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Order Statistics from the Ishita Distribution and Associated Inference

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Ishita distribution proposed by Shankar and Shukla (2017) is useful in modelling data from biomedical science and engineering and it is also useful for the analysis of Stress-strength reliability of a component. It gives the best fit than most of the distributions such as Lindley, exponential and Akash distributions. In this paper, we have derived the recurrence relations and exact expressions for single and product moments of order statistics from Ishita distribution. By using exact expressions, we have tabulated the means, variances and covariance's of order statistics from Ishita distribution. Further, using these moments, we have obtained the best linear unbiased estimators (BLUEs) for the location and scale parameters of the Ishita distribution based on the type-II censored samples.

Keywords: Order statistics, Single moment, Product moment, Type-II right censoring, Best linear unbiased estimator, Ishita distribution.

1. Introduction

Order statistics and their functions have very wide ranges of applications in applied and mathematical statistics. Order statistics are useful in characterization of probability distributions, entropy estimation, estimation of L- moments, goodness-of-fit tests, analysis of censored samples, reliability analysis, quality control, Flood frequency analysis, wireless communication system, digital image analysis and so on. Order statistics plays an important role in non-parametric inferential methods. The moments of order statistics have practical applications in many areas such as quality control, reliability analysis. Nadarajah and Pal (2008) have drawn the explicit expressions for the moments of order statistics from the Gamma distribution and also discussed application of moments for the quality control data. For more detail about order statistics see, Arnold et al. (2003) and David and Nagaraja (2003).

A one-parameter Ishita distribution was proposed by Shankar and Shukla (2017) as a lifetime distribution for analyzing lifetime data in biomedical sciences and engineering. The proposed distribution is more flexible than Lindley, Exponential, and Akash distributions for modelling lifetime data and has an increasing and decreasing hazard rate. An Ishita distribution belongs to the family of the exponential distribution. It is the mixture of exponential distribution with parameter θ and gamma (3, θ) distribution. Due to the applicability of this distribution, several researchers developed new distributions using this distribution. Anwar et al. (2019) developed a new distribution with Ishita distribution known as Poisson Ishita distribution that is useful for analyzing count data. Al-Omari et al. (2019) introduced new size biased Ishita distribution and showed its applications to real data and also draw its mathematical properties. Shukla and Shankar (2017) proposed a simulation study on Ishita distribution. Shukla and Shankar (2018) developed a Power Ishita distribution. Al-Nasser et al. (2018) developed single- acceptance Sampling Plans based on a truncated lifetime test for an Ishita distribution. Shukla and Shankar (2019) introduced weighted Ishita distribution and discussed its application in survival data.

Recently a lot of work has been carried out based on order statistics. Kumar and Goyal (2019a, b) derived the exact expressions for single and product moments of order statistics from Generalized Lindley and Power Lindley distributions and derived the best linear unbiased estimator for scale and location parameters of these distributions. Kumar et al. (2017) obtained the exact expressions of single and product moments as well as recurrence relations for single and product moments of the extended exponential distribution. Balakrishnan et al. (2014) derived some recurrence relations for single and product moments of order statistics from the complementary exponential-geometric distribution. They have also obtained the BLUE of the scale parameter based on complete and type-II right censored samples. Gupta et al. (1967) obtained the BLUEs for the parameters of the logistic distribution. Dyer and Whisenand (1973a, b) obtained the BLUEs of the parameter of Rayleigh distribution by considering two cases such as small sample theory for censored order statistics and optimum theory for selected order statistics.

Numerous recurrence relations and exact expressions for moments of order statistics have been obtained in their work. Here some of the references are made such as Joshi (1978), Khan and Khan (1983), Khan et al. (1984), Khan and Khan (1987), Mohie El-Din et al. (1991), Saran and Pushkarna (2000), Zghoul (2010), MirMostafaee (2014), Çetinkaya and Genç (2018), Akter et al. (2022).

A new one parameter lifetime distribution proposed by Shankar and Shukla (2017) known as Ishita distribution has probability distribution function (pdf)

$$f(x) = \frac{\theta^3}{\theta^3 + 2}(\theta + x^2)e^{-\theta x}, \quad x > 0, \theta > 0 \tag{1}$$

with distribution function (*df*)

$$F(x) = 1 - \left(1 + \frac{\theta x(\theta x + 2)}{\theta^3 + 2}\right) e^{-\theta x}, \quad x > 0, \theta > 0. \tag{2}$$

From (1) and (2), we get the characterizing differential equation

$$[(\theta^3 + 2) + \theta(\theta x^2 + 2x)] f(x) = \theta^3(\theta + x^2)\bar{F}(x). \tag{3}$$

Shankar and Shukla (2017) have discussed several mathematical and statistical properties. They have obtained the maximum likelihood estimate for parameter θ and by taking two data sets; they have shown that Ishita distribution gives the better fit than the exponential, Lindley and Akash distributions.

In this paper, we have derived the recurrence relations and closed-form expressions for single and product moments of order statistics from Ishita distribution. Using these closed-form expressions, we have calculated the means, variances and covariance's of order statistics from Ishita distribution. We have also discussed the application of moments by tabulating the BLUEs of location and scale parameters of Ishita distribution. The means, variances and covariance's of locations and scale parameters are also tabulated.

2. Moments of order Statistics

In this section, we have derived the exact expressions and recurrence relations for single and product moments of order statistics from the Ishita distribution.

Let $X_1, X_2, ..., X_n$ be independently identically distributed continuous random variable with pdf f(x) and df F(x). If they have arranged in the ascending order of magnitude $X_{1:n} \leq X_{2:n} \leq ... \leq X_{r:n}$

$\leq ... \leq X_{n:n}$, Then $X_{1:n} = \min\{X_1, X_2, ..., X_n\}$ and $X_{n:n} = \max\{X_1, X_2, ..., X_n\}$ are called the smallest and largest order statistics. The pdf of *r - th* order statistic $X_{r:n}$ is given by

$$f_{r:n}(X) = \frac{n!}{(r-1)!(n-r)} [F(x)]^{r-1} f(x) [\bar{F}(x)]^{n-r}, \quad 1 \leq r \leq n, \tag{4}$$

and the cdf of $X_{r:n}$ is given by

$$F_r(x) = \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1 - F(x)]^{n-i}.$$

The joint *pdf* of the *r-th* order statistic $X_{r:n}$ and the *s-th* order statistics $X_{s:n}$ is given by

$$f_{r,s:n}(x,y)=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1}f(x)[F(y)-F(x)]^{s-r-1}[\bar{F}(y)]^{n-s}f(y),\qquad (5)$$

with $1\leq r<s\leq n,\ x<y.$

The plots of *pdf* and *df* of order statistics from Ishita distribution are as follows

