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In this paper, estimations problem (point and interval) are studied for the logarithmic transformed exponential (LTE) distribution based on an adaptive progressive Type-II (APTII) censoring scheme. In point estimation; the maximum likelihood estimate (MLE) and Bayesian estimates (BE) based on squared error and Linex loss functions are obtained. The Markov Chain Monte Carlo (MCMC) technique is used to compute Bayes estimates by using gamma prior for the unknown parameter under two different loss functions. Metropolis-Hasting (M-H) algorithm has been applied to generate MCMC samples from the posterior density. In case of interval estimation; asymptotic confidence interval (CI) and two different bootstrap CI's namely; Boot-p and Boot-t are obtained. Also, we have computed posterior credible interval for the unknown parameter. The performance of these estimates are studied on the basis of their risks. Finally, a real dataset has been taken for different censoring schemes and times under APTII censoring.

Keywords: Adaptive progressive censoring, Bayes estimation, Loss function, Metropolis-Hastings algorithm, Simulated risk.

Introduction

In life testing and reliability studies, the experimenter may not have always information about the failure times for all the experimental items. In such cases, censored data have been obtained to the experimenter. Also, one of the major reason for applying censoring concept is that it reduces the total time on test and its associated costs. A censoring scheme is said to be best if it can balance between number of observed failure items, total time on test and the efficiency of the inferential procedure based on the results of the experiment. The schemes are of different types according to the problem of experimenter. As, Type-I censoring (or time censoring) is used when the life testing experiment will be terminated at the predefined time T and Type-II censoring (or failure censoring) is the most common censoring schemes where the life testing experiment is terminated at the predefined number of failures r observed. See Ng et al. (2006), Balakrishnan et al. (2007), Abd-Elfattah et al. (2008), Abdel-Hamid and AL-Hussaini (2009), Kundu and Howlader (2010), Joarder et al. (2011), Dey and Kundu (2012), Han and Kundu (2015) and Prajapati et al. (2018) etc. for more details about these two schemes.

Since, the time and failure censoring scheme do not have the flexibility of allowing the removal of items at points other than the terminal point, so that a more general censoring scheme has been introduced by Balakrishnan and Sandhu(1995) and known as progressive censoring. The progressive Type-II censoring scheme can be outlined in sort as: consider an experiment in which n items are put on life testing experiment and at the time of first failure, R_1 number of items are randomly removed from the remaining $n-1$ surviving items. Similarly, at the time of second failure, R_2 items are randomly removed from the remaining $n-2-R_1$ items. The test continues until the m^{th} failure, at which time, all the remaining $R_m = n-m-R_1-R_2-\ldots-R_{m-1}$ items are removed. Here, $(R_1, R_2, \ldots, R_m)$ is call removal scheme and all $R_i's$ (with $R_i > 0 \& \sum_{i=1}^{m} R_i + m = n$) and m are prefixed values, where these completely observed (ordered) lifetimes are by $X_{i:m:n}^{R_1,R_2,\ldots,R_m}$; $i = 1,2,\ldots,m$. For the luxury, we denote it as $X_{i:m:n}$ so, the progressively Type-II censored sample is $x_{1:m:n} < x_{2:m:n} < \ldots x_{m:m:n}$. One can follow Aggarwala and Balakrishnan(1998), Raqab et al. (2010), El-Din and Shafay (2013) and Ng and Chan (2007) etc. for more details.

Recently,Ng et al. (2007) suggested an adaptive progressive Type-II censoring scheme, which is almost mixture of Type-I and Type-II censoring schemes, described in the following Section. Since, lifetime distributions are used to describe the real life circumstances in various fields of life like; medical, engineering etc. There are lots of lifetime distributions available in the statistical literature to describe such types of situations. Weibull and exponential distributions are very useful lifetime distributions having some interesting properties. In the context of increasing flexibility in distributions, many generalization or transformation methods are available in the literature based on some baseline distribution. Recently, Maurya et al.(2016) proposed an one parameter lifetime distribution (named as LTE distribution) having increasing failure rate based on exponential distribution and discussed its statistical properties along with its suitability over other models.

A random variable X is said to have LTE distribution, if its distribution function is given by

$$F(x) = 1 - \frac{\log(1+e^{-\theta x})}{\log 2}; \quad x > 0, \theta > 0 \tag{1}$$

and its associated probability density function (pdf) is:

$$f(x) = \frac{\theta e^{-\theta x}}{(1+e^{-\theta x})\log 2}; \quad x > 0, \theta > 0 \tag{2}$$

and hazard rate is:

$$h(x) = \frac{\theta e^{-\theta x}}{(1+e^{-\theta x})\log(1+e^{-\theta x})} \tag{3}$$

where, θ is the parameter of the model.

The rest of the paper is organized as follows. In Section 2, we describe the model for APTII censoring scheme. Parametric point estimation under classical and Bayesian paradigm are presented in Section 3. Section 4, introduce the interval estimation for the parameter in both context as in classical, asymptotic confidence interval, Boot-p and Boot-t are obtained and in Bayesian set-up highest posterior density (HPD) interval are obtained. Section 6, presented MCMC simulation for the results. Numerical example using real dataset is given in the Section 6 and finally conclusions of the whole paper are summarised in Section 7.

Adaptive progressive Type-II (APTII) censoring and its likelihood function

Let n items are put on life testing experiment and suppose that $x_1, x_2, \ldots, x_n$ be their corresponding lifetimes. Here, we assume that $x_i; i = 1,2, \ldots, n$ are i.i.d. random samples with probability density function $f_X(x)$, where $x > 0$ and observed progressively Type-II censored sample as $x_{1:m:n} < x_{2:m:n} < \ldots < x_{m:m:n}$.

As illustrious byBurkschat (2008), it is assume that a progressive censoring plan has a longer test time than a conventional Type-II censoring plan in terms of gain in efficiency. The values of removals at the time of failures, depends on the objective of the experiment. Kundu and Joarder (2006) studied Type-II progressive hybrid censoring for a fixed removal scheme $(R_1, R_2, \ldots, R_m)$. But the disadvantage of this censoring scheme is that effective sample size is random and it can be possible to get a very small number even it may be zero also. So that the procedures of statistical inference will not be applicative or they will carry low efficiency. Hence, Ng et al. (2009), Cramer and Iliopoulos (2010) and El-Din et al. (2018) theorize an APTII censoring scheme. According to this scheme; the value of sample size m and the removal scheme $(R_1, R_2, \ldots, R_m)$ is fixed in advance and may change appropriately at the time of experiment. Also, the ideal time on test T is decided before the experiment but we may allow to run the experiment over the time T.

If the m^{th} progressively censored observed failure occurs before time $T(i.e. X_{m:m:n} < T)$, the experiment stops at time $X_{m:m:n}$. Otherwise, once the experimental time passes given time T but the number of failures have not reached m, we would want to put an end to the experiment as early as possible. This setting can be viewed as a design in which we are assured of getting m observed failure times for efficiency of statistical inference and at the same time, the total time on test will not be too far away from the ideal time T. From the basic properties of order statistics (say, for example, David and Nagaraja(2003)), we know that, the fewer operating items are withdrawn (i.e., the larger to the number of items on the test), the smaller the expected total time on test (Ng and Chan (2007)).

Therefore, if we want to close the experiment as soon as possible for fixed value of m, then we should leave as many surviving items on the test as possible.

Suppose j is the number of failures observed before time T, i.e.

$X_{j:m:n} < T < X_{j=1:m:n}, j = 0,1,\ldots,m,$

where $X_{0:m:n} \equiv 0$ and $X_{m+1:m:n} \equiv \infty$. After the experiment crossed the time T, we set $R_{j+1} = \ldots = R_{m-1} = 0$ and $R_m = n - m - \sum_{i=1}^{j} R_i$.

Now if $j = 0$ i.e. no failures have been occurred before time T, then APTII censoring became Type-II censoring (failure censoring).

If $j = m$ i.e. all failures occurred before time T, then it become progressive Type-II censoring.

