909	0.7266	0.8279



(a) (b)

Figure 5. Kevlar 49/epoxy strands data; (a) estimated survival functions; (b) estimated hazard functions

In order to evaluate the performance, the U-W{GL} distribution is compared with GW and EPGW distributions. All of these distributions have hazard functions with reflected-N shape. The MLEs, the log-likelihood value, the AIC, the Kolmogorov-Smirnov (K-S) statistic, and the *p*-value of the K-S for the fitted distributions are presented in Table 3. The results in Table 3 indicate that all three models fit the data equally well. However, based on the lowest AIC, K-S statistic, and highest log-likelihood value, and as shown in Table 3, the U-W{GL} distribution provides the best fit to the data, followed successively by the EPGW and GW distributions.

The plots of the empirical and estimated survival functions, and the estimated hazard rate functions of the three distributions are depicted in Figure 5. The plot in

Figure 5b reveals that the U-W{GL} hazard rate function has reflected N-shaped, and this is in agreement with the TTT-plot in Figure 4.

Old Faithful Eruption Data

The data set in this application is based on a sample of 222 interval times (in minutes) between eruptions of the Old Faithful Geyser at Yellowstone National Park, Wyoming, USA. The data is measured during August 1978 and August 1979; see Chatterjee et al. (1995) for more details. Four distributions are used to fit the data: the five-parameter U-W{GL} distribution, the four-parameter U-W{GL} distribution, the U-LL{GL} distribution, and GW distribution. All of these distributions have the ability to fit a bimodal data. The MLEs and goodness of fit statistics are given in Table 4, and the estimated PDFs and CDFs are shown in Figure 6. It can be seen that the five-parameter U-W{GL} provides the best fit followed by U-LL{GL} based on all three measures, log-likelihood, AIC and Bayesian Information Criterion (BIC). In this application, adding the location parameter δ to the four-parameter U-W{GL} distribution improves the fit with a big increase in the log-likelihood value and a big decrease in AIC. Moreover, the likelihood ratio test indicates that δ is significantly different from zero (the results are not reported). However, adding location parameter to the U-LL{GL} distribution did not improve the fit in this application, and it might work in other applications.

Based on the log-likelihood, AIC, and BIC, the five-parameter U-W{GL} provides the best fit. However, the four-parameter U-W{GL} provides the best fit based on the K-S statistic and its corresponding p-value.

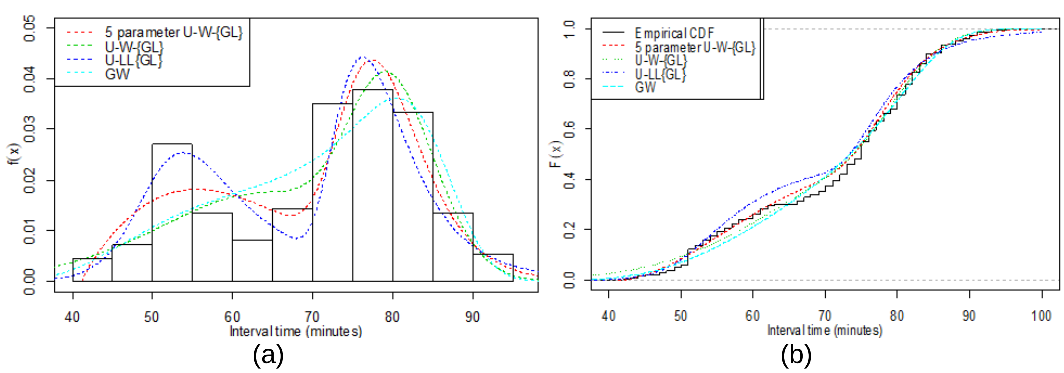


Figure 6. Old Faithful geyser data; (a) estimated PDFs; (b) estimated CDFs

Table 4. Parameter estimates and fit statistics for the Old Faithful geyser data (standard errors in parentheses)