Plots of the *pdf* of order statistics from Ishita distribution

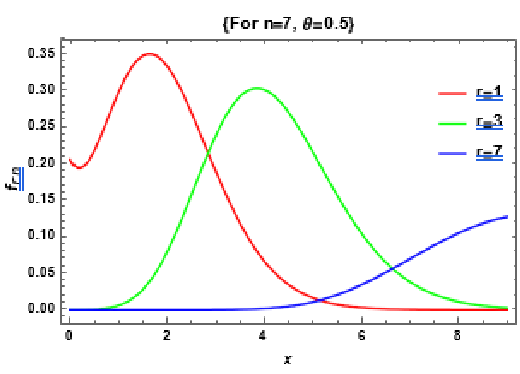

Figure 1

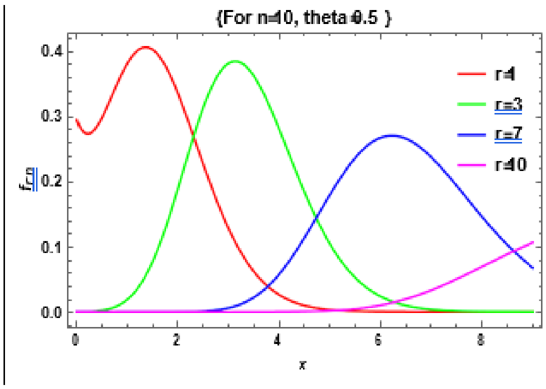

Figure 2

Plots of the *df* of order statistics from Ishita distribution

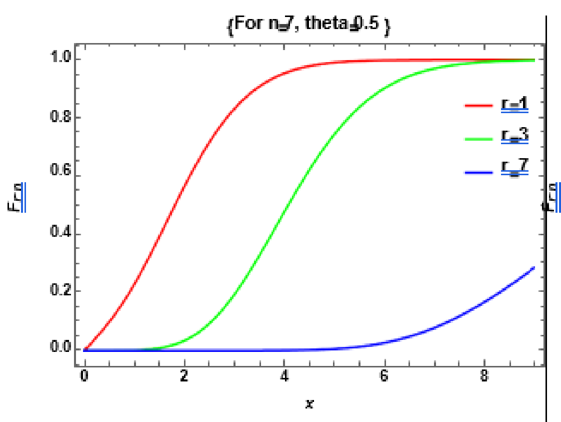

Figure 3

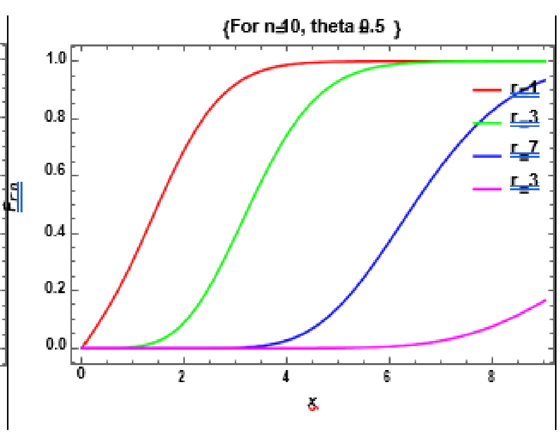

Figure 4

Figure 1, 2 and **Figure 3, 4** shows the behaviour of pdf and df of ordered random variables that follows Ishita distribution respectively.

2.1 Relations for single moments

Theorem 1 For the Ishita distribution given in (1) and $1\leq r\leq n,\ j=1,2,\ldots$

$$E(X_{r:n}^{j}) = C_{r:n}\sum_{a=0}^{r-1}\sum_{b=0}^{n-r+a}\sum_{c=0}^{b}\sum_{d=0}^{n-r+a-b+1}(-1)^{a}\binom{r-1}{a}\binom{n-r+a}{b}\binom{b}{c}\binom{n-r+a-b+1}{d}$$

$$\times\frac{2^{b}\theta^{3(n-r+a-b)+c-d+4}\Gamma(j+c+2d+1)}{(\theta^{3}+2)^{n-r+a+1}[\theta(n-r+a+1)]^{j+c+2d+1}}, \tag{6}$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.$$

Proof: From (4), we have

$$E(X_{r:n}^{j}) = C_{r:n}\int_{0}^{\infty}x^{j}[F(x)]^{r-1}f(x)[\bar{F}(x)]^{n-r}dx$$

$$= C_{r:n}\sum_{a=0}^{r-1}(-1)^{a}\binom{r-1}{a}\int_{0}^{\infty}x^{j}[\bar{F}(x)]^{n-r+a}f(x)dx$$

$$= C_{r:n}\frac{\theta^{3}}{(\theta^{3}+2)}\sum_{a=0}^{r-1}(-1)^{a}\binom{r-1}{a}\int_{0}^{\infty}x^{j}\left(\frac{2(1+\theta x)+\theta^{2}(\theta+x^{2})}{\theta^{3}+2}\right)^{n-r+a}(\theta+x^{2})e^{-\theta(n-r+a+1)x}dx$$

$$= C_{r:n}\sum_{a=0}^{r-1}\sum_{b=0}^{n-r+a}(-1)^{a}\binom{r-1}{a}\binom{n-r+a}{b}\frac{2^{b}\theta^{3(n-r+a-b)+4}}{(\theta^{3}+2)^{n-r+a+1}}$$

$$\times\int_{0}^{\infty}x^{j}(1+\theta x)^{b}\left(1+\frac{x^{2}}{\theta}\right)^{n-r+a-b+1}e^{-\theta(n-r+a+1)x}dx$$

$$= C_{r:n}\sum_{a=0}^{r-1}\sum_{b=0}^{n-r+a}\sum_{c=0}^{b}\sum_{d=0}^{n-r+a-b+1}(-1)^{a}\binom{r-1}{a}\binom{n-r+a}{b}\binom{b}{c}\binom{n-r+a-b+1}{d}$$

$$\times\frac{2^{b}\theta^{3(n-r+a-b)+c-d+4}}{(\theta^{3}+2)^{n-r+a+1}}\int_{0}^{\infty}x^{j+c+2d}e^{-\theta(n-r+a+1)x}dx. \tag{7}$$

On applying the integral of gamma function, (7) yields the result given in (6).

Remarks:

1) For $j = 1$ and r = n = 1, (6) gives the mean of Ishita distribution.

$$E(X_{1:1}) = \sum_{d=0}^{1}\binom{1}{d}\frac{\theta^{-d+4}\Gamma(2d+2)}{(\theta^{3}+2)\theta^{2d+2}},$$

$$= \frac{(\theta^{3}+6)}{\theta(\theta^{3}+2)} = E(X).$$

2) Substituting r □ 1 in (6), we obtain the exact expression for single moments of smallest order statistic given as

$$E(X^{j}_{1:n}) = n\sum_{b=0}^{n-1}\sum_{c=0}^{b}\sum_{d=0}^{n-b}\binom{n-1}{b}\binom{b}{c}\binom{n-b}{d}\frac{2^{b}\theta^{3(n-1-b)+c-d+4}\Gamma(j+c+2d+1)}{(\theta^{3}+2)^{n}(n\theta)^{j+c+2d+1}}.$$

3) Substituting r □ n in (6), we obtain the exact expression for single moment of largest order statistic give as

$$E(X^{j}_{n:n}) = n\sum_{a=0}^{n-1}\sum_{b=0}^{a}\sum_{c=0}^{b}\sum_{d=0}^{a-b+1}(-1)^{a}\binom{n-1}{a}\binom{a}{b}\binom{b}{c}\binom{a-b+1}{d}\frac{2^{b}\theta^{3(a-b)+c-d+4}\Gamma(j+c+2d+1)}{(\theta^{3}+2)^{a+1}[\theta(a+1)]^{j+c+2d+1}}.$$

Theorem 2 For the Ishita distribution given in (1) and for $1 \le r \le n$, $j = 1,2,\ldots$

$$E(X^{j+3}_{r:n}) - E(X^{j+3}_{r-1:n}) = \frac{(j+3)}{\theta^{3}(n-r+1)}\{(\theta^{3}+2)E(X^{j}_{r:n}) + \theta^{2}E(X^{j+2}_{r:n}) + 2\theta E(X^{j+1}_{r:n})\}$$
$$- \frac{\theta(j+3)}{(j+1)}\{E(X^{j+1}_{r:n}) - E(X^{j+1}_{r-1:n})\}. \tag{8}$$

