Partial Randomized Response Model for Estimating a Rare Sensitive Attribute in Probability Proportional to Size Measures Using Poisson Distribution

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In this paper motivated by Lee et al (2013) and Singh et al (2017,2020), we have suggested the estimation procedures of mean number of persons possessing sensitive attribute using Narjis and Shabbir (2020) randomized response model for the population which comprises of some clusters and the population is stratified with some clusters in each stratum. The estimator for the mean number of persons possessing sensitive attribute under a Poisson distribution, its variance, and the estimator of the variance are proposed under two-stage and stratified two-stage sampling schemes when the parameter of the rare sensitive attribute is pretended to be known and unknown. We employ the sampling scheme with probability proportional to size to select the first-stage units and simple random sampling with replacement to select the second-stage units. The performance of the suggested estimation procedures are demonstrated through numerical illustration over Singh and Suman (2019) estimators.

Keywords: Poisson distribution, Randomized response model, Probability proportional to size, Rare sensitive attribute, Unrelated non-sensitive attribute, Two-stage sampling, Stratified two-stage sampling.

Introduction

Concerns in the context of privacy and confidentiality often provide substantial non-responses and false responses, especially in surveys that ask direct questions about sensitive attributes such as use of illegal cannabis plant, cheating in exams, extra marital affairs, domestic violence, sexual behavior, mental disorder, criminal history, tax evasion, drug abuse, gambling, abortion and others. The randomized response technique is quite effective in reducing the non-response rate and inflated response bias. Warner (1965) was first to introduce randomized response technique (RRT) which uses a randomization device bearing two statements, one on sensitive character and other on its complement and sample units are drawn by simple random sampling with replacement (SRSWR) procedure.
The RRT models can be divided into three categories such as full, partial and optional. In full RRT model, all respondents are urged to use the randomization device and give the response as per the question (i.e. sensitive and non-sensitive) occurred on randomization device, whereas, in partial RRT model, an additional stage or randomization device is added to incorporate the element of truthful response. In two stage or partial RRT model, the first randomization device has two options: (i) Do you belong to the sensitive group? and (ii) Go to the second randomization device, with known probabilities U and (1-U) respectively. The second randomization device is exactly same as provided in full RRT model. In optional RRT model, respondents are urged to give the truthful answer if he/she considers the question to be non-sensitive and use the randomization device if he/she considers the question sensitive and give the answer after using randomization device [see, Narjis and Shabbir (2019), p.1].

We consider the problem where the number of persons possessing a rare sensitive attribute is very small and the large sample size is necessary for providing the enough précised estimate of this number. For instance, the proportion of AIDS patients who continue having affairs with strangers, the proportion of persons who have witnessed a murder, the proportion of persons who are told by their doctors that they will not survive long due to ghastly disease etc. Now days the communication system is increasing rapidly, so it is possible and easy to conduct such large randomized surveys over the internet or telephone etc. Land et al (2012) was first to consider the problem of estimating the mean total number of persons who possesses a rare sensitive attribute in the population. Later various authors including Singh and Tarray (2014, 2017), Tarray and Singh (2015), Singh et al (2019), Tarray et al (2019) among others have tackled this problem.

Lee et al (2013) have proposed a variant of Land et al's (2012) randomized response model when a population consists of some clusters and the population is stratified with some clusters in each stratum. In this paper following Lee et al (2013) we have made an effort to extend Narjis and Shabbir (2019) partial randomized response model when a population comprises of some clusters and population is stratified with some clusters in each stratum. We have derived the estimator for the mean number of persons who possess a rare sensitive attribute along with its variance and the variance estimator when the parameter of a rare unrelated attribute is supposed to be known and not known. The clusters are drawn with and without replacement.

**Probability proportional to size (PPS) sampling scheme for estimating the rare sensitive parameter under Poisson distribution**

We consider a finite population $\Omega = (\Omega_1, \Omega_2, ..., \Omega_N)$ of N clusters, known as first-stage units. The size of the $i^{th}$ cluster is $M_i$ $(i=1,2, ..., N)$ termed as second-stage units. Selection of n first-stage units (i.e. of n cluster) are made employing PPS sampling scheme with probabilities $p_i$ $(i=1,2, ..., n)$. At the second –stage we draw
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$m_i$ ($i = 1, 2, ..., n$) second-stage units from the ith selected first-stage unit using SRSWR scheme. We designate.

$\pi_a$ : The true proportion of persons having a rare sensitive attribute $A$,

$\pi_y$ : The true proportion of persons having a unrelated rare non-sensitive attribute $Y$,

$\pi_{ia}$ : The true proportion of persons with the rare sensitive attribute in the $i^{th}$ cluster,

$\pi_{iy}$ : The true proportion of persons with the unrelated rare non-sensitive attribute in the $i^{th}$ cluster.

We also denote

\[ M_0 = \sum_{i=1}^{N} M_i \quad \text{and} \quad m = \sum_{i=1}^{n} m_i. \]

[see, Lee et al (2013) and Singh et al (2020)].

Estimation of a rare sensitive attribute in a two-stage unrelated question RRT model

We have discussed the estimation methods for mean number of persons possessing a rare sensitive attribute using Narjis and Shabbir (2020) randomized device when the clusters are drawn with and without replacement depending on the cluster sizes and with equal probability (i.e. Simple random sampling with replacement). We have investigated the unbiased estimators for the mean number of individuals and their properties are studied when the unrelated rare innocuous attribute $Y$ is assumed to be known and unknown. Response from the elementary units in the second stage samples are obtained on employing randomized response device of Narjis and Shabbir (2020).

When the unrelated rare innocuous attribute is known

When the population proportion $\pi_y$ of persons having the unrelated rare attribute is known, respondents are requested to use the randomization device and answer without revealing their having the attribute or not. In the proposed model, each selected respondent in the sample from the $i^{th}$ cluster has been given two randomization devices $(R_1, R_2)$.

Assuming the proportion of rare non-sensitive unrelated attribute is known, the responses from the elementary units in the second stage sample were collected using the Mangat (1992) randomization device which comprises the following statements for the ith cluster (i.e. the first-stage randomization device, $R_1$ has two kinds of
statements for the $i^{th}$ cluster):
(i) Do you possess rare sensitive attribute $A$?
(ii) Go to the randomization device $R_2$
with probabilities $U_i$ and $(1-U_i)$ respectively.

The second-stage randomization device, $R_2$ consists of three statements:
(i) Do you have the rare sensitive attribute $A$?
(ii) Do you have the rare non-sensitive unrelated attribute $Y$?
(iii) Draw one more card
with corresponding probabilities $P_{1i}$, $P_{2i}$ and $P_{3i}$ respectively such that
$\sum_{j=1}^{3} P_{ji} = 1, j = 1, 2, 3$. If statement (iii) appeared then respondent repeat the process
without replacing the card. Again, if in second draw the statement (iii) appeared the
respondent is asked to report his/her actual status about sensitive attribute $A$. The
respondent should answer the question with “yes” (or “no”), if his/her actual status
matches (un-matches) with the statement on the card respectively. The investigator
does not know the respondent’s answers, whether its from the sensitive question or
from the actual status because interviewee performed randomization process
confidentialy. Thus the privacy of interviewee(s) is protected and they responds
without any fear [see, Narjis and Shabbir (2020), p.2].

The probability of obtaining answer “yes” from the respondent in the $i^{th}$ cluster is :
$$\theta_{i0} = U_i \pi_{ia} + (1-U_i) \left[ (P_{1i} \pi_{ia} + P_{2i} \pi_{iy}) \left( 1 + P_{3i} \frac{k_i}{(k_i-1)} \right) + P_{3i}^2 \frac{k_i}{(k_i-1)} \pi_{ia} \right]$$
(3.1)
where the randomization device $R_2$ consists of a deck of $k_i$ cards provided to the
respondents selected from the $i^{th}$ cluster.

Consider selecting a large sample of persons from the $i^{th}$ cluster in the population
such that $m_i \to \infty$, $\pi_{ia} \to 0$, $\pi_{iy} \to 0$, then $m_i \pi_{ia} = \lambda_{ia}$, $m_i \pi_{iy} = \lambda_{iy}$ and $\theta_{i0} \to 0$
,then $m_i \theta_{i0} = \lambda_{i0}$ (finite),
$$\lambda_{i0} = U_i \lambda_{ia} + (1-U_i) \left[ (P_{1i} \lambda_{ia} + P_{2i} \lambda_{iy}) \left( 1 + P_{3i} \frac{k_i}{(k_i-1)} \right) + P_{3i}^2 \frac{k_i}{(k_i-1)} \lambda_{ia} \right],$$
(3.2)
Let $y_{i1}, y_{i2}, ..., y_{im}$ be a random sample drawn from $i^{th}$ cluster which follows Poisson
distribution with mean $\lambda_{i0}$. Then the estimator for the mean number of individuals
with the rare sensitive characteristics, $\lambda_{ia} (\lambda_{ia} = m_i \pi_{ia})$ is defined as
\[ \lambda_{i0} = U_i \lambda_{iu} + (1-U_i) \left( P_{iu} \lambda_{iu} + P_{i2} \lambda_{iy} \right) \left( 1 + P_{3i} \frac{k_i}{(k_i-1)} \right) + P_{3i}^2 \frac{k_i}{(k_i-1)} \lambda_{iu} \]

Thus \( E(\hat{\lambda}_{appcw}) = E_1 E_2 \left( \hat{\lambda}_{appcw} \right) \), where \( E_1 \) and \( E_2 \) are the expectations over the first and second stage samples respectively. Further we have

\[ E_1 E_2 \left( \hat{\lambda}_{appcw} \right) = E_1 \left[ \frac{1}{nM_0} \sum_{i=1}^{n} \frac{M_i \hat{\lambda}_{iu}}{p_i} \right] = E_1 \left[ \frac{1}{nM_0} \sum_{i=1}^{n} M_i \hat{\lambda}_{iu} \right], \]