After discussing the APTII censoring scheme, the likelihood function under this scheme can be written as

$$L(\theta|J=j) = d_j \left[\prod_{i=1}^{m} f(x_{i:m:n})\right]\left\{\prod_{i=1}^{m} [1-F(x_{i:m:n})]^{R_i}\right\}[1-F(x_{i:m:n})]^{(n-m-\sum_{i=1}^{j} R_i)} \tag{4}$$

where

$$d_j = \prod \left[n - i + 1 - \sum_{k=1}^{max(i-1,j)} R_k\right]. \tag{5}$$

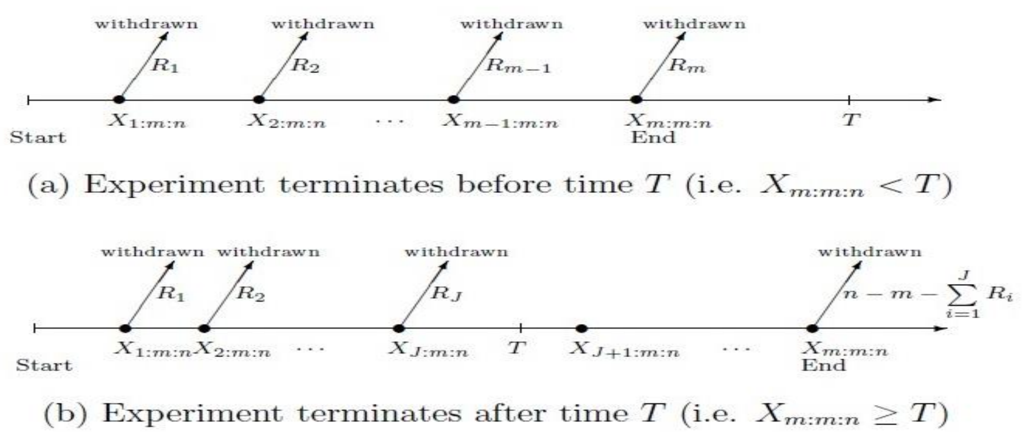

Figure 1: Schematic representation of adaptive progressive Type -II censoring

Estimation of the parameter

This section deals with estimation procedure for the LTE distribution in presence of APTII under classical and Bayesian set-up. In classical set-up, maximum likelihood estimation (MLE) is used whereas in Bayesian framework, gamma prior under two different loss function namely; squared error and Linex are used. The procedures are discussed one by one in the following subsections.

Maximum likelihood estimation

Let $\underline{x} = (x_{1:m:n} < x_{2:m:n} < \ldots < x_{m:m:n})$ be an APTII censored ordered statistics from LTE distribution, with censoring scheme $R = (R_1, R_2, \ldots, R_m)$. From equation (4), the logarithmic of the likelihood function of LTE distribution can be written as

$$\log L(\theta|J=j) = \log(d_j) + \sum_{i=1}^{m} \log\left(\frac{\theta e^{-\theta x_{i:m:n}}}{(1+e^{-\theta x_{i:m:n}})\log 2}\right) + \sum_{i=1}^{j} R_i \log\left(\frac{\log(1+e^{-\theta x_{i:m:n}})}{\log 2}\right) + \left(n-m-\sum_{i=1}^{j} R_i\right)\log\frac{\log\left(1+e^{-\theta x_{m:m:n}}\right)}{\log 2} \qquad (6)$$

After differentiating equation (6) with respect to the parameter θ and equating to zero, we get non linear equation in terms of θ and it cannot be solved analytically for θ. Hence, we need some numerical methods such as Newton's method can be employed to solve and get the MLE of the parameter θ.

Bayesian estimation

In this subsection, we provide the Bayesian inference for θ, when we have the APTII censored data as explained in Figure 1. Before proceeding further, we make selection for the prior distribution of the parameter; see Berger and Sun (1993), Raqab and Madi (2005), Singh et al. (2016). Here, we have assumed that the parameter θ is independent gamma variate, having pdf

$$\Pi(\theta) \propto \theta^{a-1} e^{-\theta b}. \qquad (7)$$

Where the hyper parameters (a,b) are assumed to be known and can be evaluated following the method suggested bySingh et al. (2013). We compute the Bayes estimator for the unknown parameter under the squared error and Linex loss function. Using the prior given in equation (7) and the likelihood function given in equation (6), the logarithmic of posterior density for the LTE is

$$\log \Pi(\theta|x) \propto L(\theta|J=j) + \log\Pi(\theta)$$

$$= (a-1)\log(\theta) - b\theta + \log(d_j) + \sum_{i=1}^{m} \log\left(\frac{\theta e^{-\theta x_{i:m:n}}}{(1+e^{-\theta x_{i:m:n}})\log 2}\right)$$

$$+\sum_{i=1}^{j} R_i \log\left(\frac{\log(1+e^{-\theta x_{i:m:n}})}{\log 2}\right) + (n - m - \sum_{i=1}^{j} R_i)\log\frac{\log(1+e^{-\theta x_{m:m:n}})}{\log 2}. \quad (8)$$

Since, the Bayesian estimation technique is based on the type of loss function. Here, we are considering two different types of loss functions, one is symmetric type of loss function named squared error loss function (SELF) and second is asymmetric loss function named Linex loss function(LLF).

The SELF is defined as $L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$, where $\hat{\theta}$ is the Bayes estimator of θ and under the SELF, the Bayes estimator is nothing but the posterior mean. The Linex loss function is defined as $L(\hat{\theta}, \theta) = e^{c(\hat{\theta}-\theta)} - c(\hat{\theta} - \theta) - 1$.

where $\delta \neq 0$. Here, constant c determines the shape of the loss function. For small values of c, the behaviour of Linex loss function is approximately same as symmetric loss function. Under the Linex loss function, the Bayes estimator is given by $\hat{\theta}_{BL} = -\frac{1}{c}\log(E_\theta(e^{-c\theta}|x))$.

Therefore, the posterior expectation of any function of θ say $g(\theta)$ is

$$E(g(\theta|x)) = \frac{\int_0^\infty g(\theta) l(\underline{x};\theta) \times \Pi(\theta)\, \partial\theta}{\int_0^\infty l(\underline{x};\theta) \times \Pi(\theta)\, \partial\theta}. \quad (9)$$

It is not possible to compute equation (9) analytically. Therefore, we propose to use the approaches of MCMC technique to approximate equation (9).

The MCMC technique provides an alternative method for parameter estimation. It is more flexible when compared with the traditional methods.

Metropolis-Hastings algorithm

A general way of constructing a MCMC sample is by Metropolis-Hastings (M-H) algorithm. Suppose, we want to simulate from a posterior density $g(\theta|data)$. M-H algorithm starts with an initial value θ^0 and specifies a rule for simulating the t^{th} value in the sequence θ^t given the $(t-1)^{st}$ value in the sequence θ^{t-1}, this rule consists of a proposal density which simulates a candidate value θ^* and acceptance probability. This algorithm can be described as follows:

1. Set the initial guess of parameter θ, say θ^0 from $U(0,1)$.
2. Simulate a candidate value θ^* from a proposal density $p(\theta^*|\theta^{t-1})$.
3. Compute the ratio $r = \frac{g(\theta^*)p(\theta^{t-1}|\theta^*)}{g(\theta^{t-1})p(\theta^*|\theta^{t-1})}$.
4. Compute acceptance probability $P = min\{r, 1\}$.
5. Sample a value θ^t such that $\theta^t = \theta^*$ with probability P; otherwise $\theta^t = \theta^{t-1}$.
6. Repeat the above steps for a long time N.

A problem for the choice of initial guess in the proposed algorithm are arises. We proposed to use classical estimate $\hat{\theta}$ obtained in subsection 3.1 as guess for MCMC process. Also, the choice of variance is an important point; one can follow Ntzoufras(2011) for more details. Another choice for variance would be the Fisher information criterion. Here, the meaning of acceptance rate, is the proportion of times a new set of values is generated at the iteration stages. For more details seeGelman et al. (1995) (pp. 334-335).

Interval estimation

This section deals with classical and Bayesian confidence intervals. In classical framework we compute, asymptotic confidence interval and Boot confidence intervals. In Bayesian framework HPD interval for parameter is obtained. A details discussion is given in subsequent subsections.

Asymptotic confidence interval

The $100(1-\alpha)\%$ confidence interval for the parameter θ can be obtained with the asymptotic normality of the maximum likelihood estimators with their variances estimated from the inverse of the observed fisher's information matrix. Thus, $100(1-\alpha)\%$ CI for for θ is

$$\left[\hat{\theta}_M - Z_{\alpha/2} * Var(\hat{\theta}_M), \quad \hat{\theta}_M + Z_{\alpha/2} * Var(\hat{\theta}_M)\right] \tag{10}$$

where $Z_{\alpha/2}$ is the right tailed probability with standard normal distribution.

Bootstrap confidence interval

Interval based on the asymptotic property or the normal theory assumption do not perform good for small samples. Through the use of bootstrap, one can obtain the accurate intervals without having normal theory assumption. The bootstrap methods make computer-based adjustments to the standard interval endpoints that are guaranteed to improve the coverage accuracy by an order of magnitude, at least asymptotically. Here, we discussed two types of CIs by using bootstrap method, percentile bootstrap (Boot-p) suggested by Efron(1982) and Studentised bootstrap (Boot-t), suggested by Hall (1988).

