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Pairwise Balanced Designs From Cyclic PBIB Designs

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A pairwise balanced designs was constructed using cyclic partially balanced incomplete block designs with either $(\lambda_1 - \lambda_2) = 1$ or $(\lambda_2 - \lambda_1) = 1$. This method of construction of Pairwise balanced designs is further generalized to construct it using cyclic partially balanced incomplete block design when $|(\lambda_1 - \lambda_2)| = p$. The methods of construction of pairwise balanced designs was supported with examples. A table consisting parameters of Cyclic PBIB designs and its corresponding constructed pairwise balanced design is also included.

Keywords: Partially balanced incomplete block designs, group divisible design, triangular design, cyclic design, pairwise balanced design

Introduction

Bose and Shrikhande (1959 A, B, C) introduced a general class of Latin square designs called pairwise balanced designs of index λ blocks. It differs from BIB designs in the sense that it does not have constant set size. Bose and Shrikhande (1960) used the pairwise balanced design in the context of constructing Mutually Orthogonal Latin Square (MOLS). In a pairwise balanced block design, any two treatments form a pair within the blocks equally often.

Definition: An arrangement of v treatments in b blocks is defined as pair-wise balanced design of index λ of type $(v; k_1, k_2, k_3, \ldots, k_m)$ provided

1. Each set contains $(k_1, k_2, k_3, \ldots, k_m)$ symbols that are all distinct.

2. $k_i \leq v$; $k_i \neq k_j$ and

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3. Every pair of distinct treatments occurs in exactly λ sets of the design along with parametric relations

$$b = \sum_{i=1}^{m} b_i \text{ and } \lambda v(v-1) = \sum_{i=1}^{m} b_i k_i (k_i - 1)$$

A characterization of pairwise balanced design in terms of NN' matrix can be expressed in the following way:

A block design D is called pairwise balanced design if all the off diagonal elements of NN' matrix are same (constant), i.e.

$$NN' = (r - \lambda)I_v + \lambda 11' \quad (1)$$

where I_v is an identity matrix and 1 is a vector of $(v \times 1)$. (1) can be rewritten as $NN' = Diag(r - \lambda) + \lambda 11'$.

Let D be a pairwise balanced design, the design found by the b_i set of size k_i is called the i^{th} equi-block component D_i of the pairwise balanced design D. Moreover a class of subsets of D_i is said to be of type I if every treatment occurs in that class exactly k_i times. A class of subsets of D_i is said to be of type II if each treatment occurs exactly once. This clarifies that the number of sets in a class of type-I is v and of type-II is v / k_i.

The concept of pairwise balanced design in block design is of combinatorial interest to Bose and Shrikhande (1960) in the context of constructing MOLS. They developed a method of construction of pairwise balanced design using BIBD and GD designs. Consider a BIBD with parameters v, b, r, k, λ, then by omitting any one treatment from all those blocks where present, gives a pairwise balanced design $(v - 1; k, k - 1)$. Omitting any x treatments $(2 \leq x \leq k)$ occurring in the same block, a pairwise balanced design $(v - 1; k, k - 1, k - x)$ of index unity is obtained. Further, they obtained pairwise balanced design $(v - 3; k, k - 1, k - 2)$ of index unity by omitting three treatments not occurring in the same set of a BIB design $(v; k)$. However, if BIB design $(v; k)$ is a resolvable design, then by adding a new symbol t_i to each set of the i^{th} replication $(i = 1, 2, 3, \ldots, x; 1 < x \leq r)$and adding a new set of symbols $(t_1, t_2, t_3, \ldots, t_i)$, a pairwise balanced design $(v + k; k + 1, k, x)$ with index unity is obtained provided $x < r$. However if $x = r$, then other class of pairwise balanced design $(v + r, k + 1, r)$ is obtained.