Distribution	5-parameter U-W{GL}	U-W{GL}	U-LL{GL}	GW
Parameter estimates	$\hat{\delta} = 41.2633$	$\hat{\gamma} = 67.7890$	$\hat{\alpha} = 51.2860$	$\hat{\beta} = 1.0464$
	(0.8888)	(2.6828)	(1.1723)	(0.1750)
	$\hat{\gamma} = 16.5181$	$\hat{c} = 6.0178$	$\hat{\beta} = 16.9121$	$\hat{\sigma} = 3.6629$
	(2.7814)	(0.6100)	(1.3172)	(0.4222)
	$\hat{c} = 1.8103$	$\hat{\lambda}_3 = 10.1946$	$\hat{\lambda}_3 = 903.11$	$\hat{\alpha} = 10.3794$
	(0.1963)	(3.9505)	(691.71)	(1.1996)
	$\hat{\lambda}_3 = 59.9887$	$\hat{\lambda}_4 = 1.2695$	$\hat{\lambda}_4 = 0.3552$	$\hat{\lambda} = 69.6984$
	(34.2785)	(0.4612)	(0.07591)	(1.8891)
	$\hat{\lambda}_4 = 0.5842$			
	(0.1520)			
Log-likelihood	– 845.15	– 857.65	–849.95	– 855.90
AIC	1700.3	1723.3	1707.9	1719.8
BIC	1717.3	1736.9	1721.5	1733.4
K-S statistic	0.0729	0.0575	0.0916	0.0703
p-value	0.1885	0.4542	0.0482	0.2226

Regression Models for Modeling Right-Censored Survival Time

Two generalized regression models are provided to fit "Times to Weaning of Breast-Fed Newborns" data ($n = 927$), which contains information regarding mothers who decide to breastfeed their first-born infants (see Klein & Moeschberger, 2003, section 1.14). This data consists of a total of 8 covariates, which include demographic variables: race of mother, education of mother, and age of mother at child's birth, behavioral variables: smoking and alcohol drinking status of mother, and other explanatory variables: year of child's birth, poverty status of mother, and lack of prenatal care status. The duration time of breast feeding (measured in weeks) is the response variable.

There were some censored cases defined by a binary variable (1 if the breast feeding was completed, and 0 if not). According to Klein and Moeschberger (2003), model selection criterion based on the AIC approach suggested that race of mother, smoking status of mother, and poverty are all significant factors related to the response variable. The predictor variable 'race of mother', which has three categories (black, white, and other), is coded by two indicator variables as v_{i1}: black (1 if the mother is black, 0 if otherwise), v_{i2}: white (1 if the mother is white, 0 if otherwise), and the referent group is when the race of mother is neither black nor white. The two binary covariates: smoking and poverty status of mother are coded so that v_{i3}: smoking (1 if smoking at birth of child, 0 otherwise), and v_{i4}: poverty (1 if mother is in poverty, 0 otherwise). We fit the U-W{GL}, U-LL{GL}, Weibull, and log-logistic models to the data. The log linear model is defined as

$$Y_i = \gamma_0 + \gamma_1 v_{i1} + \gamma_2 v_{i2} + \gamma_3 v_{i3} + \gamma_4 v_{i4} + \sigma Z_i, \quad i = 1, 2, \ldots, 927,$$

where Z_i has the appropriate distribution for each of the four models.

Provided in Table 5 are the estimates of the model parameters, the corresponding standard errors and p-values, and AIC and BIC criteria for all four models. Because the Weibull and log-logistic models are nested in the U-W{GL} and U-LL{GL} models, respectively, we use the likelihood ratio test to test the appropriateness of these models.

In the fitted U-W{GL} regression model, the parameter estimates corresponding to v_{i1}, v_{i2}, v_{i3}, and v_{i4} in the log linear model indicate that the duration of breast feeding (1) for non-white mothers is shorter than for white, (2) for mothers who were smoking at birth of child is shorter, and (3) for mothers in poverty is longer. The variable 'Mother is poor' seems to provide more impact on the duration of breast feeding than the variable 'Mother is Black.'

