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On Discriminant Analysis with some Bivariate Exponential Distributions

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On Discriminant Analysis with some Bivariate Exponential Distributions

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This study focused on obtaining allocation rules when the assumption of normality is violated. More specifically, when available data is of the bivariate exponential distributions. Three different forms of bivariate exponential distribution were considered and allocation rules arising from each of the three were suggested. Both simulated and real-life data were used to demonstrate the applicability and performance of the suggested allocation rules.

Keywords: allocation rule, bivariate exponential distribution, normality, error rate, discriminant analysis.

1. Introduction

Discriminant analysis has found relevance in many areas of science and engineering. Its purpose is to assign an observation $\boldsymbol{x}$ to one of two or more groups on the basis of the value of $\boldsymbol{x}$. Many problems of discriminant analysis have assumed that the underlying distribution is the multivariate normal in which $\boldsymbol{x}$ is assumed to have means $\boldsymbol{\mu}_i (i = 1,2, \ldots, g)$ and variance-covariance matrix$\boldsymbol{\Sigma}$which is the same for all the groups. Discrete models, Bayesian techniques and the nonparametric density estimation procedure are also available for discriminant analysis but have only been studied within the context of multivariate normal situation. The interest here is the case where the underlying distribution (or data generating process) is not multivariate normal. None of such study is known to the authors at the moment. Specifically, the bivariate exponential (BVE) distribution which has found wide application in reliability studies, competing risk, hydrology, rainfall and storm occurrence etc has not been considered in discriminant analysis. For example, on the basis of some suitably chosen vector of variables, rainfall occurrence can be classified as a particular type of various types, or failure of a system classified as either mild or severe. This study shall consider obtaining and applying allocation rule/region of classification when $(x_1, x_2) \in \boldsymbol{x}$ arecorrelated exponential random variables arising from the following distributions:

i. Freud (1961) bivariate exponential distributionwith joint pdfgiven as:
$$f(x_1, x_2) = \begin{cases} \alpha\beta' exp\{-\beta' x_2 - (\alpha + \beta - \beta')x_1\} & if\ x_1 < x_2 \\ \alpha'\beta exp\{-\alpha' x_1 - (\alpha + \beta - \alpha')x_2\} & if\ x_2 < x_1 \end{cases}$$
$x_1 > 0$, $x_2 > 0$, $\alpha > 0$, $\beta > 0$, $\alpha' > 0$, and $\beta' > 0$

ii. Marshal & Olkin (1967) bivariate exponential distributionwith joint pdf given as:
$$f(x_1, x_2) = \begin{cases} \theta_1(\theta_2 + \theta_3)exp\{-\theta_1 x_1 - (\theta_2 + \theta_3)x_2\} & if\ x_1 < x_2 \\ \theta_2(\theta_1 + \theta_3)exp\{-\theta_2 x_2 - (\theta_1 + \theta_3)x_1\} & if\ x_2 < x_1 \end{cases}$$
$x_1 > 0$, $x_2 > 0$, $\theta_1 > 0$, $\theta_2 > 0$, $\theta_3 > 0$, and $\theta = \theta_1 + \theta_2 + \theta_3$

iii. Block &Basu (1974) bivariate exponential distribution with joint pdf given as:
$$f(x_1, x_2) = \begin{cases} \dfrac{\theta_1\theta(\theta_2 + \theta_3)}{\theta_1 + \theta_2} exp\{-\theta_1 x_1 - (\theta_2 + \theta_3)x_2\} & if\ x_1 < x_2 \\ \dfrac{\theta_2\theta(\theta_1 + \theta_3)}{\theta_1 + \theta_2} exp\{-\theta_2 x_2 - (\theta_1 + \theta_3)x_1\} & if\ x_2 < x_1 \end{cases}$$
$x_1 > 0$, $x_2 > 0$, $\theta_1 > 0$, $\theta_2 > 0$, $\theta_3 > 0$, and $\theta = \theta_1 + \theta_2 + \theta_3$

The above listed distributions have been applied in real life situations. For example, see;Rai & Van (1984), Inaba &Shirahata (1986) and Gross & Lam (1981).Properties of various BVE distributions, including those considered in this study has been summarized in Balakrishnan & Lai (2009). In general, the bivariate exponential distributions assume that x_1 and x_2 are lifetimes of two independent exponential components. Dependence of x_1 and x_2 arises when a failure (or shock) occurs to either thex_1 component or the $x_2$component or even to both components. The failure of one component changes the parameter of the other component by inducing an additional burden on the other component as in Block &Basu (1974) and Freud (1961).