Proof: From (3) and (4), we have

$$(\theta^{3}+2)E(X^{j}_{r:n}) + \theta^{2}E(X^{j+2}_{r:n}) + 2\theta E(X^{j+1}_{r:n})$$
$$= \theta^{3}C_{r:n}\left[\theta\int_{\theta}^{\infty} x^{j}[F(x)]^{r-1}[\bar{F}]^{n-r+1}dx + \int_{\theta}^{\infty} x^{j+2}[F(x)]^{r-1}[\bar{F}]^{n-r+1}dx\right],$$

$$(\theta^{3}+2)E(X^{j}_{r:n}) + \theta^{2}E(X^{j+2}_{r:n}) + 2\theta E(X^{j+1}_{r:n}) = \theta^{3}C_{r:n}[\theta I_{j} + I_{j+2}], \tag{9}$$

where

$$I_{b} = \int_{\theta}^{\infty} x^{b}[F(x)]^{r-1}[\bar{F}]^{n-r+1}dx \tag{10}$$

$$C_{n:r} = \frac{n!}{(r-1)!(n-r)!}.$$

Now simplifying the integration given in (10) on integrating by parts taking x^b for integration and the remaining integrand for differentiation, we get

$$I_b = -\frac{(r-1)}{b+1}\int_{\theta}^{\infty} x^{b+1}[F(x)]^{r-2} f(x)[\bar{F}(x)]^{n-r+1}\,dx + \frac{(n-r+1)}{b+1}\int_{\theta}^{\infty} x^{b+1}[F(x)]^{r-1} f(x)[\bar{F}(x)]^{n-r}\,dx. \quad (11)$$

Substituting Ib from (11) in (9) for I_j and I_{j+2}, we get

$$(\theta^3+2)E(X_{r:n}^{j}) + \theta^2 E(X_{r:n}^{j+2}) + 2\theta E(X_{r:n}^{j+1}) = \theta^3 C_{r:n}\Big\{-\frac{\theta(r-1)}{j+1}\int_{\theta}^{\infty} x^{j+1}[F(x)]^{r-2} f(x)[\bar{F}(x)]^{n-r+1}\,dx$$

$$+\frac{\theta(n-r+1)}{j+1}\int_{\theta}^{\infty} x^{j+1}[F(x)]^{r-1} f(x)[\bar{F}(x)]^{n-r}\,dx - \frac{(r-1)}{j+3}\int_{\theta}^{\infty} x^{j+3}[F(x)]^{r-2} f(x)[\bar{F}(x)]^{n-r+1}\,dx$$

$$+\frac{(n-r+1)}{j+3}\int_{\theta}^{\infty} x^{j+3}[F(x)]^{r-1} f(x)[\bar{F}(x)]^{n-r}\,dx\Big\}.$$

After simplification, we obtain required relation as give in (8).

Relations for product moments

Theorem 3 For the Ishita distribution defined in (1) and for $1 \le r < s \le n$, $i,j = 1,2,...$

$$E(X_{r:n}^{i}X_{s:n}^{j}) = C_{rs}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}\sum_{c=0}^{n-s+b}\sum_{d=0}^{c}\sum_{e=0}^{n-s+b-c+1}\sum_{f=0}^{s-r+a-b-1}\sum_{g=0}^{j+d+2e}\sum_{h=0}^{f}\sum_{k=0}^{s-r+a-b-f}(-1)^{a+b}\binom{r-1}{a}\binom{s-r-1}{b}$$

$$\times\binom{n-s+b}{c}\binom{c}{d}\binom{n-s+b-c+1}{e}\binom{s-r-1+a-b}{f}\binom{f}{h}\binom{s-r+a-b-f}{k}$$

$$\times\frac{2^{c+f}\theta^{3(n-s+b-c+1)+d-e+1}}{(\theta^3+2)^{n-r+a+1}[\theta(n-s+b+1)]^{j+d+2e+1}}\frac{[\theta(n-s+b+1)]^{g}}{g!}$$

$$\times\frac{(j+d+2e)!\,\theta^{3(s-r+a-b-f)+h-k+1}\Gamma(i+g+h+2k+1)}{[\theta(n-r+a+1)]^{i+g+2k+h+1}}, \quad (12)$$

where

$$C_{rs} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

Proof. Using (5), we have

$$E(X_{r:n}^{i}X_{s:n}^{j}) = C_{rs}\int_0^{\infty}\int_x^{\infty} x^i y^j [F(x)]^{r-1} f(x)[F(y)-F(x)]^{s-r-1}[\bar{F}(y)]^{n-s} f(y)dydx$$

$$= C_{rs}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}(-1)^{a+b}\binom{r-1}{a}\binom{s-r-1}{b}\int_0^{\infty}\int_x^{\infty} x^i y^j [\bar{F}(x)]^{s-r-1+a-b} f(x)[\bar{F}(y)]^{n-s+b} f(y)dydx$$

$$= C_{rs}\sum_{a=0}^{r-1}\sum_{b=0}^{s-r-1}(-1)^{a+b}\binom{r-1}{a}\binom{s-r-1}{b}\int_0^{\infty} x^i [\bar{F}(x)]^{s-r-1+a-b} f(x)I(x)dx, \quad (13)$$

where

$$I(x) = \int_x^{\infty} y^j [\bar{F}(y)]^{n-s+b} f(y)dy.$$

Using (1) and (2), we have

$$I(x) = \frac{\theta^3}{(\theta^3+2)}\int_x^{\infty} y^j \left(1+\frac{\theta y(\theta y+2)}{(\theta^3+2)}\right)^{n-s+b}(\theta+y^2)(e^{-\theta y})^{n-s+b+1}dy$$

$$= \sum_{c=0}^{n-s+b}\binom{n-s+b}{c}\frac{2^c\theta^{3(n-s+b-c)+4}}{(\theta^3+2)^{n-s+b+1}}\int_x^{\infty} y^j(1+\theta y)^c\left(1+\frac{y^2}{\theta}\right)^{n-s+b-c+1}(e^{-\theta y})^{n-s+b+1}dy$$

$$= \sum_{c=0}^{n-s+b}\sum_{d=0}^{c}\sum_{e=0}^{n-s+b-c+1}\binom{n-s+b}{c}\binom{c}{d}\binom{n-s+b-c+1}{e}\frac{2^c\theta^{3(n-s+b+1)+1+d-e}}{(\theta^3+2)^{n-s+b+1}}$$

$$\times\int_x^{\infty} y^{j+d+2e}(e^{-\theta y})^{n-s+b+1}dy$$

$$= \sum_{c=0}^{n-s+b}\sum_{d=0}^{c}\sum_{e=0}^{n-s+b-c+1}\binom{n-s+b}{c}\binom{c}{d}\binom{n-s+b-c+1}{e}\frac{2^c\theta^{3(n-s+b+1)+1+d-e}}{(\theta^3+2)^{n-s+b+1}}$$

$$\times\frac{\gamma(j+d+2e+1,\theta(n-s+b+1)x)}{[\theta(n-s+b+1)]^{j+d+2e+1}}, \quad (14)$$

where

$\gamma(\alpha,x) = \int_x^{\infty} t^{\alpha-1}e^{-t}dt$ is the incomplete gamma function.