Consider a GD design $(v; k, n; 0, 1)$, where the basic parameters are $v = mn, b, r, k, \lambda_1 = 0, \lambda\_2 = 1$. By adding a new symbol θ_i, to each set of the i^{th}

replicate ($x \leq r$) and adjoining a set of new symbols if $x > 1$ to the GD design (v; k, n; 0, 1), Bose and Shrikhande (1960) obtained the pairwise balanced design of the following types:

1. Pairwise balanced design ($v + x$; $k + 1$, k, n) if $x = 1$
2. Pairwise balanced design ($v + x$, $k + 1$, k, n, x) if $1 < x < r$
3. Pairwise balanced design ($v + x$, $k + 1$, n, x) if $x = r$

Ghosh and Desai (2014) obtained pairwise balanced designs from triangular PBIBD with $v = 10$. Let $\lambda_1 = 0$ and $\lambda_2 \geq 1$, now by adding corresponding group of association scheme to the Triangular Design, they constructed pairwise balanced design (v; k, $k = n_1 + 1$). They also obtained pairwise balanced design using $TD(v = 10; k; \lambda_1 = 0, \lambda_2 \geq 1)$. Furthermore by adding corresponding group of association scheme to the $TD(v = 10; k; \lambda_1 \geq 1, \lambda_2 = 0)$, they obtained another series of pairwise balanced design (v; k, $k = n_1 + 1$). Pairwise balanced designs are applicable for combinatorial interest in block designs. Several incomplete block designs can be constructed using pairwise balanced designs.

In the current study, pairwise balanced designs are obtained from a cyclic PBIBD. Let $\lambda_1 - \lambda_2 = p$, and if a corresponding group of association scheme is added to the cyclic PBIBD, the result is a pairwise balanced design (v; k, where $k_1 = k$ and $k_2 = n_1 + 1$ or $n_2 + 1$).

Method of Construction: Consider a pairwise balanced design using cyclic PBIBD with either $|\lambda_1 - \lambda_2| = 1$ or $\lambda_1 - \lambda_2| = p$

Pairwise Balanced Designs through Cyclic Designs with $|\lambda_1 - \lambda_2| = 1$

First, construct a cyclic design with $|\lambda_1 - \lambda_2| = 1$ and then using this design construct pairwise balanced designs. This is shown in Theorem 1.

Theorem 1: Consider a cyclic design D_1 with parameters v_1, b_1, r_1, k_1, λ_1, λ_2, n_1, n_2 with $\lambda_1 \geq 1$ and $\lambda_2 \geq 2$ having $|(\lambda_2 - \lambda_1)| = 1$. Consider another cyclic design D_2 whose parameters are $v_2 = v_1$, $b_2 = v_1$, $r_2 = (n_1 + 1)$, $k_2 = n_1 + 1$, λ_{11}, λ_{12}. By adding blocks of D_1 with blocks of D_2 design, a Pairwise Balanced Design is obtained with parameters $v = v_1 = v_2$, $b = b_1 + b_2$, $r = r_1 + (n_1 + 1)$, $k = (k_1,(n_1 + 1))$ and $\lambda = \frac{1}{(v-1)}\left\{r_1\left(k_1-1\right)+r_2\left(k_2-1\right)\right\}$.

Proof: Consider cyclic design whose parameters are v_1, b_1, r_1, k_1, λ_1, λ_2, n_1, n_2 with $|(\lambda_2 - \lambda_1)| = 1$. Call this design D_1 whose incidence matrix is N_1. Note $n_1 = n_2$ for any cyclic design. Consider all those treatments of first associate class with t_i. Arrangement of these treatments in the t_i blocks will give another cyclic design. Call this design D_2 whose incidence matrix is N_2. Then by adding b_1 blocks of cyclic design D_1 with the b_2 blocks of design D_2 we obtain an incomplete block design. Further, the concurrence matrix of the resulting design shows that all the off diagonal elements are same which is given by $\lambda = \frac{1}{(v-1)}\left\{r_1\left(k_1-1\right)+r_2\left(k_2-1\right)\right\}$ and diagonal elements are another constant which happens to be $r = r_1 + (n_1 + 1)$. The given design is Pairwise Balanced Design with two unequal block sizes, whose parameters are $v = v_1 = v_2$, $b = b_1 + b_2$, $r = r_1 + (n_1 + 1)$, $k = (k_1,(n_1 + 1))$ and $\lambda = \frac{1}{(v-1)}\left\{r_1\left(k_1-1\right)+r_2\left(k_2-1\right)\right\}$, as per Appendix A.