Notations

Notation	Description
Π_i	Population i $(i = 1,2)$
$f_i(x_1, x_2, \dots, x_k)$	pdf of$\boldsymbol{x} = (x_1, x_2, \dots, x_k)$population i
$\boldsymbol{x}$	$k\ x$ 1 observation
α_1, α_1', β_1, β_1'	Parameters of $f_1(x_1, x_2)$ for Freud distribution
α_2, α_2', β_2, β_2'	Parameters of $f_2(x_1, x_2)$ for Freud distribution
θ_1, θ_{11}, θ_{21}, θ_{31}	Parameters of $f_1(x_1, x_2)$ for Marshal/Olkin and Block/Basu distributions
θ_2, θ_{12}, θ_{22}, θ_{32}	Parameters of $f_2(x_1, x_2)$ for Marshal/Olkin and Block/Basu distributions
MOBVE	Marshal and Olkin Bivariate Exponential distribution
BBBVE	Block and Basu Bivariate Exponential distribution
FBVE	Freud Bivariate Exponential distribution

2. Methodology

Definition:Let $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ be the joint densities of observations coming from group Π_1 and group Π_2 respectively, the region of classification R according to Anderson (2003)is givenas

$$R: \frac{f_1(x_1,x_2)}{f_2(x_1,x_2)} \geq c$$

Where c is a constant determined such that the critical region should have the required size.If the inequality in R holds, the observation falls in region R_1 and it is classified as belonging to population Π_1, otherwise the observation is said to fall into R_1 region and consequently is classified a belonging to population Π_2. We now derive and present allocation rules for classifying a future observation to one of population Π_1 or population Π_2 given that the underlying distribution is any of Block &Basu (1974), Freud (1961)orMarshal & Olkin (1967).

Freud BVE

If $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ are the joint densities for group Π_1 and group Π_2 respectively and (x_1,x_2) follows the Freud BVE, thenthe best region of classificationR_1 is

$$R_1: (x_2-x_1)\lambda + (\gamma+\omega)x_1 \geq c_a \qquad if\ x_1 < x_2$$

and

$$R_1: (x_1-x_2)\nu + (\gamma+\omega)x_2 \geq c_b \qquad if\ x_2 < x_1$$

Where $c_a = \log\left(\frac{\alpha_2\beta_2'}{\alpha_1\beta_1'}\right)$,$c_b = \log\left(\frac{\alpha_2'\beta_2}{\alpha_1'\beta_2}\right)$,$\lambda = \beta_2' - \beta_1'$,$\nu = \alpha_2' - \alpha_1'$,$\omega = \beta_2 - \beta_1$, and $\gamma = \alpha_2 - \alpha_1$.

Marshal and OlkinBVE

If $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ are the joint densities for populationΠ_1 and populationΠ_2 respectively and (x_1,x_2) follows the Marshal and Olkin BVE, then the best region of classification R_1 is

$$R_1: (\theta_{12}-\theta_{11})x_1 + (\lambda_2-\lambda_1)x_2 \geq d_a \qquad if\ x_1 < x_2$$

and

$$R_1: (\theta_{22}-\theta_{21})x_2 + (\delta_2-\delta_1)x_1 \geq d_b \qquad if\ x_2 < x_1$$

Where $d_a = \log\left(\frac{\theta_{12}\lambda_2}{\theta_{11}\lambda_1}\right)$, $d_b = \log\left(\frac{\theta_{22}\delta_2}{\theta_{21}\delta_1}\right)$, $\lambda_1 = \theta_{21}+\theta_{31}$, $\lambda_2 = \theta_{22}+\theta_{32}$, $\delta_1 = \theta_{11}+\theta_{31}$, $\delta_1 = \theta_{12}+\theta_{32}$

Block and Basu BVE

If $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ are the joint densities for populationΠ_1 and populationΠ_2 respectively and (x_1,x_2) follows the Block and Basu BVE, then region of classification R_1 is

$$R_1: (\theta_{21}-\theta_{11})x_1 + (\varphi_2-\varphi_1)x_2 \geq h_a \qquad if\ x_1 < x_2$$

and

$$R_1: (\theta_{22}-\theta_{21})x_2 + (\psi_2-\psi_1)x_1 \geq h_b \qquad if\ x_2 < x_1$$