Now, \( E_2 \left( \hat{\lambda}_{appcw} \right) = E_2 \left[ \left( \frac{1}{m_i} \sum_{j=1}^{m_i} y_{ij} - B_i \right) / D_i \right] \),

where \( B_i = (1-U_i) P_{i2} \left( 1 + P_{3i} \frac{k_i}{(k_i-1)} \right) \lambda_{iy} \)

and \( D_i = U_i + (1-U_i) \left( P_{i1} + P_{3i} \frac{k_i}{(k_i-1)} (P_{i1} + P_{3i}) \right). \)
We have
\[
E_2(\hat{\lambda}_{appzwr}) = \frac{1}{D_i} \left[ \frac{1}{m_i} \sum_{j=1}^{m_i} E_2(y_{ij}) - B_i \right] = \frac{1}{D_i} \left[ \frac{1}{m_i} \sum_{j=1}^{m_i} \hat{\lambda}_{ij0} - B_i \right],
\]
\[
= \frac{1}{D_i} [\hat{\lambda}_{ij0} - B_i].
\]
Thus finally, we have
\[
E_1 E_2(\hat{\lambda}_{appzwr}) = E_1 \left[ \frac{1}{nM_0} \sum_{i=1}^{n} M_i \hat{\lambda}_{ia} \right],
\]
\[
= \frac{1}{M_0} \sum_{i=1}^{n} p_i \frac{M_i \hat{\lambda}_{ia}}{p_i},
\]
\[
= \frac{1}{M_0} \sum_{i=1}^{n} M_i \hat{\lambda}_{ia} = \hat{\lambda}_a.
\]
Therefore, \( \hat{\lambda}_{appzwr} \) is an unbiased estimator of \( \lambda_a \).

Theorem 3.2 The variance of the unbiased estimator \( \hat{\lambda}_{appzwr} \) is
\[
V(\hat{\lambda}_{appzwr}) = \frac{1}{nM_0^2} \left[ \sum_{i=1}^{n} p_i \left( \frac{M_i \hat{\lambda}_{ia}}{p_i} - M_0 \lambda_a \right)^2 + \sum_{i=1}^{n} M_i^2 \Phi_i \right],
\]
where \( \Phi_i = \left[ \frac{\hat{\lambda}_{ia}}{D_i} + \frac{(1 - U_i) p_{2i} \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) \hat{\lambda}_{iy}}{D_i^2} \right]. \)

Proof- Let \( V_1 \) be the variance over the first-stage sample and \( V_2 \) be the variance over the second-stage sample. The variance of \( \hat{\lambda}_{appzwr} \) is given by
\[
V(\hat{\lambda}_{appzwr}) = V_1 E_2(\hat{\lambda}_{appzwr}) + E_1 V_2(\hat{\lambda}_{appzwr}).
\]
Now,
\[
V_1 E_2(\hat{\lambda}_{appzwr}) = V_1 \left[ \frac{1}{nM_0} \sum_{i=1}^{n} M_i \hat{\lambda}_{ia} \right],
\]
\[
= V_1 \left[ \frac{1}{nM_0} \sum_{i=1}^{n} M_i \hat{\lambda}_{ia} \right],
\]
\[
= \frac{1}{nM_0^2} \sum_{i=1}^{n} p_i \left( \frac{M_i \hat{\lambda}_{ia}}{p_i} - M_0 \lambda_a \right)^2.
\]
Next,
\[
E_1 V_2(\hat{\lambda}_{appzwr}) = E_1 \left[ \frac{1}{nM_0} \sum_{i=1}^{n} M_i \hat{\lambda}_{ia} \right],
\]
\[
= \frac{1}{nM_0^2} \sum_{i=1}^{n} p_i \left( \frac{M_i \hat{\lambda}_{ia}}{p_i} - M_0 \lambda_a \right)^2.
\]
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\[
= E_1 \left[ \frac{1}{(nM_0)^2} \sum_{i=1}^{n} \frac{M_i^2}{p_i^2} V_2 \left( \hat{\lambda}_{ia} \right) \right],
\]

\[
= E_1 \left[ \frac{1}{(nM_0)^2} \sum_{i=1}^{n} \frac{M_i^2}{p_i^2} V_2 \left( \frac{1}{m_i} \sum_{j=1}^{m_i} y_{ij} - B_i \right) \right],
\]

\[
= E_1 \left[ \frac{1}{(nM_0)^2} \sum_{i=1}^{n} \frac{M_i^2}{p_i^2 D_i^2} \left( \frac{1}{m_i} \sum_{j=1}^{m_i} \lambda_{i,0} \right) \right],
\]

\[
= E_1 \left[ \frac{1}{(nM_0)^2} \sum_{i=1}^{n} \frac{M_i^2}{p_i^2 D_i^2 m_i} \right],
\]

\[
= \left( 1 - U_i \right) \left[ \left( P_{1i} \lambda_{ia} + P_{2i} \lambda_{ia} \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) + P_{3i} \frac{k_i}{(k_i - 1)} \lambda_{ia} \right) \right] + \left( 1 - U_i \right) \left[ \frac{(1 - U_i) P_{2i} \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) \lambda_{ia}}{D_i^2} \right],
\]

\[
= \frac{1}{nM_0^2} \sum_{i=1}^{N} \frac{M_i^2}{p_i m_i} \left[ \frac{(1 - U_i) P_{2i} \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) \lambda_{ia}}{D_i^2} \right],
\]

\[
= \frac{1}{nM_0^2} \sum_{i=1}^{N} \frac{M_i^2 \Phi_{ia}}{p_i m_i}.
\]

Inserting (3.8) and (3.9) in (3.7) we get the variance of the unbiased estimator \( \hat{\lambda}_{appcwr} \).

This completes the proof of the theorem.

Theorem 3.3 The unbiased estimate of the variance of the suggested estimator \( \hat{\lambda}_{appcwr} \) is given by
\[ \hat{V}(\hat{\lambda}_{\text{appswr}}) = \frac{1}{n(n-1)M_0^2} \sum_{i=1}^{n} \left( \frac{M_i \hat{\lambda}_{ia} - \hat{\lambda}_{\text{appswr}}}{p_i} \right)^2 \]  

(3.10)

Proof is simple so omitted.

Estimation when the first-stage sample is selected with PPS sampling

The size of the cluster is known and when \( n \) clusters are selected with replacement depending on size of each cluster, \( M_i \), the probability \( p_i \) should be considered as

\[ \frac{M_i}{M_0} \text{ (i.e. } p_i = \frac{M_i}{M_0} \text{) for the i}^{\text{th}} \]

cluster. Then it is known as PPS. From the PPS, the unbiased estimator for \( \lambda_a \) is given by

\[ \hat{\lambda}_{\text{appswr}} = \frac{1}{n} \sum_{i=1}^{n} \hat{\lambda}_{ia}, \]  

(3.11)

using \( p_i = \frac{M_i}{M_0} \), and the variance is given by

\[ V(\hat{\lambda}_{\text{appswr}}) = \frac{1}{nM_0} \left[ \sum_{i=1}^{N} M_i \left( \lambda_{ia} - \lambda_a \right)^2 + \sum_{i=1}^{n} \frac{M_i \Phi_i}{m_i} \right]. \]  

(3.12)

Further the estimator for the variance of \( \hat{\lambda}_{\text{appswr}} \) is given by

\[ \hat{V}(\hat{\lambda}_{\text{appswr}}) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \left( \hat{\lambda}_{ia} - \frac{\hat{\lambda}_{\text{appswr}}}{M_0} \right)^2. \]  

(3.13)

Estimation when the first-stage sample is selected with probability proportional to size without replacement (PPSWOR)

When \( n \) clusters are selected without replacement from \( N \) clusters with size \( M_i (i = 1, 2, ..., N) \) each and \( \delta_i \) is an inclusion probability of a unit \( i \) in a sample set without replacement, then the estimator for \( \lambda_a \), which is the parameter of the rare sensitive attribute, is given by

\[ \hat{\lambda}_{\text{appswor}} = \frac{1}{M_0} \sum_{i=1}^{M} \frac{M_i \hat{\lambda}_{ia}}{\delta_i}. \]  

(3.14)

The variance of estimator \( \hat{\lambda}_{\text{appswor}} \) is given by

\[ V(\hat{\lambda}_{\text{appswor}}) = \frac{1}{M_0^2} \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \delta_i \delta_j - \delta_{ij} \right) \left( \frac{M_i \hat{\lambda}_{ia} - M_j \hat{\lambda}_{ja}}{\delta_i} \right)^2 + \sum_{i=1}^{N} \frac{M_i^2 \Phi_i}{\delta_i m_i} \right], \]  

(3.15)

where \( \delta_{ij} \) is an inclusion probability of units \( i \) and \( j \) in a sample set without replacement.

The estimator for the variance of \( \hat{\lambda}_{\text{appswor}} \) is given by
\[ \hat{V}(\hat{\lambda}_{\text{appswr}}) = \frac{1}{M_0^2} \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \delta_i \delta_j - \delta_{ij} \right) \left( \frac{M_i \hat{\lambda}_{ia} - M_j \hat{\lambda}_{ja}}{\delta_i} \right)^2 + \sum_{i=1}^{n} \frac{M_i^2}{\delta_i} \cdot \Phi_i \right], \]  
(3.16)

where \( \Phi_i = \frac{\hat{\lambda}_{ia}}{D_i^2} \left[ \frac{(1 - U_i)p_i}{D_i} \{1 + p_i \frac{k_i}{(k_i - 1)}\} \hat{\lambda}_{iy} \right] \).