Parametric Boot-p

An algorithm for the Boot-p CI is as follows:

(1) Assemble the APTII censored data x and obtain MLE for the parameter θ, denoted as $\hat{\theta}_M$.

(2) Generate an APTII censored sample by using MLE of the parameter based on pre-specified progressive censoring scheme $(R_1, R_2, \ldots, R_m)$.

(3) Generate B bootstrap samples from the above generated sample.

(4) Obtain MLE for each B bootstrap sample, denoted as $\{\hat{\theta}^*{}_1, \hat{\theta}^*{}_2, \ldots, \hat{\theta}^*{}_B\}$.

(5) Arrange these in ascending orders as $\{\hat{\theta}^*{}_{(1)}, \hat{\theta}^*{}_{(2)}, \ldots, \hat{\theta}^*{}_{(B)}\}$.

A pair of $100(1-\alpha)\%$ Boot-p CI for θ is given by

$$\left[\hat{\theta}^*{}_{(B(\alpha/2))}, \hat{\theta}^*{}_{(B(1-\alpha/2))}\right]. \tag{11}$$

Parametric Boot-t

Boot-p is very simple algorithm, though, if the sample size is small then percentile approach is not so much accurate. Thus, Boot-t, Studentized bootstrap approach can be used, as it gives more accuracy than percentile approach. The following steps are in the algorithm of Boot-t CIs.

(5) Repeat step (1)-(4) as in Boot-p approach.

(6) Compute standard errors of the parameters also, denoted as $\{\hat{s}e_1{}^*(\theta), \hat{s}e_2{}^*(\theta), \ldots, \hat{s}e_B{}^*(\theta)\}$.

(7) Compute statistics $z_B^*(\theta) = \frac{\hat{\theta}^*{}_B - \hat{\theta}_M}{\hat{s}e_B{}^*(\theta)}$, for each $b = 1,2, \ldots, B$.

(8) Arrange $z_B^*(\theta)$ in ascending orders, denoted as $z_{(B)}^*(\theta)$.

A pair of $100(1-\alpha)\%$ Boot-t CI for θ is given by

$$\left[\hat{\theta}_M - z_{(B(1-\alpha/2))}^*(\theta) * \hat{s}e(\theta), \quad \hat{\theta}_M - z_{(B(\alpha/2))}^*(\theta) * \hat{s}e(\theta), \right]. \tag{12}$$

Highest posterior density interval

The HPD credible interval (see Chen and Shao(1999)) of the parameter θ is obtained on the basis of ordered MCMC samples of θ as $\theta_{(1)}, \theta_{(2)}, \cdots, \theta_{(N)}$. After that $100(1-\alpha)\%$ credible interval for the parameter θ is obtained as $((\theta_{(1)}, \theta_{[(1-\alpha)N]+1}), \cdots, (\theta_{[N\alpha]}, \theta_N)$.) Where $[Y]$ denotes the largest integer less than or equal to Y. After that, HPD credible interval for θ is that interval for which length is shortest.

Simulation study

In order to compare the different estimators (MLE and BE's) of the parameter, we use the Monte Carlo simulation study in the following way:

1. For a given prior parameters (a, b), we generate θ from the prior density.
2. For a given θ obtained in previous step, we generate APTII censored sample from the LTE distribution with the values of parameter $\theta = 0.5, 0.8$ & 1.2 and time $T = 0.25, 0.5$ & 1 for different censoring schemes.
3. The MLE of θ computed by solving the non linear equation (6).
4. The BE for the parameter θ under squared error and Linex loss functions using MCMC method are given in equation (8).
5. The squared deviations $(\theta^* - \theta)^2$ are computed for different combinations of m, n, T and censoring schemes, where (*) stands for an estimate (MLE or BE's) and θ is actual value of the parameter.
6. The above steps are repeated 10,000 times. The estimated risks are computed by averaging the squared deviations.

In simulation study, we have used three different types of censoring schemes and the dropping schemes in the following manner: the first scheme considers the number of dropping of items zero in the earlier stages and high at the last stage. Contrary to it, second scheme is the number of dropping items is high at the first stage and zero in the remaining stages. The third scheme is constructed so that the number of dropping items at all the intermediate stages to be equal and high. It may be worthwhile to mention here that the number of dropping items are random and we are generating the APTII censored data from the complete sample data, therefore we can study the average performance of the estimators. In order to choose different censoring schemes "c*d" represents that the number "c" is repeated "d" times.

We mainly compare the performance of MLE and Bayes estimates with respect to SELF and LLF in terms of estimated risks. We also compare different confidence intervals, namely the confidence intervals obtained by using asymptotic distributions of the MLE, bootstrap confidence intervals (Boot-t and Boot-p) and the HPD intervals. Also, we have obtained absolute bias, mean squared errors (MSEs).

To find the Bayes MCMC estimates, we used the informative gamma prior for the parameter θ by using hyper parameters $(a, b) = (1,1)$. Also, we compute the Bayes estimates and 95% probability intervals based on 10,000 MCMC samples (discard the first 500 values as burn-in). We report the credible interval based on 10,000 replications. The results are summarized in Tables 2-7. In these tables, $\hat{\theta}_M$ denotes the MLE of θ, $\hat{\theta}_{BS}$ denotes the Bayes estimate under SELF, $\hat{\theta}_{BL}$ denotes the Bayes estimate under LLF, $Risk_S$ denotes the Bayes risk under SELF and $Risk_L$ denotes the Bayes risk under LLF.

Table 2-4, gives the simulation result of various estimates for different values of the true parameter as $\theta = 0.5, 0.8$&1.2 receptively. And in each table, we have reported

MLE, absolute bias, MSE, Bayes estimates and risks. All these simulations have been done for $n = 15, 25\&50$ and m= $5,15,20\&30$ with combinations $(n, m) = (15,5), (25,5), (25,15), (25,20), (50,15)\&(50,30)$ for each three proposed censoring schemes and prefixed time T= $0.25,0.5\&1$. The findings of these tables are given below:

1. The Bayes estimators are better than the MLE in terms of risks.
2. The risk of estimators decreases for the early removal, intermediate removal and late removal censoring scheme respectively.
3. From the Table 2-4, we can observe that the risk is minimum for the Linex loss function in case of Bayesian estimation.
4. The risks of the estimators (classical and Bayesian) decreases as we increases the time.
5. The risk of estimators decreases as the number of observations m increase.

Table 5-7, gives the simulation results of confidence intervals of the parameter θ. In this table, conf stands for the asymptotic confidence interval, HPD stands for HPD intervals, pboot stands for Boot-p and tboot stands for Boot-t confidence intervals. Also, the index 1 represent lower CI and 2 for upper CI. The conclusions of the tables are itemized as:

1. The lengths of the intervals are in decreasing order as Boot-p, asymptotic, Boot-t and HPD CIs respectively.
2. From Table 5-7, we observed that the length of the HPD is minimum among rest of the intervals (asymptotic, Boot-p & Boot-t).
3. Also, the length of intervals are going shorter for the intermediate removals, early removals and late removal censoring scheme respectively.
4. CIs are independent from prefixed time T.

Real Data Analysis

In this section, we considered a real dataset (taken by Stollmack and Harris(1974)) on persons released directly from correctional institutions to Parole in the district of Columbia. This dataset consist 61 failure times in days which are given as 0.01, 0.06, 0.09, 0.29, 0.30, 0.34, 0.39, 0.41, 0.44, 0.45, 0.49, 0.56, 0.84, 0.89, 0.91, 1.00, 1.03, 1.04, 1.15, 1.19, 1.24, 1.38, 1.41, 1.46, 1.56, 1.62, 1.68, 1.83, 1.85, 1.98, 2.09, 2.17, 2.17, 2.28, 2.33, 2.38, 2.41, 2.52, 2.58, 2.71, 2.75, 2.76, 2.79, 2.82, 3.05, 3.13, 3.29, 3.31, 3.34, 3.36, 3.36, 3.62, 3.84, 4.04, 4.08, 4.22, 4.38, 4.41, 4.65, 4.86, 5.56. This real life data is analysed using LTE(θ). The MLE based on complete sample is obtained as $\hat{\theta}_M = 0.5707$, reliability under MLE $\hat{R}_M(t = 1)$ = 0.6463. Bayes estimator of θ under SELF $\hat{\theta}_{BS} = 0.5724$, reliability estimate under SELF $\hat{R}_{BS}(t = 1) = 0.6454$, under LLF $\hat{\theta}_{BL} = 0.5722$, $\hat{R}_{BL}(t = 1) = 0.6455$. An APTII censored sample of size m = 15, 30 and 50 are selected randomly from n = 61 with different types of APTII censoring scheme and different time T and the results are shown in the Table 1.