Where $h_a = \log\left(\frac{\tau_2\varphi_1\theta_{11}\theta_1}{\tau_1\varphi_2\theta_{21}\theta_2}\right)$, $h_b = \log\left(\frac{\tau_2\psi_1\theta_{21}\theta_1}{\tau_1\psi_2\theta_{22}\theta_2}\right)$,$\varphi_1 = \theta_{21} + \theta_{31}$, $\varphi_2 = \theta_{22} + \theta_{32}$, $\psi_1 = \theta_{11} + \theta_{31}$, $\psi_1 = \theta_{12} + \theta_{31}$, $\tau_1 = \theta_{11} + \theta_{21}$, $\tau_2 = \theta_{12} + \theta_{22}$, $\theta_1 = \theta_{11} + \theta_{21} + \theta_{31}$, and $\theta_2 = \theta_{12} + \theta_{22} + \theta_{32}$

For the three cases considered, any observation falling into $R_1$is classified as belonging to populationΠ_1, otherwise it is classified as belonging to populationΠ_2. By replacing $\geq$ with $<$ in each of the R_1, we obtain the best region of classification R_2 satisfying the hypothesis that $\boldsymbol{x} = (x_1, x_2)$belongs to population Π_2. The right-hand side of each of the regions of classification R_1 above represents the well-known discriminant function since in each case the right-hand side of R_1 is a function of the observation vector $\boldsymbol{x} = (x_1, x_2)$. By assuming that the two populations are equally likely and the cost of misclassification$c(1/2)$ and $c(2/1)$are also equal, the right-hand side of the rules presented above will be purely a function of the parameters of the corresponding distribution.

2.1 Probability of Misclassification

Perhaps, we may wish to know the probabilities of misclassification before we draw our samples n_1 and n_2 for the purpose of determining the allocation rule. Let the probability of an observation coming from population Π_1 and Π_2 be p_1 and p_2 respectively (p_1+$p_2 = 1$).If$\boldsymbol{x} = (x_1, x_2)$ be a random observation then we wish to find the distribution of

$$U = \begin{cases} (x_2 - x_1)\lambda + (\gamma + \omega)x_1 \geq c_a & if\ x_1 < x_2 \\ (x_1 - x_2)\nu + (\gamma + \omega)x_2 \geq c_b & if\ x_2 < x_1 \end{cases}$$

on the assumption that$U$ is distributed according to the Freud bivariate exponential function. The exact distribution of $U$is very difficult to evaluate. Therefore, we resort to treating the asymptotic expansions of their probabilities as n_1 and n_2 increases. As such, the probability of wrongly classifying an observation originally from Π_1 as belonging to Π_2 is

$$P(2|1) = \begin{cases} p_1\Phi\left[\frac{c_a - E(U)}{\sqrt{var(U)}}\right], & if\ x_1 < x_2 \\ p_1\Phi\left[\frac{c_b - E(U)}{\sqrt{V(U)}}\right], & if\ x_2 < x_1 \end{cases} \tag{1}$$

Also, the probability of wrongly classifying an observation originally from Π_2 as belonging to Π_1 is

$$P(1|2) = \begin{cases} p_2\Phi\left[\frac{c_a - E(U)}{\sqrt{var(U)}}\right], & if\ x_1 < x_2 \\ p_2\Phi\left[\frac{c_b - E(U)}{\sqrt{V(U)}}\right], & if\ x_2 < x_1 \end{cases} \tag{2}$$

Where

$$E(U) = \begin{cases} \dfrac{\lambda\beta^1(\beta\beta^1 - \alpha'\beta'\alpha) + (\gamma+\omega)(\alpha'+\beta)}{\alpha'(\alpha+\beta)}, & if\ x_1 < x_2 \\ \lambda\left[\dfrac{\beta\beta' - \alpha'\alpha}{\alpha'\beta'(\alpha+\beta)}\right] + (\gamma+\omega)\left[\dfrac{\beta'+\alpha}{\beta'(\alpha+\beta)}\right], & if\ x_2 < x_1 \end{cases}$$

And

$$V(U) = \begin{cases} \dfrac{1}{(\alpha+\beta)^2}\left\{\left[\dfrac{\lambda^2(\beta^2\beta'^2 - \alpha^2\alpha'^2)}{\alpha'^2\beta'^2}\right] + (\gamma+\omega)\left[\dfrac{\alpha'^2 + 2\alpha\beta + \beta^2}{\alpha'^2}\right]\right\}, & if\ x_1 < x_2 \\ \dfrac{1}{(\alpha+\beta)^2}\left\{\left[\dfrac{\lambda^2\beta^2(\beta'^2 - \alpha'^2)}{\alpha'^2\beta'^2}\right] + (\gamma+\omega)\left[\dfrac{\beta'^2 + 2\alpha\beta + \alpha^2}{\beta'^2}\right]\right\}, & if\ x_2 < x_1 \end{cases}$$