Table 5. Parameter estimates and fit statistics for Times to Weaning of Breast-Fed Newborns data (standard errors in parentheses) and [*p*-values in brackets]

Model	λ_3	λ_3	σ	γ_0	γ_1	γ_2	γ_3	γ_4	**AIC**	**BIC**
U-W{GL}	0.9657	0.1266	0.7181	1.7452	0.0115	0.2475	-0.2890	0.2278	2838.4	2877.1
	(0.0970)	(0.0263)	(0.0403)	(0.1163)	(0.1264)	(0.0992)	(0.0778)	(0.0879)		
	[0.7236]	[<.0001]	[<.0001]	[<.0001]	[0.9276]	[0.0128]	[0.0002]	[0.0097]		
U-LL{GL}	13.2993	1.3008	0.5164	1.2875	0.3737	0.4408	-0.2878	0.0671	2840.3	2879
	(4.4876)	(0.3137)	(0.0193)	(0.1775)	(0.1265)	(0.0938)	(0.0810)	(0.1038)		
	[0.0061]	[0.3376]	[<.0001]	[<.0001]	[0.0032]	[<.0001]	[0.0004]	[0.5180]		
Weibull	1.0000	1.0000	1.0146	2.5737	0.1862	0.3770	-0.3234	0.1587	2855.1	2884.1
			(0.0258)	(0.0884)	(0.1275)	(0.0960)	(0.0768)	(0.0886)		
			[0.5715]	[<.0001]	[0.1448]	[<.0001]	[<.0001]	[0.0737]		
Log logistic	1.0000	1.0000	0.6848	2.0273	0.2522	0.4394	-0.3406	0.0878	2900.1	2929.1
			(0.0188)	(0.1013)	(0.1494)	(0.1116)	(0.0909)	(0.1041)		
			[<.0001]	[<.0001]	[0.0918]	[<.0001]	[0.0002]	[0.3990]		

Table 6. Likelihood ratio tests for the Times to Weaning of Breast-Fed Newborns data

Model	Hypotheses	LR statistic	*p*-value
U-W{GL} vs Weibull	H_0: (λ_3 , λ_4) = (1, 1) vs H_1: H_0 is false	20.7	3.2×10^{-5}
U-LL{GL} vs log-logistic	H_0: (λ_3 , λ_4) = (1, 1) vs H_1: H_0 is false	63.8	< 0.00001

A comparison of the U-W{GL} and U-LL{GL} regression models with their sub-models using likelihood ratio statistics in Table 6 indicates that the extra parameters (λ_3, λ_4) of the U-W{GL} and U-LL{GL} models are jointly significant. Thus, the U-W{GL} and U-LL{GL} models outperform the Weibull and log-logistic models in fitting the duration time of breast feeding. The two extra parameters give the flexibility for fitting real-world data.

Conclusion

The U-R{GL} family of lifetime distributions based on the T-R{GL} families of distributions is proposed. Generalizations to Weibull and log-logistic distributions, namely, the U-W{GL} and U-LL{GL} distributions are introduced and studied. The U-W{GL} can exhibit diverse and more complicated shapes such as N-shape, reflected N-shape, and W-shape hazard rate functions, whereas the U-LL{GL} can exhibit M-shape hazard rate function. These different hazard rate functions provide more flexibility to the U-W{GL} and U-LL{GL} distributions over the Weibull and log-logistic distributions, respectively. Some properties are studied, and regression models based on these distributions are presented. The distributions are applied to fit two real data sets without covariates. The survival models are applied to fit a right censored lifetime data set with covariates. The results show that the flexibility provided by the U-W{GL} and U-LL{GL} models could be very useful in describing different types of lifetime data.