From (13) and (14), we have

$$E(X_{r:n}^{i}X_{s:n}^{j}) = A\int_0^{\infty} x^i[\bar{F}]^{s-r-1+a-b} f(x)\gamma(j+d+2e+1,\theta(n-s+b+1)x)dx, \quad (15)$$

Where

$$A = C_{rs} \sum_{a=0}^{r-1} \sum_{b=0}^{s-r-1} \sum_{c=0}^{n-s+b} \sum_{d=0}^{c} \sum_{e=0}^{n-s+b-c+1} (-1)^{a+b} \binom{r-1}{a} \binom{s-r-1}{b} \binom{n-s+b}{c} \binom{c}{d} \binom{n-s+b-c+1}{e}$$

$$\times \frac{2^c \theta^{3(n-s+b-c+1)+d-e+1}}{(\theta^3+2)^{n-s+b+1} [\theta(n-s+b+1)]^{j+d+2e+1}}.$$

Thus (15) can be written as

$$E(X_{r:n}^{i} X_{s:n}^{j}) = \frac{A\theta^3}{(\theta^3+2)} \int_0^\infty x^i \left(1 + \frac{\theta x(\theta x+2)}{(\theta^3+2)}\right)^{s-r-1+a-b} (x^2+\theta)(e^{-\theta x})^{s-r+a-b}$$

$$\times \gamma(j+d+2e+1, \theta(n-s+b+1)x)dx. \qquad (16)$$

Using $\gamma(n,\theta x) = (n-1)! e^{-\theta x} \sum_{u=0}^{n-1} \frac{(\theta x)^u}{u!}$ in (16), we get

$$E(X_{r:n}^{i} X_{s:n}^{j}) = A \sum_{f=0}^{s-r-1+a-b} \sum_{g=0}^{j+d+2e} \binom{s-r-1+a-b}{f} \frac{2^f \theta^{3(s-r+a-b-f)+1}}{(\theta^3+2)^{s-r+a-b}} \frac{[\theta(n-s+b+1)]^g}{g!}$$

$$\times (j+d+2e)! \int_0^\infty x^{i+g} (1+\theta x)^f \left(1 + \frac{x^2}{\theta}\right)^{s-r+a-b-f} (e^{-\theta x})^{n-r+a+1} dx$$

$$= A^* \int_0^\infty x^{i+g} (1+\theta x)^f \left(1 + \frac{x^2}{\theta}\right)^{s-r+a-b-f} (e^{-\theta x})^{n-r+a+1} dx, \qquad (17)$$

where

$$A^* = A \sum_{f=0}^{s-r-1+a-b} \sum_{g=0}^{j+d+2e} \binom{s-r-1+a-b}{f} \frac{2^f \theta^{3(s-r+a-b-f)+1}}{(\theta^3+2)^{s-r+a-b}} \frac{[\theta(n-s+b+1)]^g (j+d+2e)!}{g!}.$$

On simplifying (17), we get

$$E(X_{r:n}^{i} X_{s:n}^{j}) = A^* \theta^{h-k} \sum_{h=0}^{f} \sum_{k=0}^{s-r+a-b-f} \binom{f}{h} \binom{s-r+a-b-f}{k} \int_0^\infty x^{i+g+2k+h} (e^{-\theta x})^{n-r+a+1} dx$$

$$= A^* \theta^{h-k} \sum_{h=0}^{f} \sum_{k=0}^{s-r+a-b-f} \binom{f}{h} \binom{s-r+a-b-f}{k} \frac{\Gamma(i+g+2k+h+1)}{[\theta(n-r+a+1)]^{i+g+2k+h+1}}. \qquad (18)$$

After substituting values of A* and A in (18) and simplifying, we get the required expression given in (12).

Note: The legitimacy of the product moments of order statistics obtained using theorem 3 can be checked through the identity given bellow

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\mu_{r,s:n}=\binom{n}{2}[E(X)]^2,$$

where

$\mu_{r,s:n}=E(X_{r,s:n})$ and $E(X)$ = Mean of the Ishita distribution.

Theorem 4 For the Ishita distribution defined in (1) and for $1\leq r<s\leq n$, $i,j=1,2,...$

$$E(X_{r:n}^{i}X_{r+1:n}^{j+3})-E(X_{r:n}^{i+j+3})$$

$$=\frac{(j+3)}{\theta^3(n-r)}\{(\theta^3+2)E(X_{r:n}^{i}X_{r+1:n}^{j})+\theta^2E(X_{r:n}^{i}X_{r+1:n}^{j+2})+2\theta E(X_{r:n}^{i}X_{r+1:n}^{j+1})\}$$

$$-\frac{\theta(j+3)}{(j+1)}\{E(X_{r:n}^{i}X_{r+1:n}^{j+2})-E(X_{r:n}^{i+j+1})\}. \tag{19}$$

Proof: Using (3) and (5), we get

$$(\theta^3+2)E(X_{r:n}^{i}X_{s:n}^{j})+\theta^2E(X_{r:n}^{i}X_{s:n}^{j+2})+2\theta E(X_{r:n}^{i}X_{s:n}^{j+1})$$
$$=C_{rs}\theta^3\left\{\theta\int_0^\infty\int_x^\infty x^iy^j[F(x)]^{r-1}f(x)[F(y)-F(x)]^{s-r-1}[\bar{F}(y)]^{n-s+1}dydx\right.$$
$$\left.+\int_0^\infty\int_x^\infty x^iy^{j+2}[F(x)]^{r-1}f(x)[F(y)-F(x)]^{s-r-1}[\bar{F}(y)]^{n-s+1}dydx\right\}$$
$$=C_{rs}\theta^3\left\{\theta\int_0^\infty x^i[F(x)]^{r-1}f(x)I_j(x)dx+\int_0^\infty x^i[F(x)]^{r-1}f(x)I_{j+2}(x)dx\right\}, \tag{20}$$

Where

$$I_b(x)=\int_x^\infty y^b[F(y)-F(x)]^{s-r-1}[\bar{F}(y)]^{n-s+1}dydx. \tag{21}$$

Now simplifying the integration given in (21) on integrating by parts taking y^b for integration and the remaining integrand for differentiation, we get

$$I_b(x)=-\frac{(s-r-1)}{b+1}\int_x^\infty y^{b+1}[F(y)-F(x)]^{s-r-2}f(y)[\bar{F}(y)]^{n-s+1}dy$$
$$+\frac{(n-s+1)}{b+1}\int_x^\infty y^{b+1}[F(y)-F(x)]^{s-r-1}f(y)[\bar{F}(y)]^{n-s}dy. \tag{22}$$

Substituting $I_b(x)$ from (22) in (20) for $I_j(x)$ and $I_{j+1}(x)$, we get

$$(\theta^3+2)E(X_{r:n}^i X_{s:n}^j)+\theta^2 E(X_{r:n}^i X_{s:n}^{j+2})+2\theta E(X_{r:n}^i X_{s:n}^{j+1})$$

$$=C_{rs}\,\theta^3\Bigg\{-\frac{\theta(s-r-1)}{j+1}\int_0^\infty\int_x^\infty x^i y^{j+1}[F(x)]^{r-1} f(x)[F(y)-F(x)]^{s-r-2}[\bar{F}(y)]^{n-s+1} f(y)dydx$$

$$+\frac{\theta(n-s+1)}{j+1}\int_0^\infty\int_x^\infty x^i y^{j+1}[F(x)]^{r-1} f(x)[F(y)-F(x)]^{s-r-1}[\bar{F}(y)]^{n-s} f(y)dydx$$

$$-\frac{(s-r-1)}{j+3}\int_0^\infty\int_x^\infty x^i y^{j+3}[F(x)]^{r-1} f(x)[F(y)-F(x)]^{s-r-2}[\bar{F}(y)]^{n-s+1} f(y)dydx$$

$$+\frac{(n-s+1)}{j+3}\int_0^\infty\int_x^\infty x^i y^{j+3}[F(x)]^{r-1} f(x)[F(y)-F(x)]^{s-r-1}[\bar{F}(y)]^{n-s} f(y)dydx\Bigg\}.$$

After simplification, we get the required result given in (19).

Corollary: For the Ishita distribution with pdf (1) for $1 \le r \le n$, the following relation is satisfied.

$$\frac{\theta^3(n-r)}{(j+3)}\{E(X_{r:n}^i X_{r+1:n}^{j+3})-E(X_{r:n}^{i+j+3})\}$$

$$=(\theta^3+2)E(X_{r:n}^i X_{r+1:n}^j)+\theta^2 E(X_{r:n}^i X_{r+1:n}^{j+2})+2\theta E(X_{r:n}^i X_{r+1:n}^{j+1})$$

$$-\frac{\theta^4(n-r)}{(j+1)}\{E(X_{r:n}^i X_{r+1:n}^{j+2})-E(X_{r:n}^{i+j+1})\}.$$

Proof: On taking

s = r +1 in (19), we obtain the result given the above corollary.