In the same way, if

$$U = \begin{cases} (\theta_{12} - \theta_{11})x_1 + (\lambda_2 - \lambda_1)x_2 \geq d_a & if\ x_1 < x_2 \\ (\theta_{22} - \theta_{21})x_2 + (\delta_2 - \delta_1)x_1 \geq d_b & if\ x_2 < x_1 \end{cases}$$

on the assumption that $U$is distributed according to the Marshal and Olkin bivariate exponential function, then $P(2|1)$ and $P(1|2)$ can be written as in (1) and (2) above by replacing c_a and c_b with d_a and d_b respectively with $E(U)$ and $V(U)$ given in the case of Marshal and Olkin as

$$E(U) = \begin{cases} \dfrac{(\theta_{12} - \theta_{11})(\theta_2 + \theta_{12}) + (\lambda_2 - \lambda_1)(\theta_1 + \theta_{12})}{(\theta_1 + \theta_{12})(\theta_2 + \theta_{12})}, & if\ x_1 < x_2 \\ \dfrac{(\theta_{22} - \theta_{21})(\theta_2 + \theta_{12}) + (\delta_2 - \delta_1)(\theta_1 + \theta_{12})}{(\theta_1 + \theta_{12})(\theta_2 + \theta_{12})}, & if\ x_2 < x_1 \end{cases}$$

And

$$V(U) = \begin{cases} \dfrac{(\theta_{12} - \theta_{11})^2(\theta_2 + \theta_{12})^2 + (\lambda_2 - \lambda_1)^2(\theta_1 + \theta_{12})^2}{(\theta_1 + \theta_{12})^2(\theta_2 + \theta_{12})^2}, & if\ x_1 < x_2 \\ \dfrac{(\theta_{22} - \theta_{21})^2(\theta_1 + \theta_{12})^2 + (\delta_2 - \delta_1)^2(\theta_2 + \theta_{12})^2}{(\theta_1 + \theta_{12})^2(\theta_2 + \theta_{12})^2}, & if\ x_2 < x_1 \end{cases}$$

Again, if

$$U = \begin{cases} (\theta_{21} - \theta_{11})x_1 + (\varphi_2 - \varphi_1)x_2 \geq h_a & if\ x_1 < x_2 \\ (\theta_{22} - \theta_{21})x_2 + (\psi_2 - \psi_1)x_1 \geq h_b & if\ x_2 < x_1 \end{cases}$$

on the assumption that$U$ is distributed according to the Block and Basu bivariate exponential function, then $P(2|1)$ and $P(1|2)$ can also be written as in (1) and (2) above by replacing c_a and c_b with h_a and h_b respectively with $E(U)$ and $V(U)$ given in the case of Block and Basu as

$$E(U) = \begin{cases} \frac{(\theta_{21}-\theta_{11})}{(\theta_1+\theta_{12})}\left[1+\frac{\theta_{12}\theta_2}{\theta(\theta_1+\theta_2)}\right]+\frac{(\varphi_2-\varphi_1)}{(\theta_2+\theta_{12})}\left[1+\frac{\theta_{12}\theta_1}{\theta(\theta_1+\theta_2)}\right], & if\ x_1 < x_2 \\ \frac{(\theta_{22}-\theta_{21})}{(\theta_2+\theta_{12})}\left[1+\frac{\theta_{12}\theta_1}{\theta(\theta_1+\theta_2)}\right]+\frac{(\psi_2-\psi_1)}{(\theta_1+\theta_{12})}\left[1+\frac{\theta_{12}\theta_2}{\theta(\theta_1+\theta_2)}\right], & if\ x_2 < x_1 \end{cases}$$

And

$$V(U = \begin{cases} \frac{\theta_{12}\theta_2(\theta_{21}-\theta_{11})^2}{(\theta_1+\theta_{12})^2}\left[\frac{1}{\theta_{12}\theta_2}+\frac{2\theta_1\theta+\theta_{12}\theta_2}{\theta^2(\theta_1+\theta_2)^2}\right]+\frac{\theta_{12}\theta_1(\varphi_2-\varphi_1)^2}{(\theta_2+\theta_{12})^2}\left[\frac{1}{\theta_{12}\theta_1}+\frac{2\theta_2\theta+\theta_{12}\theta_1}{\theta^2(\theta_1+\theta_2)^2}\right], & if\ x_1 < x_2 \\ \frac{\theta_{12}\theta_1(\theta_{22}-\theta_{21})^2}{(\theta_2+\theta_{12})^2}\left[\frac{1}{\theta_{12}\theta_1}+\frac{2\theta_2\theta+\theta_{12}\theta_1}{\theta^2(\theta_1+\theta_2)^2}\right]+\frac{\theta_{12}\theta_2(\psi_2-\psi_1)^2}{(\theta_1+\theta_{12})^2}\left[\frac{1}{\theta_{12}\theta_2}+\frac{2\theta_1\theta+\theta_{12}\theta_2}{\theta^2(\theta_1+\theta_2)^2}\right], & if\ x_2 < x_1 \end{cases}$$

2.2 Application

In other to examine the applicability of the derived allocation rules, random data were generated for $f(.)$ under various samples sizes where $f(.)$ is the bivariate exponential density function. Apparent error rate was reported for each of MOBVE, BBBVE and FBVE for both $x_1 < x_2$ and $x_2 < x_1$ as seen in Table 1.