Estimation when the first-stage sample is selected using SRSWR scheme
We consider the estimation when the first-stage sample is selected using SRSWR scheme. In this case, the selection probability for all the selected clusters in the first stage is \( p_i = \frac{1}{N} (i = 1, 2, ..., n) \). The estimator of the parameter \( \lambda_a \); when the first-stage sample units are chosen with equal probability and with replacement in two-stage sampling, is
\[ \hat{\lambda}_{\text{awr}} = \frac{N}{nM_0} \sum_{i=1}^{n} M_i \hat{\lambda}_{ia}. \]  
(3.17)

The variance of the estimator \( \hat{\lambda}_{\text{awr}} \) is
\[ V(\hat{\lambda}_{\text{awr}}) = \frac{N}{nM_0^2} \left[ \frac{N}{N-1} \sum_{i=1}^{n} \left( M_i \hat{\lambda}_{ia} - \bar{M} \hat{\lambda}_a \right)^2 + \sum_{i=1}^{n} \frac{M_i^2}{m_i} \Phi_i \right], \]  
(3.18)

and its estimate
\[ \hat{V}(\hat{\lambda}_{\text{awr}}) = \frac{1}{n(n - 1)M_0^2} \sum_{i=1}^{n} \left( NM_i \hat{\lambda}_{ia} - \hat{\lambda}_{\text{awr}} \right)^2, \]  
(3.19)

where \( \bar{M} = \frac{M_0}{N} \).

3.1.4. Comparing Probability proportional to size with replacement (PPSWR) and the equal probability two-stage sampling
From (3.12) and (3.18) we have
\[ V(\hat{\lambda}_{\text{awr}}) - V(\hat{\lambda}_{\text{appswr}}) = \frac{1}{nM_0} \left[ \frac{N}{M_0} \cdot \frac{N}{N - 1} \sum_{i=1}^{n} \left( M_i \hat{\lambda}_{ia} - \bar{M} \hat{\lambda}_a \right)^2 + \frac{N}{M_0} \sum_{i=1}^{n} \frac{M_i^2}{m_i} \Phi_i \right] \]
\[ - \sum_{i=1}^{n} M_i \left( \hat{\lambda}_{ia} - \hat{\lambda}_a \right)^2 - \sum_{i=1}^{n} \frac{M_i}{m_i} \Phi_i \].

Under the assumption \( (N - 1) \equiv N \), and \( \bar{M} = \frac{M_0}{N} \), we have
\[ V(\hat{\lambda}_{awr}) - V(\hat{\lambda}_{appswr}) = \frac{1}{nM_0M} \left[ \sum_{i=1}^{N} (M_i \lambda_{ia} - \bar{M} \lambda_a)^2 - \bar{M} \sum_{i=1}^{N} M_i (\lambda_{ia} - \lambda_a)^2 \right. \\
+ \left. \sum_{i=1}^{N} \frac{M_i^2}{m_i} \Phi_i - \bar{M} \sum_{i=1}^{N} \frac{M_i}{m_i} \Phi_i \right] \]

which will be zero when \( M_i = \bar{M} \). It follows that when the size of the clusters are all the same i.e. \( M_i = \bar{M} \), the probability \( p \left( \frac{M_i}{M_0} \right) \) in PPS should be the same as \( \frac{1}{N} \). Thereby meaning is that for \( M_i = \bar{M} \), the estimator \( \hat{\lambda}_{appswr} \) is at par with \( \hat{\lambda}_{awr} \), i.e. both the estimators \( \hat{\lambda}_{appswr} \) and \( \hat{\lambda}_{awr} \) are equally efficient. However, when the difference \( (M_i - \bar{M}) \) is larger the first term \( \sum_{i=1}^{N} (M_i - \bar{M})^2 \lambda_{ia}^2 \), and the third term, \( \sum_{i=1}^{N} \frac{(M_i - \bar{M})^2 \Phi_i}{m_i} \), are increasing in the above expression. Usually the PPS is more efficient than the equal probability two-stage sampling when clusters have different sizes.

**Estimation procedure of a rare attribute under two-stage sampling when the proportion of rare non-sensitive unrelated attribute is not known**

It is to be mentioned that here two parameters are unknown so responses are gathered twice from each individual using two randomization devices in each cluster. These randomization devices consist of the decks of \( k_i \) similar cards as described Section 3.1. Firstly, the respondents selected from \( i^{th} \) cluster are urged to response “yes” or “no” using the following two-stage randomization devices.

The first randomization device is given as follows:

First-stage randomization device \( R_{1i} \) consists of two statements:
(i) Do you have the rare sensitive attribute \( A \)?
(ii) Go to randomization device \( R_{2i} \) with corresponding probabilities \( U_{1i} \) and \( (1 - U_{1i}) \) respectively.

The second-stage randomization device consists of three statements:
(i) Do you have the rare sensitive attribute \( A \) ?
(ii) Do you have the rare non-sensitive unrelated attribute \( Y \) ?
(iii) Draw one more card
with corresponding probabilities $P_{1i}$, $P_{2i}$, and $P_{3i}$ respectively such that $\sum_{j=1}^{3} P_{ji} = 1$. If statement (iii) appeared on card then respondent repeat the process without replacing the card. Again, if in second draw the statement (iii) appeared the respondent is asked to report his/her actual status about sensitive attribute $A$.

Next, the respondent is urged again to answer one of the same questions using second randomization device.

The second randomization device $R_{12}$ consists of two statements:

(i) Do you have the rare sensitive attribute $A$?

(ii) Go to randomization device $R_{22}$

with corresponding probabilities $U_{2i}$ and $(1-U_{2i})$ respectively.

The second-stage randomization device $R_{22}$ consists of three statements:

(i) Do you have the rare sensitive attribute $A$?

(ii) Do you have the rare non-sensitive unrelated attribute $Y$?

(iii) Draw one more card

with corresponding probabilities $Q_{1i}$, $Q_{2i}$, and $Q_{3i}$ respectively such that $\sum_{j=1}^{3} Q_{ji} = 1$.

If statement (iii) appeared on card then respondent repeat the process without replacing the card. Again, if in second draw the statement (iii) appeared the respondent is asked to report his/her actual status about sensitive attribute $A$.

Based on responses gathered using two randomization devices; the probabilities that respondents in the $i^{th}$ cluster answer “yes” are

$$\theta_{i1} = U_{1i} \pi_{ia} + (1-U_{1i}) \left[ P_{1i} \pi_{ia} + P_{2i} \pi_{iy} \left( 1 + P_{3i} \frac{k_i}{(k_i-1)} \right) + P_{3i}^2 \frac{k_i}{(k_i-1)} \pi_{ia} \right]$$

and

$$\theta_{i2} = U_{2i} \pi_{ia} + (1-U_{2i}) \left[ Q_{1i} \pi_{ia} + Q_{2i} \pi_{iy} \left( 1 + Q_{3i} \frac{k_i}{(k_i-1)} \right) + Q_{3i}^2 \frac{k_i}{(k_i-1)} \pi_{ia} \right].$$

Equation (3.20) and (3.21) can be rewritten as by assuming that as $m_i \to \infty$, $\theta_{i1} \to 0$, $\theta_{i2} \to 0$, then $m_i \theta_{i1} = \lambda_{i1}$, and $m_i \theta_{i2} = \lambda_{i2}$ respectively:

$$\lambda_{i1} = U_{1i} \lambda_{ia} + (1-U_{1i}) \left[ P_{1i} \lambda_{ia} + P_{2i} \lambda_{iy} \left( 1 + P_{3i} \frac{k_i}{(k_i-1)} \right) + P_{3i}^2 \frac{k_i}{(k_i-1)} \lambda_{ia} \right]$$

and

$$\lambda_{i2} = U_{2i} \lambda_{ia} + (1-U_{2i}) \left[ Q_{1i} \lambda_{ia} + Q_{2i} \lambda_{iy} \left( 1 + Q_{3i} \frac{k_i}{(k_i-1)} \right) + Q_{3i}^2 \frac{k_i}{(k_i-1)} \lambda_{ia} \right].$$

After deriving the results, the following two equations are obtained as
\[
\frac{1}{m_i} \sum_{j=1}^{m_i} y_{i1j} = \hat{U}_i \hat{\lambda}_{sau} + (1-U_i) \left[ P_{1i} \hat{\lambda}_{sau} + P_{2i} \hat{\lambda}_{syu} \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) + P_{2i}^2 \frac{k_i}{(k_i - 1)} \right] \]

and

\[
\frac{1}{m_i} \sum_{j=1}^{m_i} y_{i2j} = \hat{U}_i \hat{\lambda}_{sau} + (1-U_{2i}) \left[ Q_{3i} \hat{\lambda}_{sau} + Q_{2i} \hat{\lambda}_{syu} \left( 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right) + Q_{2i}^2 \frac{k_i}{(k_i - 1)} \right] ,
\]

where \( y_{i1j} \) and \( y_{i2j} \) denote the observed values in the first and the second responses from the \( j^{th} (j = 1, 2, ..., m_j) \) respondents in the \( i^{th} (i = 1, 2, ..., n) \) clusters. Solving equations (3.24) and (3.25) the estimators for \( \hat{\lambda}_{sau} \) and \( \hat{\lambda}_{syu} \) are:

\[
\hat{\lambda}_{sau} = \frac{1}{C_{ii} m_i} \left[ (1-U_{2i}) Q_{2i} \left( 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right) \sum_{j=1}^{m_i} y_{i1j} - (1-U_{1i}) P_{2i} \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) \sum_{j=1}^{m_i} y_{i2j} \right] ,
\]