BLUES for α and $\boldsymbol{\sigma}$

In this section, we have calculated the best linear unbiased estimators for the location and scale parameter of Ishita distribution by using relations established in the previous sections based on Type II right censored sample. Consider an experiment in which n items are put on test and failure times are recorded. The failure times of these items follow Ishita distribution. The obtained data are arranged in order. If the final observations are lost, then the samples are said to be censored from the right. Assume that *Y*1, *Y*2 ,..., *Y*n to be a random sample of size n from Ishita distribution with *pdf*

$$f(y)=\frac{\theta^3}{\theta^3+2}\left(\theta+\left(\frac{y-\alpha}{\sigma}\right)^2\right)e^{-\theta\left(\frac{y-\alpha}{\sigma}\right)},\quad y>0,\theta>0.$$

Let $Y_{1:n} \leq Y_{2:n} \leq \ldots \leq Y_{r:n}$ be the order statistics corresponding to the above sample. Then $\underline{X} = (X_{1:n}, X_{2:n}, \ldots, X_{n:n})^T$ where $X_{r:n} = \frac{Y_{r:n} - \alpha}{\sigma},\ r = 1,2,\ldots,n$ is ordered sample from a population with the standard Ishita distribution with pdf and cdf given in (1) and (2) respectively. $E[X_{r:n}] = \mu_{r:n}$, 1 ≤ r < n – c and $\operatorname{cov}(X_{r:n}, X_{s:n}) = \mu_{r,s:n} - \mu_{r:n}\mu_{s:n}$. Here, we have considered the Type II censored sample based on the whole sample of size n given as

$$\underline{Y} = (Y_{1:n}, Y_{2:n}, \ldots, Y_{n-c:n})^T,\ 1 \leq r \leq n-c$$

$$\underline{\mu} = (\mu_{1:n}, \mu_{2:n}, \ldots, \mu_{n-c:n})^T,$$

$$\underline{1} = (\underbrace{1,\ 1,\ \ldots,\ 1}_{n-c})^T,$$

$$\mathbf{\Sigma} = ((\sigma_{r,s})),\ 1 \leq r < s \leq n-c.$$

The BLUES of α and σ is given by Arnold et al. (2003)

$$\alpha^* = \frac{\underline{\mu}^T \Sigma^{-1} \underline{\mu} \underline{1}^T \Sigma^{-1} - \underline{\mu}^T \Sigma^{-1} \underline{1} \underline{\mu}^T \Sigma^{-1}}{(\underline{\mu}^T \Sigma^{-1} \underline{\mu})(\underline{1}^T \Sigma^{-1} \underline{1}) - (\underline{\mu}^T \Sigma^{-1} \underline{1})^2} \underline{\boldsymbol{Y}} = \sum_{r=1}^{n-c} \boldsymbol{a}_r Y_{r:n},$$

$$\sigma^* = \frac{\underline{1}^T \Sigma^{-1} \underline{1} \underline{\mu}^T \Sigma^{-1} - \underline{1}^T \Sigma^{-1} \underline{\mu} \underline{1}^T \Sigma^{-1}}{(\underline{\mu}^T \Sigma^{-1} \underline{\mu})(\underline{1}^T \Sigma^{-1} \underline{1}) - (\underline{\mu}^T \Sigma^{-1} \underline{1})^2} \underline{\boldsymbol{Y}} = \sum_{r=1}^{n-c} \mathbf{b}_r Y_{r:n},$$

Where,

$$\underline{a} = \{a_1, a_2, \ldots, a_{n-c}\} = \frac{\underline{\mu}^T \Sigma^{-1} \underline{\mu} 1^T \Sigma^{-1} - \underline{\mu}^T \Sigma^{-1} \underline{1} \underline{\mu}^T \Sigma^{-1}}{(\underline{\mu}^T \Sigma^{-1} \underline{\mu})(\underline{1}^T \Sigma^{-1} \underline{1}) - (\underline{\mu}^T \Sigma^{-1} \underline{1})^2}$$

$$\underline{b} = \{b_1, b_2, \ldots, b_{n-c}\} = \frac{\underline{1}^T \Sigma^{-1} \underline{1} \underline{\mu}^T \Sigma^{-1} - \underline{1}^T \Sigma^{-1} \mu \underline{1}^T \Sigma^{-1}}{(\underline{\mu}^T \Sigma^{-1} \underline{\mu})(\underline{1}^T \Sigma^{-1} \underline{1}) - (\underline{\mu}^T \Sigma^{-1} \underline{1})^2}$$

∑ is the variance covariance matrix with $\sigma_{r,s} = \operatorname{cov}(X_{r:n}, X_{s:n})$.

The variances and covariance of these estimators are given by

$$\mathrm{var}(\alpha^*) = \sigma^2 \left\{ \frac{\underline{\mu}^T \Sigma^{-1} \underline{\mu}}{(\underline{\mu}^T \Sigma^{-1} \underline{\mu})(\underline{1}^T \Sigma^{-1} \underline{1}) - (\underline{\mu}^T \Sigma^{-1} \underline{1})^2} \right\} = \sigma^2 U_1 ,$$

$$\mathrm{var}(\sigma^*) = \sigma^2 \left\{ \frac{\underline{1}^T \Sigma^{-1} \underline{1}}{(\underline{\mu}^T \Sigma^{-1} \underline{\mu})(\underline{1}^T \Sigma^{-1} \underline{1}) - (\underline{\mu}^T \Sigma^{-1} \underline{1})^2} \right\} = \sigma^2 U_2 ,$$

and

$$\mathrm{cov}(\alpha^*, \sigma^*) = \sigma^2 \left\{ \frac{-\underline{\mu}^T \Sigma^{-1} \underline{1}}{(\underline{\mu}^T \Sigma^{-1} \underline{\mu})(\underline{1}^T \Sigma^{-1} \underline{1}) - (\underline{\mu}^T \Sigma^{-1} \underline{1})^2} \right\} = \sigma^2 U_3 .$$

The determination of BLUEs is impracticable for the larger sample size i.e. n $\geq$ 25 because the coefficients cannot be easily calculated. Therefore, it is typically useful to solely use a number of chosen observations from a sample rather than the entire or a censored sample in estimating unknown parameters and nevertheless maintain high efficiency in estimation. The coefficients of BLUES a_i's and b_i's are computed for type II censored sample of sizes n = 7, 10 and for θ = 1.5, 2.75 and using different censoring schemes c = 0(1)([n / 2] - 1). The validity of these coefficients has been checked using following conditions

$$\sum_{r=1}^{n-c} \boldsymbol{a}_r = 1 \text{ and } \sum_{r=1}^{n-c} \boldsymbol{b}_r = 0 .$$

Example. In this example, we show the usefulness of the coefficients of the BLUEs presented in Tables 6 and 7 by considering a real data set with σ = 1, μ = 0 of sample size 10 taken from 69 carbon fibers tested under tension at gauge lengths of 20mm that represent the tensile strength, measured in GPA, Bader and Priest [22],

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958.

Here, we have n = 10, c = 0. On Applying Two-sample Kolmogorov-Smirnov test, we see that this data set fit to the Ishita distribution with parameter θ=1.5. By using the values of the coefficients in the Tables 6 & 7, we calculate the BLUEs of μ and σ such as

$$\alpha^* = \sum_{r=1}^{n-c} \boldsymbol{a}_r X_{r:n}$$

$$= (1.312 \times 1.105095) - (1.314 \times 0.02064818) - (1.479 \times 0.01591712) - (1.552 \times 0.0117689)$$

$$-(1.700 \times 0.0090429) - (1.803 \times 0.00781069) - (1.861 \times 0.00777695) - (1.865 \times 0.00868086)$$

$$-(1.944 \times 0.01041263) - (1.958 \times 0.01303706)$$

$$= 1.289.$$

$$\sigma^{*} = \sum_{r=1}^{n-c} b_r X_{r:n}$$

$= -(1.312 \times 0.7070832) + (1.314 \times 0.07771449) + (1.479 \times 0.06219615) + (1.552 \times 0.05440201) + (1.700 \times 0.05432019) + (1.803 \times 0.06132909) + (1.861 \times 0.07366769) + (1.865 \times 0.08975104) + (1.944 \times 0.107579) + (1.958 \times 0.1261236)$

$= 0.314.$

3. Numerical results

Tabulations of the Means, Variances, Covariance's, Coefficients of BLUES of the location and scale parameters and Variances and Covariance's of the BLUES.