Table 1: Estimate of θ under different schemes and different times

T = 0.05					T = 5				
m	Scheme	$\hat{\theta}_M$	$\hat{\theta}_{BS}$	$\hat{\theta}_{BL}$	m	Scheme	$\hat{\theta}_M$	$\hat{\theta}_{BS}$	$\hat{\theta}_{BL}$
15	0*14, 46*1	0.4139	0.4318	0.4313	15	0*14, 46*1	0.4139	0.4248	0.4243
	46*1, 0*14	0.0645	0.0686	0.0686		46*1, 0*14	0.0981	0.1030	0.1029
	1*4, 4*9, 3*2	0.0884	0.0937	0.0936		1*4, 4*9, 3*2	0.1668	0.1727	0.1726
30	0*29, 31*1	0.4324	0.4379	0.4377	30	0*29, 31*1	0.4324	0.4394	0.4391
	31*1, 0*29	0.1455	0.1477	0.1477		31*1, 0*29	0.2433	0.2490	0.2489
	3*5, 1*3, 0*14, 1*3, 2*5	0.1973	0.1997	0.1996		3*5, 1*3, 0*14, 1*3, 2*5	0.3111	0.3144	0.3143
50	0*49, 11*1	0.5194	0.5176	0.5174	50	0*49, 11*1	0.5194	0.5180	0.5178
	11*1, 0*49	0.3355	0.3376	0.3375		11*1, 0*49	0.4751	0.4840	0.4838
	0*15, 1*5, 0*9, 1*6, 0*15	0.3625	0.3687	0.3686		0*15, 1*5, 0*9, 1*6, 0*15	0.4734	0.4782	0.4781

Conclusion

An adaptive progressive Type-II censoring suggested by Ng et al. (2009)saves both the total time on test and cost induced by failure of the items and increase the efficiency of statistical analysis. In this article, we have considered the MLE and Bayes estimates for the parameter of LTE distribution using APTII censoring scheme. Also, we develop different confidence intervals namely asymptotic confidence interval, bootstrap (Boot-p and Boot-t) confidence intervals and highest posterior density intervals for the parameter of the LTE distribution. A simulation study has been conducted to examine and compare the performance of proposed methods for different sample sizes, different times and different censoring schemes. From the results obtained in Tables 2-7, it can be concluded that the performance of the Bayes estimators are more better than the MLE. At last, a real dataset is considered to show the applicability of APTII scheme.

Table 2: Simulation results for the different estimation methods when θ = 0.5

n	m	Schemes	Time	$\hat{\theta}_M$	Bias($\hat{\theta}_M$)	MSE	$\hat{\theta}_{BS}$	$\hat{\theta}_{BL}$	$Risk_S$	$Risk_L$
15	5	0*4, 10*1	0.25	0.3156	0.3408	0.1561	0.2454	0.2444	0.1455	0.0007
		10, 0*4		0.5613	0.1947	0.0816	0.3654	0.3638	0.0823	0.0005
		2*5		0.6829	0.3019	0.1942	0.4098	0.4081	0.0640	0.0003
25	5	0*4, 20	0.25	0.3402	0.4036	0.2339	0.3092	0.3072	0.2018	0.0010
		20, 0*4		0.5475	0.1814	0.0777	0.2970	0.2957	0.1100	0.0006
		4*5		0.7602	0.4126	0.3256	0.3895	0.3877	0.0878	0.0004
25	15	0*14, 10	0.25	0.2179	0.2849	0.0885	0.2014	0.2013	0.0950	0.0005
		10, 0*14		0.5107	0.0999	0.0172	0.4588	0.4581	0.0139	0.0002
		0*2, 1*10, 0*3		0.4589	0.1463	0.0314	0.3899	0.3895	0.0211	0.0001
25	20	0*19, 5	0.25	0.3114	0.1918	0.0426	0.3044	0.3042	0.0438	0.0002
		5, 0*19		0.5134	0.0883	0.0133	0.5008	0.5002	0.0130	0.0001
		0*7, 1*5, 0*8		0.4668	0.0913	0.0125	0.4504	0.4499	0.0109	0.0001
50	15	0*14, 35	0.25	0.0902	0.4109	0.1719	0.0902	0.0902	0.1704	0.0008
		35, 0*14		0.4940	0.0962	0.0153	0.4109	0.4104	0.0121	0.0003
		1*5, 3*10		0.4553	0.2635	0.1056	0.3132	0.3129	0.0411	0.0002
50	30	0*29, 20	0.25	0.1666	0.3336	0.1137	0.1670	0.1670	0.1126	0.0006
		20, 0*29		0.5041	0.0673	0.0075	0.4928	0.4924	0.0036	0.0001
		0*5, 1*20, 0*5		0.3875	0.1401	0.0236	0.3705	0.3703	0.0204	0.0001
15	5	0*4, 10*1	0.5	0.4069	0.3464	0.1813	0.3968	0.3945	0.1534	0.0008
		10, 0*4		0.5341	0.1742	0.0731	0.4303	0.4284	0.0664	0.0004
		2*5		0.5742	0.2504	0.1411	0.4641	0.4617	0.0986	0.0005
25	5	0*4, 20	0.5	0.5421	0.3200	0.1872	0.5483	0.5448	0.1535	0.0008
		20, 0*4		0.4889	0.1639	0.0610	0.3583	0.3568	0.0569	0.0005
		4*5		0.5991	0.3083	0.1812	0.4837	0.4810	0.1182	0.0006
25	15	0*14, 10	0.5	0.2072	0.2936	0.0911	0.2130	0.2129	0.0875	0.0004
		10, 0*14		0.5124	0.0995	0.0174	0.5096	0.5088	0.0159	0.0001
		0*2, 1*10, 0*3		0.3839	0.1421	0.0254	0.3779	0.3775	0.0235	0.0001
25	20	0*19, 5	0.5	0.3046	0.1982	0.0445	0.3112	0.3110	0.0420	0.0002
		5, 0*19		0.5161	0.0872	0.0135	0.5222	0.5216	0.0097	0.0001

		0*7, 1*5, 0*8		0.4264	0.0978	0.0131	0.4313	0.4309	0.0122	0.0001
50	15	0*14, 35	0.5	0.1442	0.3899	0.1575	0.1490	0.1489	0.1536	0.0008
		35, 0*14		0.4812	0.0997	0.0161	0.4726	0.4719	0.0125	0.0001
		1*5, 3*10		0.3251	0.2118	0.0524	0.3107	0.3104	0.0447	0.0002
50	30	0*29, 20	0.5	0.1701	0.3299	0.1108	0.1738	0.1737	0.1084	0.0005
		20, 0*29		0.5057	0.0673	0.0075	0.5107	0.5104	0.0056	0.0000
		0*5, 1*20, 0*5		0.3503	0.1516	0.0260	0.3545	0.3543	0.0245	0.0001
15	5	0*4, 10*1	1	0.5751	0.2675	0.1594	0.5863	0.5829	0.1219	0.0006
		10, 0*4		0.5118	0.1903	0.1018	0.4948	0.4924	0.0457	0.0004
		2*5		0.5791	0.2697	0.1658	0.5642	0.5609	0.1266	0.0006
25	5	0*4, 20	1	0.6253	0.2363	0.1459	0.6418	0.6378	0.1124	0.0006
		20, 0*4		0.4482	0.1977	0.0975	0.4173	0.4154	0.0902	0.0004
		4*5		0.5691	0.2532	0.1705	0.5651	0.5617	0.1204	0.0006
25	15	0*14, 10	1	0.2423	0.2744	0.0832	0.2507	0.2505	0.0789	0.0004
		10, 0*14		0.5115	0.0980	0.0173	0.5222	0.5214	0.0113	0.0001
		0*2, 1*10, 0*3		0.3773	0.1504	0.0299	0.3868	0.3864	0.0273	0.0001
25	20	0*19, 5	1	0.3157	0.1889	0.0416	0.3238	0.3236	0.0388	0.0002
		5, 0*19		0.5189	0.0855	0.0128	0.5276	0.5270	0.0088	0.0001
		0*7, 1*5, 0*8		0.4048	0.1174	0.0182	0.4129	0.4125	0.0167	0.0001
50	15	0*14, 35	1	0.4752	0.1651	0.0497	0.4862	0.4854	0.0492	0.0002
		35, 0*14		0.4774	0.1035	0.0170	0.4876	0.4870	0.0123	0.0001
		1*5, 3*10		0.4034	0.1892	0.0467	0.4131	0.4126	0.0444	0.0002
50	30	0*29, 20	1	0.1853	0.3147	0.1020	0.1892	0.1891	0.0997	0.0005
		20, 0*29		0.5048	0.0672	0.0076	0.5109	0.5106	0.0055	0.0000
		0*5, 1*20, 0*5		0.3490	0.1536	0.0272	0.3543	0.3542	0.0257	0.0001