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Equation Section (Next)

Appendix A. Proof for Lemma 1

Proof. When $x \to \infty$, the U-W{GL} and U-LL{GL} density functions go to 0. The U-W{GL} density can be expressed as

$$\mathrm{f}_X(x) = \frac{\lambda_3 c}{2\gamma}\left(\frac{x}{\gamma}\right)^{c-1} \mathrm{e}^{-(x/\gamma)^c}\left(1-\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_3-1} + \frac{\lambda_4 c}{2\gamma}\left(\frac{x}{\gamma}\right)^{c-1}\left(\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_4}. \tag{A1}$$

Using Maclaurin series expansion of the exponential function $\mathrm{e}^x = \sum_{n=1}^{\infty} x^n/n!$, (A1) can be written as

$$\mathrm{f}_X(x) = \frac{\lambda_3 c}{2\gamma}\left(\frac{x}{\gamma}\right)^{c-1} \mathrm{e}^{-(x/\gamma)^c}\left(\left(\frac{x}{\gamma}\right)^{c} - \frac{1}{2!}\left(\frac{x}{\gamma}\right)^{2c} + \frac{1}{3!}\left(\frac{x}{\gamma}\right)^{3c} - \dots\right)^{\lambda_3-1}$$

$$+ \frac{\lambda_4 c}{2\gamma}\left(\frac{x}{\gamma}\right)^{c-1}\left(\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_4}$$

$$= \frac{\lambda_3 c}{2\gamma}\left(\frac{x}{\gamma}\right)^{\lambda_3 c-1} \mathrm{e}^{-(x/\gamma)^c}\left(1 - \frac{1}{2!}\left(\frac{x}{\gamma}\right)^{c} + \frac{1}{3!}\left(\frac{x}{\gamma}\right)^{2c} - \dots\right)^{\lambda_3-1} + \frac{\lambda_4 c}{2\gamma}\left(\frac{x}{\gamma}\right)^{c-1}\left(\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_4}$$

The limit of

$$\left(1 - \frac{1}{2!}\left(\frac{x}{\gamma}\right)^{c} + \frac{1}{3!}\left(\frac{x}{\gamma}\right)^{2c} - \dots\right)^{\lambda_3-1}$$

as $x \to 0$ is 1. The limit of $\mathrm{e}^{-(x/\gamma)^c}$ and $\left(\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_4}$ as $x \to 0$ is also 1. Hence, we have

$$\lim_{x\to 0}\mathrm{f}_X(x) = \lim_{x\to 0}\left(\frac{\lambda_3 c}{2\gamma}\left(\frac{x}{\gamma}\right)^{\lambda_3 c-1} + \frac{\lambda_4 c}{2\gamma}\left(\frac{x}{\gamma}\right)^{c-1}\right)$$

Therefore,

$$\lim_{x\to 0} \mathrm{f}_X(x) = \begin{cases} \infty, & c<1 \text{ or } \lambda_3 c<1 \\ \lambda_4/2\gamma, & c=1 \text{ and } \lambda_3>1 \\ 1/2\gamma, & c>1 \text{ and } \lambda_3 c=1 \\ (1+\lambda_4)/2\gamma, & c=\lambda_3=1 \\ 0, & c>1 \text{ and } \lambda_3 c>1 \end{cases}$$

The U-LL{GL} density can be expressed as

$$\mathrm{g}_X(x) = \frac{\lambda_3}{2}\left(\frac{\left(\beta/\alpha\right)\left(x/\alpha\right)^{\beta-1}}{\left[1+\left(x/\alpha\right)^{\beta}\right]^2}\right)\left(\frac{\left(x/\alpha\right)^{\beta}}{1+\left(x/\alpha\right)^{\beta}}\right)^{\lambda_3-1}$$

$$+\frac{\lambda_4}{2}\left(\frac{\left(\beta/\alpha\right)\left(x/\alpha\right)^{\beta-1}}{\left[1+\left(x/\alpha\right)^{\beta}\right]^2}\right)\left(1-\frac{\left(x/\alpha\right)^{\beta}}{1+\left(x/\alpha\right)^{\beta}}\right)^{\lambda_4-1} \qquad \text{(A2)}$$

$$=\frac{\lambda_3}{2}\frac{\left(\beta/\alpha\right)\left(x/\alpha\right)^{\lambda_3\beta-1}}{\left[1+\left(x/\alpha\right)^{\beta}\right]^{\lambda_3+1}}+\frac{\lambda_4}{2}\left(\frac{\left(\beta/\alpha\right)\left(x/\alpha\right)^{\beta-1}}{\left[1+\left(x/\alpha\right)^{\beta}\right]^2}\right)\left(1-\frac{\left(x/\alpha\right)^{\beta}}{1+\left(x/\alpha\right)^{\beta}}\right)^{\lambda_4-1}$$