(3.26)

\[
\hat{\lambda}_{syu} = \frac{1}{D_{ii} m_i} \left[ U_{2i} + (1-U_{2i}) \left[ Q_{3i} \left( 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right) + Q_{2i}^2 \frac{k_i}{(k_i - 1)} \right] \sum_{j=1}^{m_i} y_{i1j} - \left[ U_{1i} + (1-U_{1i}) \left[ P_{3i} \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) + P_{3i}^2 \frac{k_i}{(k_i - 1)} \right] \sum_{j=1}^{m_i} y_{i2j} \right] \right] ,
\]

(3.27)

where

\[
C_{ii} = U_{1i} (1-U_{2i}) Q_{2i} \left( 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right) - U_{2i} (1-U_{1i}) P_{2i} \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) + (1-U_{1i})(1-U_{2i}) \left( P_{1i} Q_{2i} - P_{1i} Q_{2i} \right) \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) \left( 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right) + (1-U_{1i})(1-U_{2i}) \frac{k_i}{(k_i - 1)} \left[ P_{3i}^2 Q_{2i} \left( 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right) - Q_{2i}^2 \frac{k_i}{(k_i - 1)} \right] - Q_{2i}^2 P_{1i} \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) \right) \neq 0 .
\]

(3.28)

\[
D_{ii} = U_{2i} (1-U_{1i}) P_{2i} \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) - U_{1i} (1-U_{2i}) Q_{2i} \left( 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right) + (1-U_{1i})(1-U_{2i}) \left( P_{2i} Q_{1i} - P_{1i} Q_{2i} \right) \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) \left( 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right) + (1-U_{1i})(1-U_{2i}) \frac{k_i}{(k_i - 1)} \left[ Q_{3i}^2 P_{1i} \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) - P_{3i} \left( 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right) \right] - P_{3i}^2 Q_{2i} \left( 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right) \right) \neq 0 .
\]

(3.29)
Thus the estimator for the parameter $\lambda_a$ of the rare sensitive attribute is given by

$$
\hat{\lambda}_{appzwru} = \frac{1}{nM_0} \sum_{i=1}^{n} M_i \hat{\lambda}_{iau}. 
$$

(3.30)

Theorem 3.4 The proposed estimator $\hat{\lambda}_{appzwru}$ for the mean number of persons who have the rare sensitive attribute is unbiased.

Proof: Since $\lambda_{i1j}$ and $\lambda_{i2j}$ are iid Poisson variates with parameters $\lambda_{i1}$ and $\lambda_{i2}$ respectively, therefore, we have

$$
E(\hat{\lambda}_{iau}) = E_1 \left[ \frac{1}{nM_0} \sum_{i=1}^{n} M_i \hat{\lambda}_{iau} \right].
$$

(3.31)

Now,

$$
E_2(\hat{\lambda}_{iau}) = E_2 \left[ \frac{1}{C_{i1}m_i} \left\{ d_{2i} \sum_{j=1}^{m} y_{i1j} - C_{2i} \sum_{j=1}^{m} y_{i2j} \right\} \right],
$$

$$
= \frac{1}{C_{i1}m_i} \left[ d_{2i} \sum_{j=1}^{m} E_2(y_{i1j}) - C_{2i} \sum_{j=1}^{m} E_2(y_{i2j}) \right],
$$

$$
= \frac{1}{C_{i1}m_i} \left[ d_{2i} \sum_{j=1}^{m} \lambda_{i1} - C_{2i} \sum_{j=1}^{m} \lambda_{i2} \right],
$$

$$
= \frac{1}{C_{i1}m_i} \left[ m_i d_{2i} \lambda_{i1} - m_i C_{2i} \lambda_{i2} \right],
$$

$$
= \frac{d_{2i} \lambda_{i1} - C_{2i} \lambda_{i2}}{C_{i1}},
$$

(3.32)

where $d_{2i} = (1 - U_{2i})Q_{2i} \left\{ 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right\}$, $C_{2i} = (1 - U_{2i})P_{2i} \left\{ 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right\}$.

Inserting the values of $\lambda_{i1}$ and $\lambda_{i2}$ from equations (3.22) and (3.23) respectively in (3.32) we get

$$
E_2(\hat{\lambda}_{iau}) = \lambda_{iau}.
$$

(3.33)

Putting (3.33) in (3.31) we have

$$
E(\hat{\lambda}_{iau}) = E_1 \left[ \frac{1}{nM_0} \sum_{i=1}^{n} M_i \lambda_{iau} \right],
$$

$$
= \frac{1}{M_0} \sum_{i=1}^{N} \frac{M_i}{p_i} \lambda_{iau},
$$

$$
= \frac{1}{M_0} \sum_{i=1}^{N} M_i \lambda_{iau} = \lambda_a.
$$
Theorem 3.5 The variance of the estimator $\hat{\lambda}_{appzwru}$ is given by
\[
V(\hat{\lambda}_{appzwru}) = \frac{1}{nM_0^2} \left[ \sum_{i=1}^n p_i \left( \frac{M_i \lambda_{ia}}{p_i} - M_0 \lambda_a \right)^2 + \sum_{i=1}^n \frac{M_i^2}{p_i} \Phi_i^{(12)} \right],
\]
(3.34)

where
\[
\Phi_i^{(12)} = \left[ d_{2i}^2 C_{3i} + C_{2i}^2 d_{3i} - 2C_{2i} C_{3i} d_{2i} \right] \lambda_{ia} + \left[ d_{2i}^2 C_{3i} + C_{2i}^2 d_{3i} - 2C_{2i}^2 d_{2i} \right] \lambda_y,
\]
\[
C_{2i} = (1 - U_{ti}) P_{2i} \left\{ 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right\},
\]
\[
C_{3i} = U_{ti} + (1 - U_{ti}) \left\{ P_{li} \left\{ 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right\} + P_{2i}^2 \frac{k_i}{(k_i - 1)} \right\},
\]
\[
d_{2i} = (1 - U_{2i}) Q_{2i} \left\{ 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right\},
\]
\[
d_{3i} = U_{2i} + (1 - U_{2i}) \left\{ Q_{li} \left\{ 1 + Q_{3i} \frac{k_i}{(k_i - 1)} \right\} + Q_{3i}^2 \frac{k_i}{(k_i - 1)} \right\}.
\]

Proof: The variance is decomposed by
\[
V(\hat{\lambda}_{appzwru}) = V_1 E_2(\hat{\lambda}_{appzwru}) + E_1 V_2(\hat{\lambda}_{appzwru}).
\]
(3.35)

Since $y_{1ij} \sim iid \text{ Poisson}(\lambda_{1i})$ and $y_{12j} \sim iid \text{ Poisson}(\lambda_{12})$,

$V(y_{1ij}) = E(y_{1ij}) = \lambda_{1i}$ and $V(y_{12j}) = E(y_{12j}) = \lambda_{12}$.

We have
\[
V_1 E_2(\hat{\lambda}_{appzwru}) = V_1 E_2 \left[ \frac{1}{nM_0^2} \sum_{i=1}^n \frac{M_i \lambda_{ia}}{p_i} \right],
\]
\[
= V_1 \left[ \frac{1}{nM_0^2} \sum_{i=1}^n \frac{M_i}{p_i} E_2(\hat{\lambda}_{ia}) \right],
\]
\[
= V_1 \left[ \frac{1}{nM_0^2} \sum_{i=1}^n \frac{M_i \lambda_{ia}}{p_i} \right],
\]
\[
= \frac{1}{nM_0^2} \sum_{i=1}^n p_i \left( \frac{M_i \lambda_{ia}}{p_i} - M_0 \lambda_a \right)^2
\]
(3.36)

and
\[
E_1 V_2(\hat{\lambda}_{appzwru}) = E_1 V_2 \left[ \frac{1}{nM_0} \sum_{i=1}^n \frac{M_i \lambda_{ia}}{p_i} \right],
\]
\[
= E_1 V_2 \left[ \frac{1}{nM_0} \sum_{i=1}^n \frac{M_i}{p_i} \frac{1}{C_i m_i} \left\{ d_{2i} \sum_{j=1}^{m_i} y_{1ij} - C_{2i} \sum_{j=1}^{m_i} y_{12j} \right\} \right],
\]
PARTIAL RANDOMIZED RESPONSE MODEL FOR ESTIMATING A RARE SENSITIVE ATTRIBUTE IN PROBABILITY PROPORTIONAL TO SIZE MEASURES USING POISSON DISTRIBUTION

\[
\hat{\lambda}_{\text{appcswru}}(\lambda_i) = V(y_{ij}) = \left[ U_{ij} \lambda_{ia} + (1 - U_{ij}) \left( P_{ij} \lambda_{ia} + P_{2i} \lambda_{iy} \right) \left( 1 + P_{3i} \frac{k_i}{(k_i - 1)} \right) + P_{3i}^2 \frac{k_i}{(k_i - 1)} \lambda_{ia} \right],
\]

(3.38)

and

\[
\lambda_{ij} = Cov(y_{ij}, y_{ij}) = E(y_{ij} y_{ij}) - E(y_{ij})E(y_{ij}),
\]

(3.39)

Inserting the values from (3.36) and (3.37) in (3.35), we obtained the expression of the variance of the estimator \( \hat{\lambda}_{\text{appcswru}} \) as given in (3.34).

The estimator of an unbiased estimator of the variance of \( \hat{\lambda}_{\text{appcswru}} \) is given by

\[
\hat{V}(\hat{\lambda}_{\text{appcswru}}) = \frac{1}{n(n-1)M_0^2} \sum_{i=1}^{n} \left( \frac{M_i \hat{\lambda}_{iaa} - \hat{\lambda}_{\text{appcswru}}}{p_i} \right)^2.
\]

(3.41)

Estimation when the first-stage sample is selected using PPSWR.

In the PPS, the selecting probability \( p_i \) for cluster \( i \) is defined as \( \frac{M_i}{M_0} \) (i.e. \( p_i = \frac{M_i}{M_0} \)).