Table 3: Simulation results for the different estimation methods when θ = 0.8

n	m	Schemes	Time	$\hat{\theta}_M$	Bias ($\hat{\theta}_M$)	MSE	$\hat{\theta}_{BS}$	$\hat{\theta}_{BL}$	$Risk_S$	$Risk_L$
15	5	0*4, 10*1	0.25	0.5803	0.5497	0.4477	0.5145	0.5105	0.3378	0.0017
		10, 0*4		0.8775	0.2928	0.2185	0.6319	0.6276	0.2154	0.0011
		2*5		0.9647	0.4167	0.4099	0.6794	0.6747	0.1907	0.0009
25	5	0*4, 20	0.25	0.7574	0.5805	0.5232	0.7004	0.6939	0.3532	0.0017
		20, 0*4		0.8257	0.2732	0.2499	0.5261	0.5227	0.2298	0.0013
		4*5		1.0156	0.5375	0.5671	0.6749	0.6697	0.2402	0.0012
25	15	0*14, 10	0.25	0.3213	0.4707	0.2348	0.3301	0.3298	0.2323	0.0011
		10, 0*14		0.8235	0.1631	0.0474	0.7882	0.7863	0.0396	0.0003
		0*2, 1*10, 0*3		0.6255	0.2295	0.0688	0.5999	0.5989	0.0617	0.0003
25	20	0*19, 5	0.25	0.4883	0.3152	0.1128	0.4914	0.4909	0.1100	0.0005
		5, 0*19		0.8265	0.1356	0.0317	0.8185	0.8171	0.0250	0.0002
		0*7, 1*5, 0*8		0.7013	0.1491	0.0312	0.6942	0.6932	0.0300	0.0001
50	15	0*14, 35	0.25	0.1668	0.6452	0.4244	0.1729	0.1727	0.4150	0.0020
		35, 0*14		0.7802	0.1564	0.0392	0.7315	0.7299	0.0561	0.0004
		1*5, 3*10		0.5600	0.3521	0.1535	0.4910	0.4903	0.1139	0.0006
50	30	0*29, 20	0.25	0.2676	0.5324	0.2884	0.2720	0.2719	0.2836	0.0014
		20, 0*29		0.8104	0.1097	0.0201	0.8071	0.8062	0.0147	0.0001
		0*5, 1*20, 0*5		0.5747	0.2347	0.0631	0.5724	0.5719	0.0604	0.0003
15	5	0*4, 10*1	0.5	0.8714	0.4903	0.4668	0.8209	0.8137	0.2728	0.0013
		10, 0*4		0.8278	0.2912	0.2396	0.7213	0.7162	0.1832	0.0009
		2*5		0.9327	0.4522	0.4992	0.8123	0.8053	0.2621	0.0013
25	5	0*4, 20	0.5	0.9746	0.3841	0.3624	0.9330	0.9248	0.2007	0.0010
		20, 0*4		0.7358	0.2917	0.1857	0.6221	0.6181	0.1496	0.0009
		4*5		0.8984	0.4121	0.3465	0.8082	0.8015	0.2132	0.0010
25	15	0*14, 10	0.5	0.3524	0.4541	0.2219	0.3622	0.3618	0.2126	0.0010
		10, 0*14		0.8228	0.1614	0.0459	0.8230	0.8210	0.0434	0.0002
		0*2, 1*10, 0*3		0.5957	0.2357	0.0701	0.6007	0.5997	0.0657	0.0003
25	20	0*19, 5	0.5	0.4983	0.3079	0.1091	0.5067	0.5062	0.1008	0.0005
		5, 0*19		0.8280	0.1382	0.0339	0.8307	0.8293	0.0319	0.0002

		0*7, 1*5, 0*8		0.6511	0.1764	0.0410	0.6569	0.6560	0.0384	0.0002
50	15	0*14, 35	0.5	0.5866	0.4350	0.2414	0.5871	0.5856	0.2316	0.0011
		35, 0*14		0.7630	0.1650	0.0431	0.7639	0.7622	0.0307	0.0002
		1*5, 3*10		0.5629	0.3278	0.1318	0.5655	0.5644	0.1222	0.0006
50	30	0*29, 20	0.5	0.2856	0.5145	0.2710	0.2906	0.2905	0.2660	0.0013
		20, 0*29		0.8048	0.1095	0.0196	0.8074	0.8065	0.0189	0.0001
		0*5, 1*20, 0*5		0.5513	0.2507	0.0711	0.5565	0.5560	0.0684	0.0003
15	5	0*4, 10*1	1	0.9803	0.3595	0.3256	0.9400	0.9321	0.1799	0.0009
		10, 0*4		0.8524	0.3819	0.4119	0.8068	0.8006	0.2113	0.0010
		2*5		0.9700	0.4061	0.4084	0.9144	0.9066	0.2188	0.0011
25	5	0*4, 20	1	1.0064	0.3684	0.3619	0.9638	0.9553	0.1842	0.0009
		20, 0*4		0.7102	0.3781	0.3459	0.6828	0.6781	0.1227	0.0010
		4*5		0.8973	0.3814	0.3511	0.8638	0.8567	0.1886	0.0009
25	15	0*14, 10	1	0.5744	0.3698	0.1642	0.6012	0.5999	0.1509	0.0008
		10, 0*14		0.8186	0.1571	0.0439	0.8221	0.8202	0.0231	0.0002
		0*2, 1*10, 0*3		0.6655	0.2572	0.0993	0.6997	0.6981	0.0894	0.0004
25	20	0*19, 5	1	0.5463	0.2778	0.0947	0.5543	0.5536	0.0896	0.0004
		5, 0*19		0.8344	0.1390	0.0335	0.8371	0.8357	0.0218	0.0002
		0*7, 1*5, 0*8		0.7073	0.2003	0.0573	0.7118	0.7106	0.0536	0.0003
50	15	0*14, 35	1	0.8566	0.1707	0.0543	0.8575	0.8552	0.0487	0.0002
		35, 0*14		0.7548	0.1663	0.0432	0.7607	0.7591	0.0306	0.0002
		1*5, 3*10		0.7570	0.2053	0.0691	0.7603	0.7585	0.0628	0.0003
50	30	0*29, 20	1	0.4224	0.4268	0.2006	0.4269	0.4265	0.1961	0.0010
		20, 0*29		0.8119	0.1113	0.0204	0.8146	0.8137	0.0197	0.0001
		0*5, 1*20, 0*5		0.6181	0.2243	0.0636	0.6224	0.6218	0.0610	0.0003