The expressions obtained in the previous sections are used to calculate means, variances and covariance's for order statistics for different sample sizes 1 to 10 on choosing various values of the parameter. The means, variances and covariance's are also used to obtain the coefficients of the BLUEs of location and scale parameters. In Tables 1 & 2 and Tables 3 & 4, we have presented the means and variances of order statistics for □ □ 1.5, 2.75 . The product moments, covariance's of order statistics have presented in Table 5. One can see that the variances of the order statistics given in the Tables 3 & 4 decreases as the sample size increases. On the other hand, we have also concluded that the variances of order statistics decrease as the value of θ increases.

Table 1. Means of order statistics for $\theta = 1.5$

n	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$
1	1.16279									
2	0.59437	1.73121								
3	0.39362	0.99589	2.09886							
4	0.29205	0.69833	1.29344	2.36734						
5	0.23123	0.53528	0.94291	1.52713	2.57739					
6	0.19097	0.43254	0.74077	1.14505	1.71817	2.74923				
7	0.16245	0.3621	0.60862	0.91697	1.31612	1.87899	2.89427			
8	0.14124	0.31096	0.51554	0.76377	1.07017	1.46368	2.01743	3.01954		
9	0.12486	0.27222	0.44654	0.65353	0.90158	1.20505	1.593	2.13869	3.12964	

10	0.11186	0.24192	0.39345	0.57042	0.77819	1.02496	1.3251	1.70781	2.2464	3.22778

Table 2. Means of order statistics for θ = 2.75

n	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8	r = 9	r = 10
1	0.42744									
2	0.20802	0.64686								
3	0.13694	0.35018	0.7952							
4	0.10198	0.24179	0.45857	0.90741						
5	0.08123	0.18501	0.32698	0.54629	0.99769					
6	0.06749	0.14994	0.25515	0.3988	0.62004	1.07322				
7	0.05772	0.12609	0.20956	0.31595	0.46097	0.68368	1.13815			
8	0.05042	0.10881	0.17793	0.26226	0.36963	0.51574	0.73966	1.19507		
9	0.04476	0.0957	0.15467	0.22445	0.30952	0.41772	0.56475	0.78964	1.24575	
10	0.04024	0.08542	0.13683	0.19631	0.26667	0.35237	0.46128	0.60909	0.83477	1.29142

The validity of the means of order statistics tabulated in the Tables 1 & 2 have checked by using the identity

$$\sum_{r=1}^{n} \mu_{r:n} = nE[X]$$ (David and Nagaraja (2003)),

where

$$E[X_{r:n}] = \mu_{r:n}.$$

Table 3. Variances of order statistics for θ = 1.5

n	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8	r = 9	r = 10
1	1.19055									
2	0.35449	1.38043								
3	0.16462	0.4924	1.41892							
4	0.09292	0.25593	0.55179	1.41966						
5	0.05882	0.15533	0.30714	0.57836	1.40937					
6	0.04022	0.10323	0.19618	0.33638	0.58987	1.39609				
7	0.02906	0.07298	0.13544	0.22284	0.35325	0.594	1.38251			
8	0.0219	0.05401	0.09853	0.15845	0.2403	0.36295	0.59435	1.36958		
9	0.01705	0.0414	0.07451	0.11802	0.1748	0.25177	0.36837	0.59275	1.35757	
10	0.01362	0.03264	0.05807	0.09094	0.13273	0.18643	0.25931	0.37117	0.59013	1.34653

Table 4. Variances of order statistics for θ = 2.75

n	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$
1	0.197765									
2	0.04639	0.25285								
3	0.01979	0.06926	0.27863							
4	0.01086	0.03194	0.08308	0.29345						
5	0.00684	0.01834	0.04025	0.09239	0.30296					
6	0.00469	0.01189	0.02385	0.04634	0.09911	0.30949				
7	0.00342	0.00834	0.01581	0.0281	0.05101	0.10417	0.31421			
8	0.0026	0.00617	0.01126	0.01894	0.0315	0.0547	0.10813	0.31773		
9	0.00204	0.00475	0.00843	0.01366	0.02152	0.03428	0.05771	0.1113	0.32042	
10	0.00165	0.00377	0.00655	0.01034	0.01569	0.02368	0.0366	0.0602	0.11388	0.32251

Plots of the variances of order statistics

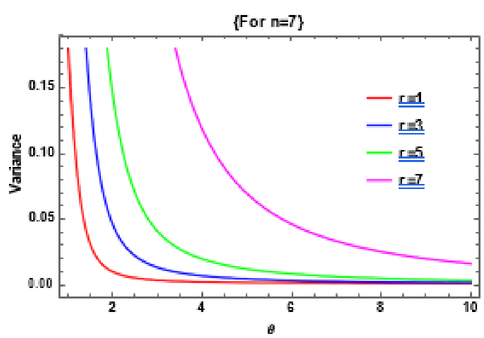

Figure 5

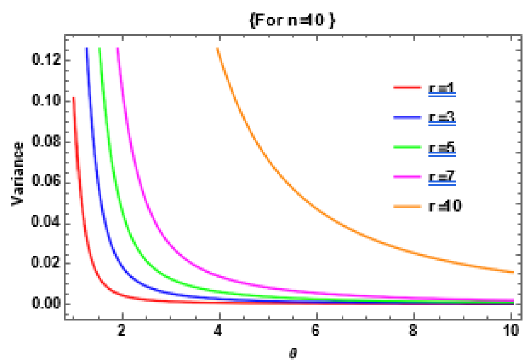

Figure 6

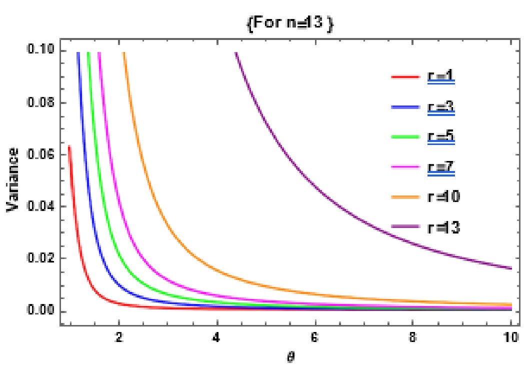

Figure 7

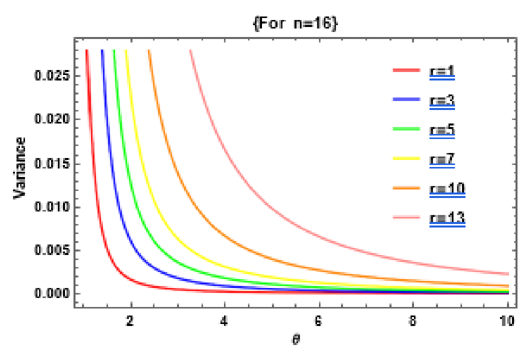

Figure 8

From the above Figure. 5, 6, 7 and 8, we conclude that the variance of order statistics decreases as the values of θ as well as sample sizes increases.