The estimator for the parameter of the rare sensitive attribute; \( \lambda_a \) is defined as
\[ \hat{\lambda}_{\text{appswru}} = \frac{1}{n} \sum_{i=1}^{n} \hat{\lambda}_{iau}, \]  
(3.42)

and its variance is obtained as
\[ V(\hat{\lambda}_{\text{appswru}}) = \frac{1}{nM_0} \left[ \sum_{i=1}^{N} M_i (\hat{\lambda}_{iau} - \hat{\lambda}_a)^2 + \sum_{i=1}^{N} \frac{M_i \Phi^{(12)}_i}{C^2_i m_i} \right]. \]  
(3.43)

Further the estimator for the variance of \( \lambda_{\text{appswru}} \) is given by
\[ \hat{V}(\hat{\lambda}_{\text{appswru}}) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \left( \hat{\lambda}_{iau} - \frac{\hat{\lambda}_{\text{appswru}}}{M_0} \right)^2. \]  
(3.44)

### 3.2.2. Estimation when the first-stage sample is selected using SRSWR.

In this situation, the probability of selecting the clusters in the first stage is
\[ p_i = \frac{1}{N} (i = 1, 2, ..., n). \]

The estimator of the parameter \( \lambda_a \) when the first-stage sample units are selected with equal probability and with replacement in two-stage sampling, is
\[ \hat{\lambda}_{\text{awru}} = \frac{N}{nM_0} \sum_{i=1}^{n} M_i \hat{\lambda}_{iau}. \]  
(3.45)

The variance of the estimator \( \hat{\lambda}_{\text{awru}} \) is
\[ V(\hat{\lambda}_{\text{awru}}) = \frac{N}{nM_0^2} \left[ \frac{N}{N-1} \sum_{i=1}^{N} (M_i \hat{\lambda}_{iau} - M \hat{\lambda}_a)^2 + \sum_{i=1}^{N} \frac{M_i^2 \Phi^{(12)}_i}{C^2_i m_i} \right]. \]  
(3.46)

and its estimate
\[ \hat{V}(\hat{\lambda}_{\text{awru}}) = \frac{1}{n(n-1)M_0^2} \sum_{i=1}^{n} (NM_i \hat{\lambda}_{iau} - \hat{\lambda}_{\text{awru}})^2, \]  
(3.47)

where \( \overline{M} = \frac{M_0}{N} \).

**Estimation by PPSWOR**

After \( n \) clusters are drawn without replacement (WOR) from \( N \) clusters with size \( M_i \) each, \( m_i \) samples are selected in cluster \( i \) randomly with replacement. Let \( \delta_i \) is an inclusion probability of a unit \( i \) in a sample set without replacement, and \( \delta_{ij} \) is an inclusion probability of units \( i \) and \( j \) in a sample set without replacement. When the parameter of the rare unrelated attribute is unknown, the estimator for the rare sensitive attribute is defined by
\[ \hat{\lambda}_{\text{appswwor}} = \frac{1}{M_0} \sum_{i=1}^{n} \frac{M_i \hat{\lambda}_{iau}}{\delta_i}. \]  
(3.48)

The variance of estimator \( \hat{\lambda}_{\text{appswwor}} \) is given by
The variance of estimator $\hat{\lambda}_{\text{appsworu}}$ is given by

$$V(\hat{\lambda}_{\text{appsworu}}) = \frac{1}{M_0^2} \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} (\delta_i \delta_j - \delta_{ij}) \left( \frac{M_i \hat{\lambda}_{ia} - M_j \hat{\lambda}_{ja}}{\delta_i} \right)^2 + \sum_{i=1}^{N} M_i^2 \Phi_i^{(12)} C_{li}^2 m_i \right]. \quad (3.49)$$

The estimator for the variance of $\hat{\lambda}_{\text{appsworu}}$ is given by

$$\hat{V}(\hat{\lambda}_{\text{appsworu}}) = \frac{1}{M_0^2} \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} (\delta_i \delta_j - \delta_{ij}) \left( \frac{M_i \hat{\lambda}_{iau} - M_j \hat{\lambda}_{jau}}{\delta_i} \right)^2 + \sum_{i=1}^{n} M_i^2 \Phi_i^{(12)} C_{li}^2 (m_i - 1) \right], \quad (3.50)$$

where $\Phi_i^{(12)} = \left\{ \left( d_{i2}^2 C_{3i} + C_{2i}^2 d_{3i} - 2 C_{2i} C_{3i} d_{2i} d_{3i} \right) \hat{\lambda}_{iau} + \left( d_{i2}^2 C_{3i} + C_{2i}^2 d_{2i} - 2 C_{2i}^2 d_{2i}^2 \right) \hat{\lambda}_{jau} \right\}$. \quad (3.51)

Comparing PPSWR and the equal probability two-stage sampling

Assuming $(N-1) \equiv N$, From (3.46) and (3.49) we have

$$V(\hat{\lambda}_{\text{awru}}) - V(\hat{\lambda}_{\text{appsworu}}) = \frac{N}{nM_0} \sum_{i=1}^{N} \left( M_i \hat{\lambda}_{ia} - \overline{M} \lambda_a \right)^2 + \sum_{i=1}^{N} M_i^2 \Phi_i^{(12)} \frac{C_{li}^2 m_i}{C_{li}^2 m_i} - \frac{1}{nM_0} \left[ \sum_{i=1}^{N} M_i \left( \hat{\lambda}_{ia} - \lambda_a \right)^2 + \sum_{i=1}^{N} M_i \Phi_i^{(12)} \frac{C_{li}^2 m_i}{C_{li}^2 m_i} \right],$$

$$= \frac{1}{nM_0} \left[ \sum_{i=1}^{N} \left( M_i \hat{\lambda}_{ia} - \overline{M} \lambda_a \right)^2 + \sum_{i=1}^{N} M_i^2 \Phi_i^{(12)} \frac{C_{li}^2 m_i}{C_{li}^2 m_i} - \overline{M} \sum_{i=1}^{N} M_i \left( \hat{\lambda}_{ia} - \lambda_a \right)^2 - \overline{M} \sum_{i=1}^{N} M_i \Phi_i^{(12)} \frac{C_{li}^2 m_i}{C_{li}^2 m_i} \right],$$

$$= \frac{1}{nM_0 \overline{M}} \left[ \sum_{i=1}^{N} \left( M_i - \overline{M} \right)^2 \lambda_a^2 + \overline{M} \sum_{i=1}^{N} \left( M_i - \overline{M} \right) \left( \lambda_a^2 - \lambda_a^2 \right) + \sum_{i=1}^{N} \left( M_i - \overline{M} \right)^2 \Phi_i^{(12)} \frac{C_{li}^2 m_i}{C_{li}^2 m_i} + \overline{M} \sum_{i=1}^{N} \left( M_i - \overline{M} \right) \Phi_i^{(12)} \frac{C_{li}^2 m_i}{C_{li}^2 m_i} \right]. \quad (3.52)$$

From (3.52) it is observed that the variance $\hat{\lambda}_{\text{awru}}$ is same as the variance of $\hat{\lambda}_{\text{appsworu}}$ when $M_i = \overline{M} = \frac{M_0}{N}$, that is, when all clusters have the same size. The PPS with $p_i = \frac{1}{N}$ has the same variance as the equal probability two-stage sampling even when the parameter of the rare unrelated attribute is not known. If the difference in the cluster sizes is large, the first term $\sum_{i=1}^{N} \left( M_i - \overline{M} \right)^2 \lambda_a^2$, and the third term .
\[
\sum_{i=1}^{N} \left( M_i - \bar{M} \right)^2 \frac{\Phi_{i2}}{C_i/m_i}
\]

are increasing. Usually the estimation by PPS is better than the equal probability two-stage sampling when clusters have different sizes.

Estimation with PPS method of a rare sensitive attribute under stratified two-stage sampling using a randomized response model.

We consider a population which is supposed to be stratified into \( L \) strata such that the \( h \)th stratum has \( N_h \), \( h = 1, 2, ..., L \), cluster, which are the first-stage units. Each cluster has size \( M_{hi} \) \((i = 1, 2, ..., N)\) in stratum \( h \). In stratum \( h \), \( n_h \) clusters (which are at the first units) are selected from \( N_h \) clusters with probability \( p_{hi} \). At the second stage, we select \( m_{hi} \) \((i = 1, 2, ..., n_h)\), second-stage units from the \( i \)th cluster drawn from the \( h \)th stratum, using SRSWR scheme.

PPS and Equal Probability Two-Stage Sampling for Stratification (when the rare unrelated attribute is known).

In this section, we assume that \( n_h \) clusters are drawn from \( h \)th stratum using probability proportional to size with replacement (PPS) sampling scheme and \( \pi_{h} \), the proportion of the rare non-sensitive unrelated attribute in the \( h \)th stratum is supposed to be known. When the randomized response model described in Section 3 is applied, the probability that respondents answer ‘yes’ in the \( i \)th cluster of the stratum \( h \) is defined as

\[
\theta_{hi0} = U_{hi} \pi_{hia} + (1 - U_{hi}) \left( P_{1hi} \pi_{hia} + P_{2hi} \pi_{hia} \left( 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) + P_{3hi}^2 \frac{k_{hi}}{(k_{hi} - 1)} \pi_{hia} \right),
\]

(4.1)

where \( U_{hi} \) is the probability of the question being asked; Question (i): ‘Do you possess the rare sensitive attribute \( A \)? in the first stage randomization device of Section 3.1 and the symbols \( P_{1hi}, P_{2hi} \) and \( P_{3hi} \) are the probabilities of presenting the statements (i), (ii) and (iii) in the randomization device used in the \( i \)th cluster of the \( h \)th stratum \( \left( P_{1hi} + P_{2hi} + P_{3hi} = 1 \right) \). \( \pi_{hia} \) and \( \pi_{hia} \) are the population proportions of the rare sensitive attribute, \( A \), and the rare unrelated attribute, \( Y \), respectively in the \( i \)th cluster of the \( h \)th stratum. \( \pi_{hia} \) is assumed to be known. In the \( i \)th cluster of stratum \( h \) since the two attribute \( A \) and \( Y \) are rare, \( m_{hi} \theta_{hi0} = \lambda_{h0i} > 0 \), \( m_{hi} \pi_{hia} = \lambda_{hia} > 0 \) and \( m_{hi} \pi_{hia} = \lambda_{hia} > 0 \) are finite for \( m_{hi} \to \infty \) as \( \theta_{hi0} \to 0 \), \( \pi_{hia} \to 0 \) and \( \pi_{hia} \to 0 \).