Example 1: Consider a C19 design with parameters $v_1 = 13$, $b_1 = 39$, $r_1 = 9$, $k_1 = 3$, $\lambda_1 = 1$, $\lambda_2 = 2$, $n_1 = 6$, $n_2 = 6$. Call this design D_1. Note $|(\lambda_2 - \lambda_1)| = 1$. The 39 blocks of C19 are as follows:

1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	1	2	3	4	5	6	7	8	9	10	11
13	1	2	3	4	5	6	7	8	9	10	11	12

3	4	5	6	7	8	9	10	11	12	13	1	2
10	11	12	13	1	2	3	4	5	6	7	8	9
13	1	2	3	4	5	6	7	8	9	10	11	12

4	5	6	7	8	9	10	11	12	13	1	2	3
9	10	11	12	13	1	2	3	4	5	6	7	8
13	1	2	3	4	5	6	7	8	9	10	11	12

The first associate treatments of t_i along with treatment t_i are following:

1	2	3	4	5	6	7	8	9	10	11	12	13
3	4	1	2	3	1	1	1	1	2	3	1	2
6	7	5	6	7	4	2	2	2	3	4	4	5
7	8	8	9	10	8	5	3	3	4	5	5	6
8	9	9	10	11	11	9	6	4	5	6	6	7
9	10	10	11	12	12	12	10	7	8	9	7	8
12	13	11	12	13	13	13	13	11	12	13	10	11

Consider column as a block. Call this design D_2 with parameters $v_2 = 13$, $b_2 = 13$, $r_2 = 7$, $k_2 = 7$, $\lambda_{11} = 4$, $\lambda_{12} = 3$.

Now adding blocks of D_2 with the blocks of design D_1, we obtain incomplete design with two unequal block sizes. The 52 blocks with 13 treatments of this design are following

1	2	3	4	5	6	7	8	9	10	11	12	13
12	13	1	2	3	4	5	6	7	8	9	10	11
13	1	2	3	4	5	6	7	8	9	10	11	12

3	4	5	6	7	8	9	10	11	12	13	1	2
10	11	12	13	1	2	3	4	5	6	7	8	9
13	1	2	3	4	5	6	7	8	9	10	11	12

4	5	6	7	8	9	10	11	12	13	1	2	3
9	10	11	12	13	1	2	3	4	5	6	7	8
13	1	2	3	4	5	6	7	8	9	10	11	12

1	2	3	4	5	6	7	8	9	10	11	12	13
3	4	1	2	3	1	1	1	1	2	3	1	2
6	7	5	6	7	4	2	2	2	3	4	4	5
7	8	8	9	10	8	5	3	3	4	5	5	6
8	9	9	10	11	11	9	6	4	5	6	6	7
9	10	10	11	12	12	12	10	7	8	9	7	8
12	13	11	12	13	13	13	13	11	12	13	10	11

This incomplete design is a pairwise balanced design with parameters $v = 13$, $b = 52$, $r = 16$, $k = (3,7)$, $\lambda = 5$. In Table 1, the parameters of cyclic PBIB design will be shown along with parameters of pairwise balanced designs as v, $b = b_1 + b_2$, $r = r_1 + r_2$, k = (k_1, k_2) and λ.

Table 1. Table shows the parameters of Cyclic designs and corresponding Pairwise Balanced Designs.