Table 4: Simulation results for the different estimation methods when θ = 1.2

n	m	Schemes	Time	$\hat{\theta}_M$	Bias ($\hat{\theta}_M$)	MSE	$\hat{\theta}_{BS}$	$\hat{\theta}_{BL}$	$Risk_S$	$Risk_L$
15	5	0*4, 10*1	0.25	1.0835	0.8315	1.0835	0.9266	0.9161	0.5891	0.0029
		10, 0*4		1.2606	0.4233	0.4988	0.9567	0.9478	0.3691	0.0018
		2*5		1.3711	0.6305	0.9546	1.0389	1.0280	0.3871	0.0019
25	5	0*4, 20	0.25	1.3876	0.6581	0.8872	1.2104	1.1958	0.4003	0.0019
		20, 0*4		1.1509	0.4069	0.3970	0.8247	0.8176	0.4319	0.0021
		4*5		1.3799	0.6823	0.9315	1.0601	1.0484	0.3944	0.0019
25	15	0*14, 10	0.25	0.4892	0.7007	0.5202	0.5111	0.5103	0.5068	0.0025
		10, 0*14		1.2311	0.2378	0.0972	1.1920	1.1880	0.0668	0.0005
		0*2, 1*10, 0*3		0.8841	0.3455	0.1506	0.8804	0.8782	0.1500	0.0007
25	20	0*19, 5	0.25	0.7323	0.4722	0.2526	0.7367	0.7355	0.2466	0.0012
		5, 0*19		1.2408	0.2086	0.0752	1.2206	1.2175	0.0471	0.0003
		0*7, 1*5, 0*8		0.9997	0.2445	0.0804	0.9915	0.9895	0.0795	0.0004
50	15	0*14, 35	0.25	0.4861	0.8814	0.8226	0.4815	0.4800	0.7877	0.0038
		35, 0*14		1.1340	0.2452	0.0945	1.0921	1.0887	0.0741	0.0006
		1*5, 3*10		0.7766	0.5041	0.3013	0.7540	0.7521	0.2812	0.0014
50	30	0*29, 20	0.25	0.4131	0.7869	0.6315	0.4187	0.4185	0.6226	0.0030
		20, 0*29		1.2109	0.1654	0.0440	1.2003	1.1982	0.0413	0.0002
		0*5, 1*20, 0*5		0.8287	0.3740	0.1575	0.8288	0.8279	0.1559	0.0008
15	5	0*4, 10*1	0.5	1.4503	0.5767	0.7817	1.2775	1.2628	0.3014	0.0015
		10, 0*4		1.2579	0.5096	0.8124	1.0835	1.0726	0.3327	0.0016
		2*5		1.4281	0.6362	0.9840	1.2220	1.2080	0.3610	0.0018
25	5	0*4, 20	0.5	1.5184	0.5709	0.9553	1.3267	1.3109	0.2777	0.0013
		20, 0*4		1.0636	0.5144	0.6227	0.9323	0.9238	0.3501	0.0017
		4*5		1.3375	0.5649	0.8296	1.1774	1.1643	0.3029	0.0015
25	15	0*14, 10	0.5	0.6750	0.6334	0.4568	0.6747	0.6731	0.4299	0.0021
		10, 0*14		1.2303	0.2387	0.1000	1.2072	1.2031	0.0822	0.0004
		0*2, 1*10, 0*3		0.9379	0.3622	0.1797	0.9306	0.9280	0.1653	0.0008
25	20	0*19, 5	0.5	0.7751	0.4416	0.2301	0.7786	0.7773	0.2232	0.0011
		5, 0*19		1.2445	0.2058	0.0735	1.2271	1.2240	0.0628	0.0003

		0*7, 1*5, 0*8		0.9805	0.2994	0.1194	0.9754	0.9733	0.1133	0.0006
50	15	0*14, 35	0.5	1.2457	0.2860	0.1632	1.2151	1.2105	0.1342	0.0007
		35, 0*14		1.1418	0.2525	0.1016	1.1246	1.1210	0.0690	0.0004
		1*5, 3*10		1.0494	0.3863	0.2173	1.0317	1.0282	0.1919	0.0009
50	30	0*29, 20	0.5	0.4755	0.7312	0.5572	0.4806	0.4802	0.5487	0.0027
		20, 0*29		1.2159	0.1641	0.0460	1.2059	1.2039	0.0419	0.0002
		0*5, 1*20, 0*5		0.8575	0.3578	0.1506	0.8579	0.8568	0.1485	0.0007
15	5	0*4, 10*1	1	1.4791	0.5341	0.7083	1.3091	1.2942	0.2509	0.0012
		10, 0*4		1.3188	0.6095	0.9915	1.1684	1.1560	0.3390	0.0017
		2*5		1.4804	0.6086	1.0495	1.2901	1.2751	0.3177	0.0015
25	5	0*4, 20	1	1.4972	0.5442	0.8359	1.3186	1.3031	0.2546	0.0012
		20, 0*4		1.0814	0.5896	0.7077	0.9904	0.9810	0.3415	0.0017
		4*5		1.3289	0.5596	0.7928	1.1865	1.1734	0.2786	0.0014
25	15	0*14, 10	1	1.2229	0.3114	0.1741	1.1934	1.1892	0.1457	0.0007
		10, 0*14		1.2304	0.2456	0.1079	1.2075	1.2034	0.0604	0.0004
		0*2, 1*10, 0*3		1.2592	0.3724	0.2258	1.2252	1.2207	0.1804	0.0009
25	20	0*19, 5	1	0.9696	0.3799	0.1873	0.9634	0.9612	0.1754	0.0009
		5, 0*19		1.2464	0.2106	0.0783	1.2287	1.2255	0.0427	0.0003
		0*7, 1*5, 0*8		1.2225	0.2803	0.1390	1.2031	1.1999	0.1191	0.0006
50	15	0*14, 35	1	1.2800	0.2594	0.1268	1.2487	1.2439	0.0979	0.0005
		35, 0*14		1.1356	0.2520	0.1031	1.1193	1.1157	0.0633	0.0004
		1*5, 3*10		1.1566	0.2901	0.1384	1.1343	1.1304	0.1153	0.0006
50	30	0*29, 20	1	1.1525	0.2534	0.1235	1.1413	1.1393	0.1170	0.0006
		20, 0*29		1.2102	0.1645	0.0439	1.2007	1.1986	0.0402	0.0002
		0*5, 1*20, 0*5		1.1780	0.2828	0.1192	1.1661	1.1641	0.1095	0.0005

Table 5: Class intervals for different estimation methods when θ=0.5

n	m	Scheme	Time	conf [1]	conf[2]	pboot[1]	pboot[2]	tboot[1]	tboot[2]	HPD[1]	HPD[2]
15	5	0*4, 10*1	0.25	0.0561	0.5752	0.2225	1.8196	0.0508	0.2297	0.1470	0.3220
		10, 0*4		0.1137	1.0088	0.3163	1.3283	0.3106	1.7764	0.1202	0.6417
		2*5		0.1341	1.2316	0.6709	2.1856	0.4337	1.1029	0.1323	0.7228
25	5	0*4, 20	0.25	0.0527	0.6277	0.3504	2.6694	0.1062	0.4346	0.0675	0.3848
		20, 0*4		0.1105	0.9845	0.2924	1.1531	0.3072	2.2408	0.0971	0.5230
		4*5		0.1453	1.3752	0.8231	2.2460	0.6062	1.3377	0.1226	0.6919
25	15	0*14, 10	0.25	0.1156	0.3202	0.1082	0.3631	0.0547	0.1525	0.1623	0.2584
		10, 0*14		0.2741	0.7473	0.2273	0.7568	0.3614	1.3801	0.2629	0.6682
		0*2, 1*10, 0*3		0.2464	0.6715	0.3855	0.8625	0.2344	0.4805	0.2223	0.4682
25	20	0*19, 5	0.25	0.1854	0.4374	0.1823	0.4751	0.0964	0.2371	0.2832	0.4123
		5, 0*19		0.3072	0.7195	0.2900	0.7443	0.3679	0.9732	0.3133	0.6998
		0*7, 1*5, 0*8		0.2804	0.6531	0.3890	0.7745	0.2494	0.4836	0.3352	0.5515
50	15	0*14, 35	0.25	0.0465	0.1339	0.0695	0.1865	0.0471	0.1083	0.0604	0.1128
		35, 0*14		0.2647	0.7233	0.1505	0.6572	0.3638	2.6734	0.2347	0.5990
		1*5, 3*10		0.2414	0.6692	0.4127	1.0671	0.2557	0.4999	0.2251	0.4606
50	30	0*29, 20	0.25	0.1110	0.2221	0.0744	0.2263	0.0364	0.0898	0.1443	0.1922
		20, 0*29		0.3386	0.6697	0.1742	0.6228	0.4163	1.5990	0.3391	0.6533
		0*5, 1*20, 0*5		0.2596	0.5154	0.3077	0.6712	0.1650	0.3392	0.2999	0.4333
15	5	0*4, 10*1	0.5	0.0678	0.7460	0.3853	2.5502	0.1179	0.4768	0.0736	0.4040
		10, 0*4		0.1064	0.9619	0.2305	1.1629	0.3053	1.9643	0.1422	0.7548
		2*5		0.1064	1.0421	0.5323	1.6604	0.3645	0.9305	0.2496	0.7180
25	5	0*4, 20	0.5	0.0809	1.0032	0.6270	4.4956	0.1542	0.6949	0.2931	0.8299
		20, 0*4		0.0960	0.8818	0.1957	0.9422	0.2797	2.3592	0.1174	0.6302
		4*5		0.1024	1.0958	0.5898	1.5432	0.4491	1.0135	0.1127	0.6347
25	15	0*14, 10	0.5	0.1097	0.3048	0.0892	0.3512	0.0352	0.1135	0.1912	0.2612
		10, 0*14		0.2747	0.7501	0.1918	0.7374	0.3770	1.5289	0.2921	0.7417
		0*2, 1*10, 0*3		0.2050	0.5629	0.2836	0.6815	0.1525	0.3467	0.2758	0.4206
25	20	0*19, 5	0.5	0.1811	0.4281	0.1700	0.4667	0.0791	0.2073	0.2536	0.3256
		5, 0*19		0.3088	0.7235	0.2761	0.7378	0.3797	1.0234	0.3263	0.7296