Let \( y_{hi1}, y_{hi2}, ..., y_{himhi} \) be \( m_{hi} \) random samples from the Poisson distribution with mean \( \lambda_{h0i} \) in the cluster of stratum \( h \). Then the estimator \( \hat{\lambda}_{hia} \) of the mean total number of persons bearing the rare sensitive attribute in the \( i \)th cluster of the \( h \)th stratum is defined as
\[
\hat{\lambda}_{hi} = \frac{(1/m_{hi}) \sum_{j=1}^{m_{hi}} y_{hij} - (1 - U_{hi}) P_{2hi} \left\{ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} \lambda_{hiy}}{U_{hi} + (1 - U_{hi}) \left\{ P_{1hi} + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} (P_{1hi} + P_{3hi}) \right\}}.
\] (4.2)

Now, an estimator for the rare sensitive attribute, \( \hat{\lambda}_{ha} \) in stratum \( h \) is given by

\[
\hat{\lambda}_{ha} = \frac{1}{n_{h}M_{h0}} \sum_{i=1}^{n_{h}} M_{hi} \hat{\lambda}_{hia},
\] (4.3)

where \( M_{h0} = \sum_{i=1}^{n_{h}} M_{hi} \).

Under the stratified two-stage sampling design, the final estimator \( \hat{\lambda}_{asppzwr} \) for the rare sensitive attribute, \( \hat{\lambda}_{ha} \), is given by

\[
\hat{\lambda}_{asppzwr} = \frac{E}{W_{h}} \frac{1}{n_{h}M_{h0}} \sum_{i=1}^{n_{h}} M_{hi} \hat{\lambda}_{hia},
\] (4.4)

where \( W_{h} = \frac{N_{h}}{N} \), \( N = \sum_{h=1}^{L} N_{h} \) and \( p_{hi} \) is the initial probability of selecting the \( i \)th cluster, which is a first-stage unit in the \( h \)th stratum.

Properties of the estimator \( \hat{\lambda}_{asppzwr} \)

Theorem 4.1 - The estimator \( \hat{\lambda}_{asppzwr} \) is unbiased.

Proof: Since \( y_{hij} \sim iid \ Poisson(\lambda_{h0}) \), \( E(y_{hij}) = \lambda_{h0} \),

\[
\lambda_{h0} = U_{hi} \lambda_{hia} + (1 - U_{hi}) \left\{ P_{1hi} \lambda_{hia} + P_{2hi} \lambda_{hia} \right\} \left\{ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \lambda_{hia},
\] (4.5)

and the estimator \( \hat{\lambda}_{hia} \) is an unbiased estimator of \( \lambda_{hia} (i = 1, 2, ..., n_{h}; j = 1, 2, ..., m_{hi}) \). Therefore,

\[
E(\hat{\lambda}_{asppzwr}) = E_{1} E_{2} \left[ \sum_{h=1}^{L} W_{h} \frac{1}{n_{h}M_{h0}} \sum_{i=1}^{n_{h}} M_{hi} \hat{\lambda}_{hia} \right],
\]

\[
= E_{1} \left[ \sum_{h=1}^{L} W_{h} \frac{1}{n_{h}M_{h0}} \sum_{i=1}^{n_{h}} M_{hi} \hat{\lambda}_{hia} \right],
\]
This completes the proof of the theorem.

**Theorem 4.2** The variance of the estimator \( \hat{\lambda}_{\text{appczwr}} \) is given by

\[
V(\hat{\lambda}_{\text{appczwr}}) = \sum_{h=1}^{L} W_h \frac{1}{n_h M_{h0}} \left[ \sum_{i=1}^{N_h} p_{hi} \left( \frac{M_{hi}^2 \lambda_{hia}}{p_{hi}} - M_{h0}^2 \lambda_{hia} \right) \right]^2 + \sum_{i=1}^{N_h} \frac{M_{hi}^2}{p_{hi} m_{hi}} \Phi_{hi}.
\]

(4.6)

where \( \Phi_{hi} = \left[ \frac{\lambda_{hia}}{D_{hi}} \right] + \frac{(1 - U_{hi}) P_{h2i} \left[ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right] \lambda_{hiy}}{D_{hi}^2} \),

and \( D_{hi} = \left[ U_{hi} + (1 - U_{hi}) \left( P_{1hi} + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} (P_{1hi} + P_{3hi}) \right) \right] \).

**Proof:** The variance of \( \hat{\lambda}_{\text{appczwr}} \) is decomposed by

\[
V(\hat{\lambda}_{\text{appczwr}}) = V_1 E_2 (\hat{\lambda}_{\text{appczwr}}) + E_1 V_2 (\hat{\lambda}_{\text{appczwr}}).
\]

(4.7)

We have

\[
V_1 E_2 (\hat{\lambda}_{\text{appczwr}}) = V_1 \left[ \sum_{h=1}^{L} W_h \frac{1}{n_h M_{h0}} \sum_{i=1}^{N_h} M_{hi} \hat{\lambda}_{hia} \right].
\]

\[
= V_1 \left[ \sum_{h=1}^{L} W_h \frac{1}{n_h M_{h0}} \sum_{i=1}^{N_h} M_{hi} \lambda_{hia} \right],
\]

\[
= \sum_{h=1}^{L} W_h^2 \frac{1}{n_h M_{h0}^2} \sum_{i=1}^{N_h} p_{hi} \left( \frac{M_{hi}^2 \lambda_{hia}}{p_{hi}} - M_{h0}^2 \lambda_{hia} \right)^2.
\]

(4.8)

As \( y_{hij} \sim \text{iid Poisson} (\lambda_{h0}) \), \( V(y_{hij}) = \lambda_{h0} \), the second term is

\[
E_1 V_2 (\hat{\lambda}_{\text{appczwr}}) = E_1 \left[ \sum_{h=1}^{L} W_h \frac{1}{n_h M_{h0}} \sum_{i=1}^{N_h} M_{hi} \hat{\lambda}_{hia} \right],
\]

\[
= E_1 \left[ \sum_{h=1}^{L} W_h^2 \frac{1}{(n_h M_{h0})^2} \sum_{i=1}^{N_h} \frac{M_{hi}^2}{p_{hi}^2} V_2 (\hat{\lambda}_{hia}) \right],
\]

\[
= E_1 \left[ \sum_{h=1}^{L} W_h^2 \frac{1}{(n_h M_{h0})^2} \sum_{i=1}^{N_h} \frac{M_{hi}^2}{p_{hi}^2} V_2 \left( \frac{1}{D_{hi}} \left( V_m \sum_{j=1}^{m_{hi}} y_{hij} - B_{hi} \right) \right) \right].
\]
It is known as PPSWR. Hence the unbiased estimator of $\lambda(\Theta)$, is

$$
\hat{\lambda}(\Theta) = \sum_{h=1}^{L} \frac{W_h^2}{n_h M_{h0}^2} \sum_{i=1}^{n_h} \left( \frac{M_{hi}}{p_{hi} D_{hi}^2} \right) \left( \frac{1}{m_{hi}} \sum_{j=1}^{m_{hi}} V_2(y_{hij}) \right),
$$

$$
= E_h \left[ \sum_{h=1}^{L} \frac{W_h^2}{n_h M_{h0}^2} \sum_{i=1}^{n_h} \frac{M_{hi}^2}{p_{hi} D_{hi}^2} \lambda_{h0} \right],
$$

$$
= E_h \left[ \sum_{h=1}^{L} \frac{W_h^2}{n_h M_{h0}^2} \sum_{i=1}^{n_h} \frac{M_{hi}^2}{p_{hi} D_{hi}^2} \lambda_{h0} \right],
$$

$$
= \left[ \sum_{h=1}^{L} \frac{W_h^2}{n_h M_{h0}^2} \sum_{i=1}^{n_h} \frac{M_{hi}^2}{p_{hi} D_{hi}^2} \Phi_{hi} \right] \cdots \frac{\lambda_{h0}}{D_{hi}^2} = \Phi_h; \quad (4.9)
$$

where $B_{hi} = (1 - U_{hi})P_{2hi} \left( 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) \lambda_{hi}$. 

Adding (4.8) and (4.9), we get the variance of the unbiased estimator $\hat{\lambda}_{asppzwr}$ as given in (4.6). Thus the theorem is proved.