	Parameters of Cyclic Designs								*Parameters of PBD*							
	v	***b***	***r***	***k***	$\boldsymbol{\lambda_1}$	$\boldsymbol{\lambda_2}$	$\boldsymbol{n_1}$	$\boldsymbol{n_2}$	***v***	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{r_1}$	$\boldsymbol{r_2}$	$\boldsymbol{k_1}$	$\boldsymbol{k_2}$	$\boldsymbol{\lambda}$
C_1	5	5	2	2	1	0	2	2	5	5	5	2	3	2	3	2
C_2	5	10	4	2	2	0	2	2	5	10	10	4	6	2	3	4
C_3	5	15	6	2	3	0	2	2	5	15	15	6	3	2	3	6
C_4	5	20	8	2	4	0	2	2	5	20	20	8	12	2	3	8
C_5	5	25	10	2	5	0	2	2	5	25	25	10	15	2	3	10
C_6	5	15	6	2	2	1	2	2	5	15	5	6	3	2	3	3
C_7	5	20	8	2	3	1	2	2	5	20	10	8	6	2	3	5
C_8	5	25	10	2	4	1	2	2	5	25	15	10	9	2	3	7
C_9	5	25	10	2	3	2	2	2	5	25	5	10	3	2	3	4
C_{10}	13	39	6	2	1	0	6	6	13	39	13	6	7	2	7	4
C_{11}	17	68	8	2	1	0	8	8	17	68	17	8	9	2	9	5
C_{12}	5	5	3	3	2	1	2	2	5	5	5	3	3	3	3	3
C_{13}	5	10	8	3	4	2	2	2	5	10	10	8	6	3	3	7
C_{14}	5	15	9	3	6	3	2	2	5	15	15	9	9	3	3	9
C_{15}	5	15	9	3	5	4	2	2	5	15	5	9	3	3	3	6
C_{16}	13	13	3	3	1	0	6	6	13	13	13	3	7	3	7	4
C_{17}	13	26	6	3	2	0	6	6	13	26	26	6	14	3	7	8
C_{18}	13	39	9	3	3	0	6	6	13	39	39	9	21	3	7	12
C_{19}	13	39	9	3	1	2	6	6	12	39	13	9	7	3	7	5
C_{20}	37	111	9	3	1	0	18	18	37	111	37	9	19	3	19	10
C_{21}	13	26	8	4	1	3	6	6	13	26	26	8	14	4	7	9
C_{22}	17	34	8	4	1	2	8	8	17	34	17	8	8	4	8	5
C_{23}	13	13	6	6	3	2	6	6	13	13	13	6	7	6	7	6
C_{24}	13	13	7	7	4	3	6	6	13	13	13	7	7	7	7	7
C_{25}	29	29	7	7	2	1	14	14	29	29	29	7	15	7	15	9
C_{26}	17	17	8	8	4	3	8	8	17	17	17	8	9	8	9	8
C_{27}	29	29	8	8	3	1	14	14	29	29	58	8	30	8	15	17
C_{28}	17	17	9	9	5	4	8	8	17	17	17	9	9	9	9	9
C_{29}	13	13	10	10	8	7	6	6	13	13	13	10	7	10	7	11

Note that $r = r_1 + r_2$ and $b = b_1 + b_2$ for pairwise balanced designs.

Pairwise Balanced Designs through Cyclic Designs with $|\lambda_1 - \lambda_2| = p$

The method discussed in Theorem 1 is extended for $|(\lambda_1 - \lambda_2)| = p$ here. Construct a cyclic design with $|(\lambda_1 - \lambda_2)| = p$ and then, using this design, construct pairwise balanced designs. This is shown in Theorem 2.

Theorem 2: Consider a cyclic design D_1 with parameters v_1, b_1, r_1, k_1, λ_1, λ_2, n_1, n_2 with $|(\lambda_1 - \lambda_2)| = p$, where $p = 1, 2, 3, \ldots$. Consider the treatments of second associate class of this design D_1 with t_i. Arranging these treatments in t_i blocks another cyclic design will be obtained. Next repeat this cyclic design p times and call this design D_2 whose parameters are $v_2 = v_1$, $b_2 = pv_1$, $r_2 = p(n_2 + 1)$, $k_2 = n_2 + 1$, λ_{11}, λ_{12}. By adding blocks of D_1 to blocks of D_2 design, we obtain a Pairwise Balanced Design with parameters

$$v = v_1 = v_2, b = b_1 + pv_1,$$
$$r = r_1 + (n_2 + 1)p, k = (k_1, (n_2 + 1)), \text{ and}$$
$$\lambda = \frac{1}{(v-1)}\left\{r_1(k_1 - 1) + r_2(k_2 - 1)\right\}$$