		0*7, 1*5, 0*8		0.2559	0.5970	0.3324	0.6840	0.1999	0.4038	0.3901	0.5517
50	15	0*14, 35	0.5	0.0743	0.2140	0.0264	0.5698	0.0776	0.2148	0.0832	0.2194
		35, 0*14		0.2573	0.7051	0.0902	0.5847	0.3614	2.9059	0.2706	0.6888
		1*5, 3*10		0.1691	0.4810	0.2398	0.7219	0.1172	0.2831	0.2993	0.4446
50	30	0*29, 20	0.5	0.1134	0.2269	0.0620	0.2304	0.0201	0.0658	0.1590	0.2011
		20, 0*29		0.3396	0.6718	0.1606	0.6168	0.4229	1.6590	0.3519	0.6771
		0*5, 1*20, 0*5		0.2344	0.4663	0.2462	0.5750	0.1121	0.2625	0.2907	0.4155
15	5	0*4, 10*1	1	0.0946	1.0557	0.6448	3.6706	0.1797	0.7459	0.3779	0.9101
		10, 0*4		0.1002	0.9234	0.1798	1.0595	0.3171	2.0167	0.1640	0.8677
		2*5		0.1015	1.0567	0.5177	1.4603	0.3872	0.9722	0.2815	0.8150
25	5	0*4, 20	1	0.0932	1.1575	0.7464	5.1076	0.1700	0.8045	0.4922	1.0968
		20, 0*4		0.0850	0.8114	0.1254	0.7340	0.2608	2.2014	0.1367	0.7336
		4*5		0.0894	1.0488	0.4866	1.2496	0.3862	0.8954	0.4979	1.0044
25	15	0*14, 10	1	0.1281	0.3565	0.1032	0.4815	0.0303	0.1187	0.2026	0.2864
		10, 0*14		0.2741	0.7488	0.1784	0.7262	0.3783	1.5650	0.2995	0.7603
		0*2, 1*10, 0*3		0.2013	0.5534	0.2460	0.6078	0.1360	0.3354	0.3210	0.4635
25	20	0*19, 5	1	0.1875	0.4439	0.1763	0.5048	0.0769	0.2105	0.3014	0.4136
		5, 0*19		0.3104	0.7273	0.2764	0.7464	0.3850	1.0445	0.3298	0.7367
		0*7, 1*5, 0*8		0.2425	0.5672	0.2922	0.6238	0.1766	0.3759	0.3585	0.5263
50	15	0*14, 35	1	0.2448	0.7055	0.1780	0.5960	0.0956	0.6175	0.2714	0.7161
		35, 0*14		0.2550	0.6998	0.0720	0.5659	0.3597	2.9787	0.2791	0.7108
		1*5, 3*10		0.2090	0.5977	0.2936	0.7030	0.1683	0.3652	0.3317	0.5073
50	30	0*29, 20	1	0.1234	0.2472	0.0634	0.2656	0.0157	0.0638	0.1594	0.2018
		20, 0*29		0.3391	0.6705	0.1595	0.6151	0.4221	1.6557	0.3521	0.6774
		0*5, 1*20, 0*5		0.2337	0.4643	0.2264	0.5143	0.1106	0.2549	0.3135	0.4201

Table 6: Class intervals for different estimation methods when θ=0.8

n	m	Scheme	Time	conf [1]	conf[2]	pboot[1]	pboot[2]	tboot[1]	tboot[2]	HPD[1]	HPD[2]
15	5	0*4, 10*1	0.25	0.0979	1.0626	0.4811	3.4630	0.1557	0.6186	0.2636	0.7118
		10, 0*4		0.1757	1.5793	0.4189	1.9477	0.4978	3.0747	0.2072	1.1104
		2*5		0.1820	1.7474	0.8947	2.8106	0.5983	1.5148	0.2180	1.1097
25	5	0*4, 20	0.25	0.1137	1.4012	0.8519	6.2061	0.2264	0.9701	0.3180	1.0493
		20, 0*4		0.1638	1.4877	0.3670	1.6416	0.4742	3.8313	0.1716	0.9274
		4*5		0.1805	1.8507	1.0322	2.7724	0.7726	1.7516	0.3121	1.1999
25	15	0*14, 10	0.25	0.1661	0.4765	0.1459	0.5555	0.0616	0.1908	0.2878	0.4021
		10, 0*14		0.4415	1.2055	0.3226	1.1911	0.6015	2.4058	0.4514	1.1481
		0*2, 1*10, 0*3		0.3258	0.9252	0.4890	1.1617	0.2690	0.5974	0.4822	0.7744
25	20	0*19, 5	0.25	0.2905	0.6861	0.2775	0.7472	0.1340	0.3438	0.4157	0.6078
		5, 0*19		0.4945	1.1584	0.4520	1.1956	0.6117	1.6406	0.5116	1.1435
		0*7, 1*5, 0*8		0.4211	0.9816	0.5631	1.1385	0.3460	0.6862	0.4943	0.8283
50	15	0*14, 35	0.25	0.0859	0.2477	0.0791	0.3384	0.0411	0.1134	0.1366	0.2046
		35, 0*14		0.4173	1.1432	0.1701	0.9765	0.5848	4.6328	0.4184	1.0669
		1*5, 3*10		0.2927	0.8273	0.4524	1.2800	0.2460	0.5400	0.3747	0.6226
50	30	0*29, 20	0.25	0.1783	0.3569	0.0984	0.3572	0.0357	0.1083	0.2362	0.3019
		20, 0*29		0.5443	1.0766	0.2611	0.9910	0.6786	2.6515	0.5560	1.0706
		0*5, 1*20, 0*5		0.3846	0.7647	0.4178	0.9610	0.1979	0.4480	0.4930	0.7099
15	5	0*4, 10*1	0.5	0.1435	1.5993	0.9329	5.5826	0.2611	1.0893	0.4596	1.2542
		10, 0*4		0.1626	1.4929	0.3103	1.7300	0.5029	3.2151	0.2378	1.2664
		2*5		0.1657	1.6997	0.8479	2.4160	0.6222	1.5486	0.3597	1.2357
25	5	0*4, 20	0.5	0.1453	1.8039	1.1643	8.2734	0.2664	1.2584	0.3903	1.3643
		20, 0*4		0.1406	1.3310	0.2334	1.2642	0.4351	3.6296	0.2033	1.0957
		4*5		0.1437	1.6530	0.7871	2.0356	0.6126	1.4228	0.5525	1.3383
25	15	0*14, 10	0.5	0.1864	0.5184	0.1398	0.6325	0.0432	0.1643	0.3060	0.3990
		10, 0*14		0.4409	1.2046	0.2886	1.1708	0.6063	2.5204	0.4720	1.1983
		0*2, 1*10, 0*3		0.3177	0.8738	0.3995	0.9904	0.2072	0.5068	0.5429	0.8156
25	20	0*19, 5	0.5	0.2961	0.7004	0.2766	0.7833	0.1219	0.3307	0.3149	0.5096
		5, 0*19		0.4954	1.1607	0.4418	1.1898	0.6153	1.6655	0.5193	1.1605

		0*7, 1*5, 0*8		0.3903	0.9118	0.4825	1.0100	0.2869	0.5930	0.5710	0.8168
50	15	0*14, 35	0.5	0.3022	0.8711	0.4863	2.8439	0.0892	0.5110	0.4877	0.8656
		35, 0*14		0.4076	1.1183	0.1206	0.9105	0.5786	4.7620	0.4368	1.1139
		1*5, 3*10		0.2918	0.8341	0.3768	1.0754	0.1812	0.4567	0.4965	0.7317
50	30	0*29, 20	0.5	0.1902	0.3809	0.0955	0.3989	0.0244	0.0967	0.2588	0.3165
		20, 0*29		0.5405	1.0691	0.2543	0.9828	0.6728	2.6448	0.5563	1.0708
		0*5, 1*20, 0*5		0.3691	0.7335	0.3661	0.8497	0.1697	0.4003	0.4824	0.6388
15	5	0*4, 10*1	1	0.1613	1.7993	1.1414	6.2530	0.3136	1.3109	0.5991	1.5657
		10, 0*4		0.1667	1.5380	0.2940	1.6649	0.5870	3.3608	0.2667	1.4150
		2*5		0.1680	1.7720	0.8582	2.3414	0.6664	1.6553	0.5929	1.5354
25	5	0*4, 20	1	0.1500	1.8628	1.2053	8.1839	0.2804	1.2972	0.7003	1.6196
		20, 0*4		0.1330	1.2873	0.1730	1.1049	0.4418	3.4802	0.2236	1.2021
		4*5		0.1391	1.6556	0.7470	1.8978	0.6015	1.4075	0.5714	1.3371
25	15	0*14, 10	1	0.2935	0.8552	0.4232	1.5732	0.1425	0.4753	0.4413	0.7279
		10, 0*14		0.4388	1.1985	0.2865	1.1787	0.6123	2.5628	0.4712	1.1975
		0*2, 1*10, 0*3		0.3409	0.9900	0.4784	1.0666	0.3633	0.8063	0.5797	1.0099
25	20	0*19, 5	1	0.3243	0.7683	0.3203	0.9551	0.1355	0.3818	0.4444	0.6770
		5, 0*19		0.4992	1.1695	0.4442	1.2056	0.6196	1.6913	0.5234	1.1698
		0*7, 1*5, 0*8		0.4229	0.9917	0.4826	1.0600	0.3619	0.8134	0.5443	0.9946
50	15	0*14, 35	1	0.4413	1.2719	0.8959	4.6110	0.1586	0.9283	0.5783	1.2634
		35, 0*14		0.4032	1.1064	0.1103	0.8957	0.5726	4.7853	0.4349	1.1087
		1*5, 3*10		0.3921	1.1218	0.6289	1.1237	0.4492	0.7841	0.6857	1.0083
50	30	0*29, 20	1	0.2814	0.5635	0.2020	0.8129	0.0571	0.2040	0.3821	0.5181
		20, 0*29		0.5454	1.0785	0.2561	0.9940	0.6781	2.6855	0.5615	1.0806
		0*5, 1*20, 0*5		0.4140	0.8223	0.3993	0.8228	0.2609	0.5382	0.5278	0.7861