The unbiased estimate of the variance of $\hat{\lambda}_{asppzwr}$ is given by

$$
\hat{V}(\hat{\lambda}_{asppzwr}) = \sum_{h=1}^{L} \frac{W_h^2}{n_h (n_h - 1) M_{h0}^2} \sum_{i=1}^{n_h} \left( \frac{M_{hi} \hat{\lambda}_{hia}}{p_{hi}} - \hat{\lambda}_{ha} \right)^2. \quad (4.10)
$$

Estimation when the first-stage sample is selected using PPSWR

When $n_h$ clusters are selected with replacement (WR) from $N_h$ clusters depending on the cluster size, $M_{hi}$ in stratum $h$, with selection probabilities $p_{hi}(i = 1, 2, ..., n_h)$, where $p_{hi} = \frac{M_{hi}}{M_{h0}}$. It is known as PPSWR. Hence the unbiased estimator of $\lambda_a$ is given by

$$
\hat{\lambda}_{asppswr} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sum_{i=1}^{n_h} \hat{\lambda}_{hia}, \quad (4.11)
$$

and its variance is obtained as

$$
\hat{V}(\hat{\lambda}_{asppswr}) = \sum_{h=1}^{L} \frac{W_h^2}{n_h (n_h - 1) M_{h0}^2} \sum_{i=1}^{n_h} \frac{M_{hi} (\hat{\lambda}_{hia} - \hat{\lambda}_{ha})^2}{m_{hi}}. \quad (4.12)
$$

An estimator for the variance of the estimator $\hat{\lambda}_{asppswr}$, is

$$
\hat{V}(\hat{\lambda}_{asppswr}) = \sum_{h=1}^{L} \frac{W_h^2}{n_h (n_h - 1)} \sum_{i=1}^{n_h} \left( \hat{\lambda}_{hia} - \hat{\lambda}_{ha} \right)^2. \quad (4.13)
$$
Estimation when the first-stage sample is selected using SRSWR scheme.

When the first-stage sample units are drawn with equal probability and with replacement in two-stage sampling, the selection probabilities for all the selected clusters at first-stage from the $h^{th}$ stratum is $p_{hi} = \frac{1}{N_h}$ and the estimator of $\lambda_a$ is

$$\hat{\lambda}_{assrwr} = \sum_{h=1}^{L} W_h \frac{N_h}{n_h M_{h0}} \sum_{i=1}^{n_h} M_{hi} \hat{\lambda}_{hia},$$

$$= \sum_{h=1}^{L} W_h \hat{\lambda}_{ha}, \quad (4.14)$$

where $\hat{\lambda}_{ha} = \frac{N_h}{n_h M_{h0}} \sum_{i=1}^{n_h} M_{hi} \hat{\lambda}_{hia}.$

The variance of the estimator $\hat{\lambda}_{assrwr}$ is

$$V(\hat{\lambda}_{assrwr}) = \sum_{h=1}^{L} W_h^2 \frac{N_h}{n_h M_{h0}^2} \left[ \frac{N_h}{(N_h - 1)} \sum_{i=1}^{N_h} \left( M_{hi} \hat{\lambda}_{hia} - \overline{M}_h \hat{\lambda}_{ha} \right)^2 + \sum_{i=1}^{N_h} \frac{M_{hi}^2}{m_{hi}} \Phi_{hi} \right], \quad (4.15)$$

and its estimate is given by

$$\hat{V}(\hat{\lambda}_{assrwr}) = \sum_{h=1}^{L} W_h^2 \frac{1}{n_h (n_h - 1) M_{h0}^2} \sum_{i=1}^{n_h} \left( N_h M_{hi} \hat{\lambda}_{hia} - \hat{\lambda}_{ha} \right)^2, \quad (4.16)$$

where $\overline{M}_h = \frac{M_{hi}}{N_h}.$

Estimation by PPSWOR for stratification

Let $\delta_{hi}$ be the probability that the $i^{th}$ unit belongs to the first-stage sample and $\delta_{hij}$ the probability that both the $i^{th}$ and $j^{th}$ units belong to this sample, using PPSWOR from the $h^{th}$ stratum. The estimator for the rare sensitive attribute in stratum $h$ is given by

$$\hat{\lambda}_{appswor} = \frac{1}{M_{h0}} \sum_{i=1}^{n_h} M_{hi} \hat{\lambda}_{hia}. \quad (4.17)$$

The unbiased estimator for the rare sensitive attribute (i.e. of $\lambda_a$) is given by

$$\hat{\lambda}_{appswor} = \sum_{h=1}^{L} W_h \frac{1}{M_{h0}} \sum_{i=1}^{n_h} \frac{M_{hi}}{\delta_{hi}} \hat{\lambda}_{hia}. \quad (4.18)$$

The variance of the estimator $\hat{\lambda}_{appswor}$ is given by

$$V(\hat{\lambda}_{appswor}) = \sum_{h=1}^{L} W_h^2 \frac{1}{M_{h0}^2} \left[ \sum_{i=1}^{n_h} \sum_{j=1}^{n_h} (\delta_{hi} \delta_{hj} - \delta_{hij}) \left( \frac{M_{hi} \hat{\lambda}_{hia}}{\delta_{hi}} - \frac{M_{hj} \hat{\lambda}_{hja}}{\delta_{hj}} \right)^2 + \sum_{i=1}^{n_h} \frac{M_{hi}^2}{m_{hi}} \Phi_{hi} \right]. \quad (4.19)$$

The estimator for the variance of $\hat{\lambda}_{appswor}$ is given by
\[ V(\hat{\lambda}_{aspswr}) = \sum_{h=1}^{H} W_h^2 \left[ \frac{1}{M_{h0}^2} \left( \sum_{i=1}^{n_h} \sum_{j=1}^{n_h} \left( \delta_{hi} - \delta_{hj} \right) \left( \frac{M_{hi} \hat{\lambda}_{hia} - M_{hj} \hat{\lambda}_{hia}}{\delta_{hi}} \right) \right)^2 + \sum_{i=1}^{n_h} \frac{M_{hi}^2}{p_{hi}} \frac{\hat{\Phi}_{hi}}{m_{hi} - 1} \right], \]

where \( \hat{\Phi}_{hi} = \left[ \frac{\hat{\lambda}_{hia}}{D_{hi}} + \frac{(1 - U_{hi}) P_{2hi} \left\{ 1 + P_{3hi} \left( k_{hi} / (k_{hi} - 1) \right) \right\} \hat{\lambda}_{hia}}{D_{hi}^2} \right] \). 

Comparing PPSWR and the equal probability two-stage sampling for stratification. From (4.12) and (4.15) we have 
\[ V(\hat{\lambda}_{asssr}) - V(\hat{\lambda}_{aspswr}) = \frac{1}{n_h M_{h0}^2} \left[ \sum_{i=1}^{N_h} \left( M_{hi} - \bar{M}_h \right)^2 \hat{\lambda}_{hia}^2 + \sum_{i=1}^{N_h} \left( M_{hi} - \bar{M}_h \right) \left( \hat{\lambda}_{hia}^2 - \hat{\lambda}_{hia}^2 \right) \right] + \frac{N_h}{m_{hi}} \Phi_{hi} \]

where \( \Phi_{hi} = \left[ \frac{\hat{\lambda}_{hia}}{m_{hi}} + \frac{P_{1hi} \left( k_{hi} / (k_{hi} - 1) \right) \hat{\lambda}_{hia}}{m_{hi}^2} \right] \). 

For \( (N_h - 1) \approx N_h \), (4.21) boils down to 
\[ V(\hat{\lambda}_{asssr}) - V(\hat{\lambda}_{aspswr}) = \sum_{h=1}^{H} W_h^2 \left[ \sum_{i=1}^{N_h} \left( M_{hi} - \bar{M}_h \right)^2 \hat{\lambda}_{hia}^2 + \sum_{i=1}^{N_h} \left( M_{hi} - \bar{M}_h \right) \hat{\lambda}_{hia}^2 - \hat{\lambda}_{hia}^2 \right] + \frac{N_h}{m_{hi}} \Phi_{hi} \] 

It is observed from (4.22) that 
\[ V(\hat{\lambda}_{asssr}) = V(\hat{\lambda}_{aspswr}) \]

when each cluster has the same size, \( M_{hi} = \bar{M}_h = \frac{M_{h0}}{N_h} \) in stratum \( h \), let \( p_{hi} = \frac{1}{N_h} \) in the PPS. Also in (4.22) the first term \( \sum_{i=1}^{N_h} \left( M_{hi} - \bar{M}_h \right)^2 \hat{\lambda}_{hia}^2 \) is increasing in case the clusters have different sizes. Usually the PPS is better than the equal probability two-stage sampling in stratified two-stage sampling.
Estimation method of rare-attribute when the rare non-sensitive unrelated attribute is unknown in stratified population.

Estimation by PPSWR for Stratification.
In this section, it is supposed that \( n_h \) clusters are selected from \( h^{\text{th}} \) stratum \((h=1, 2, \ldots, L)\) using PPSWR sampling design and \( \pi_{h1y} \) is supposed to be unknown. The randomization device in the \( i^{\text{th}} \) cluster of the \( h^{\text{th}} \) stratum is the same as in Section 3.2. The probabilities that respondents in the \( i^{\text{th}} \) cluster of the \( h^{\text{th}} \) stratum answer “yes” are

\[
\theta_{hi1} = U_{h1} \pi_{h1a} + (1 - U_{h1}) \left[ P_{h1} \pi_{h1a} + P_{2h} \pi_{h1y} \left(1 + P_{3h} \frac{k_{hi}}{k_{hi} - 1}\right) + P_{3h}^2 \frac{k_{hi}}{k_{hi} - 1} \pi_{h1a}\right] \tag{4.23}
\]

and

\[
\theta_{hi2} = U_{h2} \pi_{h2a} + (1 - U_{h2}) \left[ Q_{1h} \pi_{h2a} + Q_{2h} \pi_{h2y} \left(1 + Q_{3h} \frac{k_{hi}}{k_{hi} - 1}\right) + Q_{3h}^2 \frac{k_{hi}}{k_{hi} - 1} \pi_{h2a}\right], \tag{4.24}
\]