Proof: Consider a cyclic design whose parameters are v_1, b_1, r_1, k_1, λ_1, λ_2, n_1, n_2 with $|(\lambda_1 - \lambda_2)| = p$. Let call this design D_1 whose incidence matrix is N_1. Consider the treatments of second associate class of this design D_1 with t_i. Arrange these treatments in t_i blocks. This gives another cyclic design. Call this design D_0. Further, take p times of design D_0, call this design D_2 whose incidence matrix is N_2. Then by adding b_1 blocks of cyclic design D_1 with the b_2 blocks of design D_2 we obtain an incomplete block design. Further the concurrence matrix of the resulting design shows that all the off diagonal elements are same which is given by $\lambda = \frac{1}{(v-1)}\left\{r_1(k_1 - 1) + r_2(k_2 - 1)\right\}$ and diagonal elements are another constant which happens to be $r = r_1 + (n_2 + 1)p$. Hence, the resulting design is Pairwise Balanced Design with two unequal block sizes, whose parameters are $v = v_1 = v_2$, $b = b_1 + pv_1$, $r = r_1 + (n_2 + 1)p$, $k = k_1, (n_2 + 1))$, and $\lambda = \frac{1}{(v-1)}\left\{r_1(k_1 - 1) + r_2(k_2 - 1)\right\}$ (as per Appendix A).

Example 2: Consider a C7 design with parameters $v_1 = 5$, $b_1 = 20$, $r_1 = 8$, $k_1 = 2$, $\lambda_1 = 3$, $\lambda_2 = 1$, $n_1 = 2$, $n_2 = 2$. Call this design D_1. Note $|(\lambda_1 - \lambda_2)| = 2$. The blocks of C7 are as follows:

1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	2	3	4	5	1

The second associate treatments of t_i along with treatment t_i are following

1	2	3	4	5
2	1	2	3	1
5	3	4	5	4

Consider column as a block and then repeat it 2 times. Call this design D_2 with parameters $v_2 = 5$, $b_2 = 10$, $r_2 = 6$, $k_2 = 3$, $\lambda_{11} = 2$, $\lambda_{12} = 4$.

Now adding D_2 with the design D_1, we obtain incomplete design with two unequal block sizes. The blocks of this design are following

1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	2	3	4	5	1

1	2	3	4	5	1	2	3	4	5
2	1	2	3	1	2	1	2	3	1
5	3	4	5	4	5	3	4	5	4

This incomplete design is a pairwise balanced design with parameters $v = 5$, $b = 30$, $r = 14$, $k = (2,3)$, $\lambda = 5$. Note that $b = b_1 + pv_1 = b_1 + b_2$.

Remarks

It is possible to construct another cyclic design separately from each of the first associate and second associate treatment along with i^{th} treatment of a cyclic design. That is, from one cyclic design one can construct another two cyclic designs. For both the cyclic designs $v = b$ and $r = k$ will be same however value of λ_1 and λ_2 will be different. The number of blocks will be smaller than the original cyclic design but r and k will be same as of original design. The difference is that if the cyclic design is constructed using first associate treatment along with t_i then all the second associate treatment will occur λ_2 times with treatment t_i. Similarly, if the cyclic design is constructed using second associate treatment along with t_i then all the first associate treatment will occur λ_2 times with treatment t_i. However other parameters along with p_{ij}^k matrix remain same.

If blocks of cyclic design constructed from first associate treatments of original design are added with the blocks of the cyclic design obtained from second associate treatments of the original design, a balanced incomplete block design is always constructed.

Note for the cyclic design $n_1 = n_2$, λ will be same for both theorems.

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Appendix A

We know that

$$\lambda v(v-1) = \sum_{i=1}^{m} b_i k_i (k_i - 1) \quad \text{or, } \lambda = \frac{b_1 k_1 (k_1 - 1)}{v(v-1)} + \frac{b_2 k_2 (k_2 - 1)}{v(v-1)}$$

$$\text{or, } \lambda = \frac{vr_1(k_1 - 1)}{v(v-1)} + \frac{vr_2(k_2 - 1)}{v(v-1)} \quad \text{or, } \lambda = \frac{r_1(k_1 - 1)}{(v-1)} + \frac{r_2(k_2 - 1)}{(v-1)}$$

$$\therefore \lambda = \frac{1}{(v-1)}\left\{r_1(k_1 - 1) + r_2(k_2 - 1)\right\}$$