The limit of

$$\left(1-\frac{\left(x/\alpha\right)^{\beta}}{1+\left(x/\alpha\right)^{\beta}}\right)^{\lambda_4-1}$$

as $x \to 0$ is 1.. Thus, we have

$$\lim_{x\to 0}\mathrm{g}_X(x)=\lim_{x\to 0}\left(\frac{\lambda_3}{2}\frac{\left(\beta/\alpha\right)\left(x/\alpha\right)^{\lambda_3\beta-1}}{\left[1+\left(x/\alpha\right)^{\beta}\right]^{\lambda_3+1}}+\frac{\lambda_4}{2}\left(\frac{\left(\beta/\alpha\right)\left(x/\alpha\right)^{\beta-1}}{\left[1+\left(x/\alpha\right)^{\beta}\right]^2}\right)\right).$$

Therefore,

$$\lim_{x\to 0} g_X(x)=\begin{cases}\infty, & \beta<1 \text{ or } \lambda_3\beta<1\\ \lambda_4/2\alpha, & \beta=1 \text{ and } \lambda_3>1\\ 1/2\alpha, & \beta>1 \text{ and } \lambda_3\beta=1\\ (1+\lambda_4)/2\alpha, & \beta=\lambda_3=1\\ 0, & \beta>1 \text{ and } \lambda_3\beta>1\end{cases}$$

This completes the proof.

Equation Section (Next)

Appendix B. Proof for Theorem 1

Proof. (i) First, we will find the expected value of $(x / \gamma)^r$:

$$\mathrm{E}\left[\left(\frac{X}{\gamma}\right)^r\right] = \int_0^\infty \left(\frac{x}{\gamma}\right)^r \mathrm{f}_X(x)\,dx$$

$$= \frac{\lambda_3 c}{2\gamma}\int_0^\infty \left(\frac{x}{\gamma}\right)^{r+c-1} \mathrm{e}^{-(x/\gamma)^c}\left(1-\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_3-1} dx \tag{B1}$$

$$+ \frac{\lambda_4 c}{2\gamma}\int_0^\infty \left(\frac{x}{\gamma}\right)^{r+c-1} \left(\mathrm{e}^{-(x/\gamma)^c}\right)^{\lambda_4} dx$$

Using the substitution $t = (x / \gamma)^r$, equation (B1) can be written as

$$\mathrm{E}\left[\left(\frac{X}{\gamma}\right)^r\right] = \frac{\lambda_3}{2}\int_0^\infty t^{r/c}\mathrm{e}^{-t}\left(1-\mathrm{e}^{-t}\right)^{\lambda_3-1} dt + \frac{\lambda_4}{2}\int_0^\infty t^{r/c}\mathrm{e}^{-\lambda_4 t}dt \tag{B2}$$

Using the substitution $u = \lambda_4 t$, integral I_2 in equation (B2) becomes

$$I_2 = \frac{\lambda_4^{-r/c}}{2}\int_0^\infty u^{r/c}\mathrm{e}^{-u}du = \frac{\lambda_4^{-r/c}}{2}\Gamma\left(\frac{r}{c}+1\right).$$

By applying the generalized binomial expansion to integral I_1 in equation (B2), we obtain

$$I_1 = \frac{\lambda_3}{2}\int_0^\infty t^{r/c}\mathrm{e}^{-t}\sum_{k=0}^\infty \frac{(-1)^k\,\Gamma(\lambda_3)\mathrm{e}^{-kt}}{k!\Gamma(\lambda_3-k)}dt = \frac{\lambda_3}{2}\sum_{k=0}^\infty \frac{(-1)^k\,\Gamma(\lambda_3)}{k!\Gamma(\lambda_3-k)}\int_0^\infty t^{r/c}\mathrm{e}^{-(k+1)t}dt$$