Table 5. Product moments and Covariance's of order statistics

θ =1.5					θ = 2.75				
n	s	r	$\mu_{r,s:n}$	$\sigma_{r,s:n}$	n	s	r	$\mu_{r,s:n}$	$\sigma_{r,s:n}$
2	2	1	1.35208	0.3231	2	2	1	0.1827	0.04815
3			0.55165	0.15965	3			0.06838	0.02043
4			0.29681	0.09286	4			0.03579	0.01113
5			0.18353	0.05976	5			0.02201	0.00698
6			0.12383	0.04123	6			0.01489	0.00477
7			0.08876	0.02994	7			0.01075	0.00347
8			0.06653	0.02261	8			0.00812	0.00263
9			0.05161	0.01762	9			0.00635	0.00207
10			0.04115	0.01408	10			0.0051	0.00167
3	3	1	0.9701	0.14395	3	3	1	0.12998	0.02109
4			0.46622	0.08848	4			0.05822	0.01146
5			0.27664	0.0586	5			0.0337	0.00714
6			0.18261	0.04114	6			0.02209	0.00487
7			0.12907	0.0302	7			0.01562	0.00353
8			0.09578	0.02297	8			0.01164	0.00267
9			0.07374	0.01798	9			0.00902	0.00209
10			0.05842	0.01441	10			0.00719	0.00168
3		2	2.53449	0.44426	3		2	0.34975	0.07128
4			1.14676	0.2435	4			0.14371	0.03283
5			0.65681	0.15209	5			0.07925	0.01876
6			0.42328	0.10286	6			0.05037	0.01212
7			0.29392	0.07354	7			0.03489	0.00847
8			0.21512	0.05481	8			0.02561	0.00625
9			0.16376	0.0422	9			0.01976	0.00481
10			0.12855	0.03337	10			0.0155	0.00381
4	4	1	0.77093	0.07956	4	4	1	0.10431	0.01177
5			0.40845	0.05532	5			0.0517	0.00733
6			0.25855	0.03987	6			0.03189	0.00497
7			0.17874	0.02977	7			0.02183	0.00359
8			0.13078	0.0229	8			0.01594	0.00271
9			0.09967	0.01807	9			0.01217	0.00212

10			0.07838	0.01456	10			0.00961	0.00171
4		2	1.8723	0.21911	4		2	0.25309	0.03368
5			0.96095	0.1435	5			0.12031	0.01924
6			0.59491	0.09963	6			0.07218	0.01238
7			0.40447	0.07244	7			0.04846	0.00863
8			0.29211	0.05461	8			0.03489	0.00635
9			0.22029	0.04238	9			0.02635	0.00487
10			0.17169	0.0337	10			0.02062	0.00385
4		3	3.55946	0.49744	4		3	0.5011	0.08499
5			1.72961	0.28966	5			0.21986	0.04123
6			1.03807	0.18984	6			0.12611	0.02435
7			0.69139	0.13329	7			0.0823	0.01609
8			0.49185	0.09809	8			0.0581	0.01143
9			0.3666	0.07477	9			0.04326	0.00854
10			0.28303	0.0586	10			0.03349	0.00663
5	5	1	0.64578	0.0498	5	5	1	0.08854	0.0075
6			0.36559	0.03747	6			0.04694	0.00509
7			0.24246	0.02865	7			0.03027	0.00367
8			0.17354	0.02239	8			0.0214	0.00276
9			0.13044	0.01787	9			0.01601	0.00215
10			0.10156	0.01451	10			0.01246	0.00173
5		2	1.50888	0.12924	5		2	0.20425	0.01967
6			0.83679	0.09361	6			0.10563	0.01267
7			0.54626	0.06969	7			0.06692	0.0088
8			0.38615	0.05337	8			0.04668	0.00646
9			0.28732	0.04189	9			0.03457	0.00494
10			0.22183	0.03357	10			0.02668	0.0039
5		3	2.6914	0.26115	5		3	0.36832	0.0421
6			1.45114	0.17837	6			0.1831	0.0249
7			0.92921	0.12818	7			0.11301	0.01641
8			0.64753	0.09582	8			0.0774	0.01163
9			0.47645	0.07386	9			0.05655	0.00867
10			0.36454	0.05836	10			0.04321	0.00672
5		4	4.45876	0.52275	5		4	0.63911	0.09407
6			2.28346	0.31606	6			0.29461	0.04734
7			1.42104	0.2142	7			0.17428	0.02865
8			0.97206	0.15469	8			0.1162	0.01926
9			0.70572	0.11651	9			0.08334	0.01386
10			0.53442	0.09052	10			0.06282	0.01047
6	6	1	0.55886	0.03382	6	6	1	0.07762	0.00519
7			0.33213	0.02688	7			0.04321	0.00374

8			0.22819	0.02146	8			0.02882	0.00281
9			0.16784	0.01737	9			0.02089	0.00219
10			0.12892	0.01426	10			0.01593	0.00175
6		2	1.27368	0.08453	6		2	0.17382	0.01291
7			0.74576	0.06537	7			0.09519	0.00898
8			0.50628	0.05114	8			0.0627	0.00658
9			0.36876	0.04072	9			0.045	0.00503
10			0.28094	0.03299	10			0.03406	0.00396
6		3	2.19772	0.16116	6		3	0.29919	0.02535
7			1.26385	0.12025	7			0.16002	0.01675
8			0.84638	0.0918	8			0.10361	0.01184
9			0.60989	0.07179	9			0.07342	0.00881
10			0.46059	0.05732	10			0.05503	0.00681
6		4	3.43394	0.28592	6		4	0.47614	0.04814
7			1.92396	0.20098	7			0.24523	0.02923
8			1.26609	0.14817	8			0.15487	0.01962
9			0.90074	0.11321	9			0.10785	0.01409
10			0.67354	0.08888	10			0.07979	0.01062
6		5	5.25879	0.53514	6		5	0.76597	0.10053
7			2.80454	0.33158	7			0.36712	0.05198
8			1.79652	0.23013	8			0.2227	0.03207
9			1.25622	0.16978	9			0.15116	0.02186
10			0.9279	0.13028	10			0.10987	0.01591
7	7	1	0.49452	0.02434	7	7	1	0.0695	0.00381
8			0.30507	0.02013	8			0.04016	0.00287
9			0.21552	0.016612	9			0.02751	0.00223
10			0.16205	0.01382	10			0.02034	0.00178
7		2	1.10724	0.05921	7		2	0.15264	0.00913
8			0.6753	0.04796	8			0.08719	0.00671
9			0.47258	0.03893	9			0.05916	0.00511
10			0.35252	0.03196	10			0.04343	0.00402
7		3	1.87051	0.10898	7		3	0.25552	0.01701
8			1.12616	0.08611	8			0.14368	0.01207
9			0.77996	0.06863	9			0.09632	0.00897
10			0.57688	0.05553	10			0.07004	0.00692
7		4	2.83623	0.18226	7		4	0.38925	0.02966
8			1.67986	0.13900	8			0.21397	0.01998
9			1.14929	0.10822	9			0.14109	0.01433
10			0.84195	0.08608	10			0.10134	0.01078
7		5	4.1103	0.3011	7		5	0.57732	0.05269
8			2.37494	0.21595	8			0.30605	0.03264
9			1.5985	0.16229	9			0.19704	0.02223