Table 7: Class intervals for different estimation methods when θ=1.2

n	m	Scheme	Time	conf [1]	conf[2]	pboot[1]	pboot[2]	tboot[1]	tboot[2]	HPD[1]	HPD[2]
15	5	0*4, 10*1	0.25	0.1794	1.9876	1.1169	7.0858	0.3251	1.3437	0.2926	1.2460
		10, 0*4		0.2500	2.2712	0.5162	2.6983	0.7435	4.7770	0.3135	1.6823
		2*5		0.2489	2.4934	1.2631	3.7172	0.8902	2.2309	0.3302	1.6386
25	5	0*4, 20	0.25	0.2070	2.5683	1.6221	11.4807	0.3845	1.7726	0.3758	1.7631
		20, 0*4		0.2238	2.0780	0.4178	2.1002	0.6657	5.6276	0.2684	1.4529
		4*5		0.2278	2.5319	1.3021	3.3942	1.0015	2.2832	0.3315	1.5890
25	15	0*14, 10	0.25	0.2522	0.7262	0.2050	0.8698	0.0720	0.2512	0.3903	0.5466
		10, 0*14		0.6598	1.8024	0.4442	1.7512	0.9026	3.6952	0.6827	1.7369
		0*2, 1*10, 0*3		0.4633	1.3050	0.6376	1.5584	0.3321	0.7815	0.6021	0.9838
25	20	0*19, 5	0.25	0.4354	1.0292	0.4055	1.1271	0.1837	0.4890	0.4581	0.7318
		5, 0*19		0.7423	1.7393	0.6646	1.7826	0.9206	2.4871	0.7628	1.7055
		0*7, 1*5, 0*8		0.5997	1.3997	0.7710	1.5939	0.4602	0.9363	0.6201	1.0838
50	15	0*14, 35	0.25	0.2504	0.7218	0.2812	1.7343	0.0664	0.3114	0.3685	0.6094
		35, 0*14		0.6060	1.6620	0.1993	1.3622	0.8532	6.9118	0.6240	1.5926
		1*5, 3*10		0.4029	1.1502	0.5469	1.6428	0.2613	0.6473	0.6924	1.0097
50	30	0*29, 20	0.25	0.2752	0.5509	0.1403	0.5604	0.0401	0.1456	0.4065	0.5071
		20, 0*29		0.8133	1.6085	0.3838	1.4775	1.0136	3.9728	0.8265	1.5918
		0*5, 1*20, 0*5		0.5546	1.1028	0.5705	1.3339	0.2594	0.6121	0.5690	0.9006
15	5	0*4, 10*1	0.5	0.2386	2.6620	1.6435	9.2120	0.4477	1.8930	0.6833	2.0689
		10, 0*4		0.2459	2.2700	0.4383	2.4940	0.8261	4.9917	0.3551	1.9037
		2*5		0.2485	2.6077	1.2627	3.5372	0.9583	2.4157	0.7883	2.1645
25	5	0*4, 20	0.5	0.2263	2.8104	1.8069	12.8109	0.4071	1.9466	0.5900	2.0701
		20, 0*4		0.1997	1.9276	0.2746	1.6709	0.6174	5.1362	0.3032	1.6432
		4*5		0.2082	2.4668	1.1151	2.8354	0.8899	2.0624	0.9668	2.0993
25	15	0*14, 10	0.5	0.3571	0.9929	0.3385	1.5040	0.1023	0.3836	0.7831	0.9868
		10, 0*14		0.6593	1.8012	0.4225	1.7462	0.9014	3.7857	0.6920	1.7589
		0*2, 1*10, 0*3		0.5002	1.3755	0.6067	1.4699	0.3675	0.8916	0.7315	1.1565
25	20	0*19, 5	0.5	0.4604	1.0898	0.4393	1.2867	0.1879	0.5240	0.7836	1.0910
		5, 0*19		0.7445	1.7444	0.6637	1.7981	0.9246	2.5160	0.7668	1.7151

		0*7, 1*5, 0*8		0.5868	1.3741	0.6879	1.4844	0.4353	0.9445	0.6997	1.1622
50	15	0*14, 35	0.5	0.6418	1.8497	1.2598	6.6389	0.2207	1.3052	0.8771	1.7914
		35, 0*14		0.6099	1.6737	0.1663	1.3513	0.8544	7.2215	0.6426	1.6403
		1*5, 3*10		0.5438	1.5550	0.8131	1.6955	0.5220	1.0171	0.7973	1.1979
50	30	0*29, 20	0.5	0.3166	0.6343	0.1645	0.7261	0.0394	0.1649	0.4286	0.5296
		20, 0*29		0.8165	1.6152	0.3867	1.4857	1.0169	3.9772	0.8302	1.5990
		0*5, 1*20, 0*5		0.5744	1.1406	0.5497	1.2161	0.2898	0.6483	0.6891	1.0384
15	5	0*4, 10*1	1	0.2433	2.7148	1.7267	9.4677	0.4731	1.9815	0.6123	2.1045
		10, 0*4		0.2583	2.3793	0.4635	2.5154	0.9548	5.0627	0.3830	2.0523
		2*5		0.2556	2.7051	1.2837	3.4248	1.0140	2.5040	0.8105	2.2846
25	5	0*4, 20	1	0.2232	2.7713	1.8048	12.3944	0.4149	1.9427	0.7088	2.1549
		20, 0*4		0.2015	1.9614	0.2431	1.6178	0.6893	5.1198	0.3214	1.7444
		4*5		0.2057	2.4521	1.1018	2.8129	0.8916	2.1000	0.9698	2.1136
25	15	0*14, 10	1	0.6493	1.7964	1.1737	3.5742	0.4416	1.3174	0.9768	1.7399
		10, 0*14		0.6595	1.8013	0.4375	1.7678	0.9263	3.8123	0.6913	1.7590
		0*2, 1*10, 0*3		0.6749	1.8436	1.0120	1.9360	0.9461	1.8093	0.8876	1.7125
25	20	0*19, 5	1	0.5771	1.3621	0.7096	1.8978	0.3318	0.8413	0.7496	1.2361
		5, 0*19		0.7456	1.7472	0.6724	1.8093	0.9375	2.5418	0.7680	1.7170
		0*7, 1*5, 0*8		0.7322	1.7128	0.9339	1.9077	0.8823	1.8195	0.7241	1.6189
50	15	0*14, 35	1	0.6595	1.9006	1.3517	6.8777	0.2419	1.4007	0.6959	1.8410
		35, 0*14		0.6067	1.6644	0.1688	1.3530	0.8697	7.1890	0.6401	1.6322
		1*5, 3*10		0.5992	1.7140	1.0028	1.6757	0.7494	1.2456	0.9325	1.4089
50	30	0*29, 20	1	0.7699	1.5351	1.0665	2.9915	0.3967	1.1000	0.8822	1.5171
		20, 0*29		0.8128	1.6076	0.3834	1.4898	1.0145	4.0260	0.8273	1.5924
		0*5, 1*20, 0*5		0.7905	1.5655	0.9401	1.5614	0.8722	1.4480	0.8918	1.4572

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