Using the substitution $u = (k + 1)t$, integral I_1 becomes

$$I_1 = \frac{\lambda_3}{2}\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(\lambda_3)(k+1)^{-((r/c)+1)}}{k!\,\Gamma(\lambda_3-k)}\int_0^{\infty}u^{r/c}\mathrm{e}^{-u}du$$
$$= \frac{\lambda_3}{2}\Gamma(\lambda_3)\Gamma\left(\frac{r}{c}+1\right)\sum_{k=0}^{\infty}\frac{(-1)^k\,(k+1)^{-((r/c)+1)}}{k!\,\Gamma(\lambda_3-k)}$$

By adding I_1 and I_2, and considering $\mathrm{E}[(X / \gamma)^r] = \gamma^r\mathrm{E}[X^r]$, then we have

$$\mathrm{E}\left(X^r\right) = \frac{\gamma^r}{2}\Gamma\left(\frac{r}{c}+1\right)\left[\lambda_4^{-r/c} + \lambda_3\Gamma(\lambda_3)\sum_{k=0}^{\infty}\frac{(-1)^k\,(k+1)^{-((r/c)+1)}}{k!\,\Gamma(\lambda_3-k)}\right]. \quad \text{(B3)}$$

The summation in (B3) will be finite when $\lambda_3 > 1$ and an integer, and the summation will stop at $\lambda_3 - 1$. Thus, equation (B3) becomes

$$\mathrm{E}\left(X^r\right) = \frac{\gamma^r}{2}\Gamma\left(\frac{r}{c}+1\right)\left[\lambda_4^{-r/c} + \lambda_3\sum_{k=0}^{\lambda_3-1}(-1)^k\binom{\lambda_3-1}{k}(k+1)^{-((r/c)+1)}\right]$$

(ii)

$$\mathrm{E}\left[X^r\right] = \int_0^{\infty}x^r\,\mathrm{g}_X(x)dx$$
$$= \frac{\lambda_3}{2}\int_0^{\infty}x^r\left(\frac{(\beta/\alpha)(x/\alpha)^{\beta-1}}{\left[1+(x/\alpha)^{\beta}\right]^2}\right)\left(\frac{(x/\alpha)^{\beta}}{1+(x/\alpha)^{\beta}}\right)^{\lambda_3-1}dx \quad \text{(B4)}$$
$$+\frac{\lambda_4}{2}\int_0^{\infty}x^r\left(\frac{(\beta/\alpha)(x/\alpha)^{\beta-1}}{\left[1+(x/\alpha)^{\beta}\right]^2}\right)\left(1-\frac{(x/\alpha)^{\beta}}{1+(x/\alpha)^{\beta}}\right)^{\lambda_4-1}dx$$

Using the substitution $u = (x / \alpha)^{\beta} / [1 + (x / \alpha)^{\beta}]$, then equation (B4) can be written as

$$\begin{aligned}\mathrm{E}\left[X^r\right] &= \frac{\lambda_3}{2}\alpha^r\int_0^1 u^{\lambda_3+\frac{r}{\beta}-1}(1-u)^{-\frac{r}{\beta}}\,du + \frac{\lambda_4}{2}\alpha^r\int_0^1 u^{-\frac{r}{\beta}}(1-u)^{\lambda_4+\frac{r}{\beta}-1}\,du \\ &= \frac{\alpha^r}{2}\left[\lambda_3\,\mathrm{B}\left(\lambda_3+\frac{r}{\beta},1-\frac{r}{\beta}\right)+\lambda_4\,\mathrm{B}\left(1+\frac{r}{\beta},\lambda_4-\frac{r}{\beta}\right)\right]\end{aligned}$$

This completes the proof.