Table 1. Apparent error rate for the three models using simulated data

	$x_1 < x_2$			$x_2 < x_1$		
Sample sizes	MOBVE	BBBVE	FBVE	MOBVE	BBBVE	FBVE
10	0.3500	0.4500	0.4000	0.4000	0.4500	0.4000
20	0.5250	0.5250	0.4750	0.5750	0.5750	0.4750
50	0.5400	0.5400	0.5200	0.5400	0.5200	0.4500
100	0.4600	0.4550	0.4250	0.5500	0.5300	0.3800
250	0.4220	0.4460	0.4120	0.4720	0.4720	0.3780
500	0.4720	0.4860	0.4520	0.4950	0.5100	0.3980
750	0.4600	0.4713	0.4453	0.5033	0.5220	0.3807
1000	0.4605	0.4680	0.4450	0.5020	0.5130	0.3765
1500	0.4473	0.4580	0.4343	0.4893	0.5023	0.3727
2000	0.4525	0.4680	0.4450	0.4875	0.5035	0.3813

Apparent error rate as presented in Table 1 indicates the applicability of the allocation rules presented above. MOBVE and BBBVE generally produced less error rate when $x_1 < x_2$ than when $x_2 < x_1$ while the FBVE produced less error rate when $x_2 < x_1$. In the normal distribution function, error rate mostly reduces (or stabilizes) with increase in sample size,however, such was not the case in the bivariate exponential functions studied here.Only few larger sample sizes produced less error rate.

2.3 Illustrative examples

The allocation rules presented above were used to examine twosets of real-life data. The first set consist of n=60 observations representing the number of days between observed successive lowest temperature, denoted as x_1 and number of days between observed successive highest precipitation denoted byx_2 recorded in two different meteorological stations of Nigeria Institute for Oil Palm Research, Edo State. Bivariate exponential distribution has been applied and reported to be a good fit distribution for studying weather data. The grouping variable are the two different stations (both in same state of Nigeria) of the Institute. The data can be made available on request.

The second set is a summarized extract of reported observations representing time (in days) from exposure to illness onset denoted here asx_1,and, the number of days from illness onset to confirmation of infection status denoted by x_2 of the novel coronavirus (covid-19) of n=126 individuals in Wuhan Chinaas at January 2020. Only individuals with complete recorded information was included in the analysis. The data is available at http://www.mdpi.com/2077-0383/9/2/538/s1. The grouping variable are; Π_1: Wuhan Residents (those who Lives Works and Studies in Wuhan) and Π_2: Other Residents (those who travelled to Wuhan).

Since both x_1 and x_2 in the two sets are from same sampling unit, there are obvious correlation between the two variables. Given that these data follows the bivariate exponential distribution and the allocation rules above holds, the essence is that, in the first set of data, an observation could be classified as either belonging to the main station or the sub-station; and in the second set, an individual case of covid-19 could be classified as either a Wuhan resident or other resident. Results obtained are reported in Table 2.

Table 2. Apparent error rate for the three models using real-life data

	$x_1 < x_2$			$x_2 < x_1$		
Dataset	MOBVE	BBBVE	FBVE	MOBVE	BBBVE	FBVE
Set I (n=60)	0.4688	0.5000	0.4500	0.5000	0.5000	0.3833
Set II (n=126)	0.4400	0.4800	0.4240	0.4400	0.5600	0.4080

Reported apparent error rates for the real-life data do not differ markedly from those reported for the simulated data. Whereas, each of the originating exponential functions considered above may have different forms, their applicability in any appropriate real-life situation is not out of place. The choice of which to use would depend on availability and reported performance. As in this study, the choice of which allocation rule to use could be enhanced by considering their respective apparent error rates.

3. Conclusion

Discriminant analysis in the presence of violation of normality assumption of the distributions of the study variables has been discussed. This becomes necessary because such scenario often arise in many real-life situations as seen in the two illustrative examples above.This study has provided a framework for discriminant analysis when the assumption of normality is violated and the resulting data follows a bivariate exponential function.

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