10			1.15734	0.12615	10			0.13916	0.01615
7		6	5.97934	0.54103	7		6	0.88348	0.10535
8			3.29367	0.3408	8			0.43711	0.05564
9			2.1603	0.24066	9			0.27075	0.03485
10			1.53867	0.18048	10			0.18658	0.02404
8	8	1	0.44476	0.15982	8	8	1	0.06316	0.00291
9			0.28263	0.08373	9			0.03761	0.00227
10			0.20424	0.05601	10			0.02632	0.00181
8		2	0.98254	0.04359	8		2	0.13683	0.0068
9			0.61874	0.03654	9			0.08077	0.0052
10			0.44367	0.03052	10			0.05612	0.00409
8		3	1.63496	0.07828	8		3	0.22487	0.01223
9			1.01942	0.06441	9			0.13126	0.00912
10			0.72496	0.05303	10			0.09037	0.00703
8		4	2.43267	0.12643	8		4	0.33366	0.02023
9			1.49928	0.10158	9			0.19182	0.01458
10			1.05638	0.08221	10			0.13053	0.01096
8		5	3.42799	0.19657	8		5	0.47477	0.03304
9			2.08057	0.15237	9			0.26701	0.02261
10			1.44948	0.12047	10			0.17883	0.01641
8		6	4.73029	0.31065	8		6	0.67259	0.05624
9			2.80327	0.22605	9			0.36526	0.03541
10			1.92281	0.17237	10			0.23904	0.02442
8		7	6.63512	0.54343	8		7	0.99302	0.10907
9			3.7532	0.34626	9			0.50454	0.05859
10			2.5107	0.24768	10			0.31813	0.03717
9	9	1	0.40499	0.01421	9	9	1	0.05805	0.00229
10			0.26368	0.0124	10			0.03543	0.00184
9		2	0.88527	0.03331	9		2	0.12448	0.00526
10			0.57212	0.02867	10			0.07546	0.00415
9		3	1.45625	0.05874	9		3	0.20191	0.00922
10			0.93367	0.04982	10			0.12136	0.00714
9		4	2.13799	0.09268	9		4	0.29435	0.01473
10			1.35865	0.07725	10			0.17501	0.01113
9		5	2.9607	0.13908	9		5	0.40842	0.02284
10			1.86137	0.11323	10			0.23927	0.01666
9		6	3.97785	0.20649	9		6	0.55613	0.03576
10			2.46453	0.16204	10			0.31894	0.02479
9		7	5.30228	0.31675	9		7	0.76263	0.05909
10			3.20967	0.23295	10			0.42278	0.03771
9		8	7.23723	0.54389	9		8	1.09572	0.11202

10			4.18584	0.34939	10			0.56948	0.06103
10	10	1	0.37239	0.01134	10	10	1	0.05383	0.00186
10		2	0.80708	0.02622	10		2	0.11451	0.00419
10		3	1.31554	0.04558	10		3	0.18391	0.00721
10		4	1.91188	0.07068	10		4	0.26475	0.01123
10		5	2.61548	0.10364	10		5	0.36118	0.0168
10		6	3.45674	0.14839	10		6	0.48005	0.02498
10		7	4.49063	0.21349	10		7	0.6337	0.03799
10		8	5.83312	0.32067	10		8	0.84801	0.06143
10		9	7.79417	0.54325	10		9	1.19246	0.11441

Table 6. Coefficient of BLUEs of location parameter

θ	n	c	a_i , $i = 1,2,...,(n-c)$						
1.5	7	0	1.140300	-0.032988	-0.023148	-0.017986	-0.017508	-0.020869	-0.027799
		1	1.166361	-0.036981	-0.026604	-0.021660	-0.022001	-0.059114	
		2	1.205509	-0.043131	-0.031919	-0.027327	-0.103130		
	10	0	1.105095	-0.020648	-0.015917	-0.011769	-0.009043	-0.00781	-0.007777
			-0.008681	-0.010412	-0.013037				
		1	1.116225	-0.022005	-0.016993	-0.012711	-0.009986	-0.008872	-0.009046
			-0.010205	-0.026407					
		2	1.130588	-0.023795	-0.018418	-0.013962	-0.011250	-0.010278	-0.010726
			-0.042158						
		3	1.149838	-0.026211	-0.020352	-0.01568232	-0.012941	-0.012207	-0.062443
		4	1.177081	-0.029670	-0.023103	-0.01808546	-0.015357	-0.090865	
2.75	7	0	1.143696	-0.028654	-0.024607	-0.02526752	-0.022597	-0.021139	-0.021432
		1	1.172212	-0.033956	-0.029646	-0.03005154	-0.027110	-0.051448	
		2	1.213895	-0.041463	-0.036911	-0.03701803	-0.098503		
	10	0	1.103461	-0.018484	-0.009484	-0.01050767	-0.014270	-0.010263	-0.010865
			-0.010032	-0.009550	-0.010005				
		1	1.116223	-0.020146	-0.011046	-0.01198404	-0.015735	-0.011667	-0.012180
			-0.011325	-0.022139					
		2	1.131956	-0.022081	-0.012887	-0.01386162	-0.017593	-0.013286	-0.013832
			-0.038413						
		3	1.152915	-0.024926	-0.015231	-0.01624263	-0.019876	-0.015514	-0.061125
		4	1.182200	-0.028352	-0.018949	-0.01937578	-0.023132	-0.092391	

Table 7. Coefficient of the BLUEs of Scale parameter

θ	n	c	$b_i, i = 1,2,...,(n-c)$						
1.5	7	0	-0.712583	0.098589	0.085198	0.090692	0.111977	0.144029	0.182097
		1	-0.883290	0.124746	0.107831	0.114754	0.141412	0.394547	
		2	-1.144579	0.165792	0.143312	0.152583	0.682891		
	10	0	-0.707083	0.077714	0.062196	0.054402	0.054320	0.061329	0.073668
			0.089751	0.107579	0.126124				
		1	-0.69615	0.077613	0.062037	0.054269	0.054208	0.061174	0.073435
			0.089287	0.224129					
		2	1.995758	-0.226432	-0.180849	-0.158309	-0.158429	-0.178360	-0.213931
			-0.879447						
		3	-1.150077	0.132805	0.106120	0.093159	0.092921	0.104871	0.620201
		4	-1.420654	0.167155	0.133434	0.117027	0.116917	0.886121	
2.75	7	0	-2.430610	0.458757	0.431219	0.410431	0.382643	0.362649	0.384909
		1	-2.942757	0.553980	0.521719	0.496351	0.463712	0.906994	
		2	-3.677600	0.686321	0.649798	0.619166	1.722315		
	10	0	-2.471679	0.327719	0.298634	0.287346	0.288978	0.266283	0.256960
			0.2447657	0.239039	0.260187				
		1	-2.803794	0.370954	0.339268	0.325741	0.327063	0.302797	0.291155
			0.278389	0.566424					
		2	-3.206636	0.420463	0.386374	0.373779	0.374614	0.344232	0.333425
			0.971451						
		3	-3.737048	0.492409	0.4456227	0.433993	0.432359	0.400569	1.529429
		4	-4.470248	0.578144	0.538649	0.512386	0.513815	2.324106	

Table 8. Variances and co-variances of BLUEs

θ	N	c	$var(\alpha *)$	$var(\sigma *)$	$cov(\alpha *, \sigma *)$
1.5	7	0	0.03299	0.13861	-0.02349
		1	0.03386	0.17611	-0.02921
		2	0.03518	0.23482	-0.03801
	10	0	0.01482	0.09257	-0.01057
		1	0.01499	0.10832	-0.01220
		2	0.01521	0.12988	-0.01437
		3	0.01551	0.15926	-0.01731
		4	0.01593	0.20017	-0.02176

2.75	7	0	0.00396	0.17889	-0.00983
		1	0.00407	0.21604	-0.01190
		2	0.00424	0.26837	-0.01487
	10	0	0.00182	0.11946	-0.00455
		1	0.00185	0.13539	-0.00516
		2	0.00187	0.15432	-0.00591
		3	0.00191	0.17899	-0.00688
		4	0.00197	0.21292	-0.00823

The coefficients of BLUEs of location and scale parameters have displayed in the Tables 6 & 7 for type II censored sample of sizes n = 7, 10, θ = 1.5, 2.75 by considering different censoring cases c = 0(1)([n/2] -1) . In Table 8, the variances and covariance's of BLUEs have presented. We see that the variances of BLUEs increase as the censoring level increases while the variances decrease as the sample sizes increases. On the other hand, covariance's of BLUEs decrease as the censoring level increases while covariance's of the BLUEs increases as the sample sizes increases.

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