where \( U_{h1} \) is the probability of the question being asked as described in (i) of Section 3.1 in first-stage randomization device and \( \{P_{1h}, P_{2h}, P_{3h}\} \) are the probabilities of presenting the statements (i), (ii), and (iii) in the second stage randomization device, when this device is used in the \( i^{\text{th}} \) cluster of the \( h^{\text{th}} \) stratum, such that \( P_{1h} + P_{2h} + P_{3h} = 1 \), \( \pi_{hia} \) is the true proportion of rare sensitive attribute \( A_h \) and \( \pi_{h1y} \) is the true proportion of rare non-sensitive unrelated attribute \( Y_h \) in the \( i^{\text{th}} \) cluster of the \( h^{\text{th}} \) stratum. Further \( U_{h2} \) is the probability of the question being asked as described in (i) of Section 3.1 in first-stage randomization device (of second randomization device) and \( \{Q_{1h}, Q_{2h}, Q_{3h}\} \) are the probabilities of presenting the statements (i), (ii), and (iii) in the second-stage randomization device (of second randomization device), when this device is used in the \( i^{\text{th}} \) cluster of the \( h^{\text{th}} \) stratum, such that \( Q_{1h} + Q_{2h} + Q_{3h} = 1 \), and \( \pi_{h2a} \) and \( \pi_{h2y} \) are the population proportions of the rare sensitive and unrelated attributes, respectively. Now \( \pi_{h1y} \) is assumed to be unknown. Since the two attribute are in the population, therefore, \( m_{h} \theta_{h1} = \lambda_{h1} \), and \( m_{h} \theta_{h2} = \lambda_{h2} \) are finite as \( m_{h} \to \infty \), \( \theta_{h1} \to 0 \), \( \theta_{h2} \to 0 \), where

\[
\lambda_{hi1} = U_{h1} \lambda_{h1a} + (1 - U_{h1}) \left[ P_{h1} \lambda_{h1a} + P_{2h} \lambda_{h1y} \left(1 + P_{3h} \frac{k_{hi}}{k_{hi} - 1}\right) + P_{3h}^2 \frac{k_{hi}}{k_{hi} - 1} \lambda_{h1a}\right] \tag{4.25}
\]

and

\[
\lambda_{hi2} = U_{h2} \lambda_{h2a} + (1 - U_{h2}) \left[ Q_{1h} \lambda_{h2a} + Q_{2h} \lambda_{h2y} \left(1 + Q_{3h} \frac{k_{hi}}{k_{hi} - 1}\right) + Q_{3h}^2 \frac{k_{hi}}{k_{hi} - 1} \lambda_{h2a}\right]. \tag{4.26}
\]

Similar to the previous section, the following equations are obtained:
\[
\frac{1}{m_{hi}} \sum_{j=1}^{m_{hi}} y_{hi1j} = U_{hi} \hat{\lambda}_{hia} + (1 - U_{hi}) \left[ P_{1hi} \hat{\lambda}_{hia} + P_{2hi} \hat{\lambda}_{hhy} \left( 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) + P_{3hi}^2 \frac{k_{hi}}{(k_{hi} - 1)} \hat{\lambda}_{hia} \right]
\]

and
\[
\frac{1}{m_{hi}} \sum_{j=1}^{m_{hi}} y_{hi2j} = U_{hi} \hat{\lambda}_{hia} + (1 - U_{hi}) \left[ Q_{1hi} \hat{\lambda}_{hia} + Q_{2hi} \hat{\lambda}_{hhy} \left( 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) + Q_{3hi}^2 \frac{k_{hi}}{(k_{hi} - 1)} \hat{\lambda}_{hia} \right],
\]

where \( y_{hi1j} \) and \( y_{hi2j} \) are the first and the second answers of the \( j^{th} \) \((j = 1, 2, \ldots, m_{hi})\) respondents in the \( i^{th} \) cluster of the \( h^{th} \) stratum.

From (4.27) and (4.28), the estimators for the mean number of persons who possess the rare sensitive and unrelated attribute in the \( i^{th} \) cluster of stratum \( h \) are given by

\[
\hat{\lambda}_{hia} = \frac{1}{C_{hi} m_{hi}} \left[ (1 - U_{hi}) P_{2hi} \left( 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) \sum_{j=1}^{m_{hi}} y_{hi1j} \right. \\
- \left. (1 - U_{hi}) P_{3hi} \left( 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) \sum_{j=1}^{m_{hi}} y_{hi2j} \right],
\]

\[
\hat{\lambda}_{hhy} = \frac{1}{D_{hi} m_{hi}} \left[ \left( U_{hi} + (1 - U_{hi}) \left[ Q_{1hi} \left( 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) + Q_{3hi}^2 \frac{k_{hi}}{(k_{hi} - 1)} \right] \right) \sum_{j=1}^{m_{hi}} y_{hi1j} \right. \\
- \left. \left( U_{hi} + (1 - U_{hi}) \left[ P_{1hi} \left( 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) + P_{3hi}^2 \frac{k_{hi}}{(k_{hi} - 1)} \right] \right) \sum_{j=1}^{m_{hi}} y_{hi2j} \right],
\]

where

\[
C_{hi} = U_{hi} \left( 1 - U_{hi} \right) Q_{2hi} \left( 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) - U_{hi} \left( 1 - U_{hi} \right) P_{2hi} \left( 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) \\
+ \left( 1 - U_{hi} \right) \left( 1 - U_{hi} \right) P_{3hi} Q_{2hi} - P_{2hi} Q_{3hi} \left( 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) \left( 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) \neq 0
\]

\[
D_{hi} = U_{hi} \left( 1 - U_{hi} \right) P_{2hi} \left( 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) - U_{hi} \left( 1 - U_{hi} \right) Q_{2hi} \left( 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) \\
+ \left( 1 - U_{hi} \right) \left( 1 - U_{hi} \right) P_{3hi} Q_{2hi} - P_{3hi} Q_{3hi} \left( 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) \left( 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right) \neq 0.
\]
The estimator of the mean total number \( \hat{\lambda}_{ha} \) of persons having rare sensitive attribute in \( h \)-th stratum is

\[
\hat{\lambda}_{hasppzwru} = \frac{1}{n_h M_{h0}} \sum_{i=1}^{n_h} \frac{M_{hi} \hat{\lambda}_{hiau}}{p_{hi}},
\]

(4.33)

where \( p_{hi} \) is the selection probability of the \( i \)-th cluster in \( h \)-th stratum under PPSWR scheme and \( M_{h0} = \sum_{i=1}^{n_h} M_{hi} \). Thus the final estimator \( \hat{\lambda}_{asppzwru} \) of the mean total number \( \lambda_a \) of persons having a rare sensitive attribute in the population under stratified two-stage sampling scheme is

\[
\hat{\lambda}_{asppzwru} = \sum_{h=1}^{L} W_h \frac{1}{n_h M_{h0}} \sum_{i=1}^{n_h} \frac{M_{hi} \hat{\lambda}_{hiau}}{p_{hi}},
\]

(4.34)

where \( W_h = \frac{N_h}{N} \) and \( N = \sum_{h=1}^{L} N_h \).

Properties of the estimator \( \hat{\lambda}_{asppzwru} \)

Theorem 4.3 The estimator \( \hat{\lambda}_{asppzwru} \) of the mean total number of persons is having the rare sensitive attribute.

Proof: Since \( \hat{\lambda}_{h1j} \) and \( \hat{\lambda}_{h2j} \) are iid Poisson variates with parameters \( \hat{\lambda}_{h1} \) and \( \hat{\lambda}_{h2} \), respectively, therefore, we have

\[
E(\hat{\lambda}_{asppzwru}) = E_1 E_2 \left( \hat{\lambda}_{asppzwru} \right),
\]

\[
= E_1 E_2 \left[ \sum_{h=1}^{L} W_h \frac{1}{n_h M_{h0}} \sum_{i=1}^{n_h} \frac{M_{hi} \hat{\lambda}_{hiau}}{p_{hi}} \right],
\]

Now,

\[
E_2(\hat{\lambda}_{hiau}) = E_2 \left[ \frac{1}{C_{hi} m_{hi}} \left\{ d_{h2i} \sum_{j=1}^{m_{hi}} y_{h1j} - C_{h2i} \sum_{j=1}^{m_{hi}} y_{h2j} \right\} \right],
\]

\[
= \frac{1}{C_{hi} m_{hi}} \left[ d_{h2i} \sum_{j=1}^{m_{hi}} E_2(y_{h1j}) - C_{h2i} \sum_{j=1}^{m_{hi}} E_2(y_{h2j}) \right],
\]

\[
= \frac{1}{C_{hi} m_{hi}} \left[ d_{h2i} \sum_{j=1}^{m_{hi}} \hat{\lambda}_{h1j} - C_{h2i} \sum_{j=1}^{m_{hi}} \hat{\lambda}_{h2j} \right],
\]

\[
= \frac{1}{C_{hi} m_{hi}} \left[ m_{hi} d_{h2i} \hat{\lambda}_{h1} - m_{hi} C_{h2i} \hat{\lambda}_{h2} \right],
\]

\[
= \frac{d_{h2i} \hat{\lambda}_{h1} - C_{h2i} \hat{\lambda}_{h2}}{C_{hi}},
\]

(4.35)
where \[ d_{h2i} = (1 - U_{h2i})Q_{2hi} \left\{ 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} \]
and
\[ C_{h2i} = (1 - U_{hli})P_{2hi} \left\{ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\}. \]

Inserting the values of \( \lambda_{h1} \) and \( \lambda_{h2} \) from equations (4.25) and (4.26) respectively in (4.35) we get
\[ E_2(\hat{\lambda}_{hiau}) = \hat{\lambda}_{hia}. \]

Thus, \( E_1E_2(\hat{\lambda}_{asppzwru}) = E_1E_2 \left[ \sum_{h=1}^{L} W_h \frac{1}{n_h} \sum_{i=1}^{\bar{N}_h} M_{hi} \hat{\lambda}_{hiau} \right] \]
\[ = E_1 \left[ \sum_{h=1}^{L} W_h \frac{1}{n_h} \sum_{i=1}^{\bar{N}_h} M_{hi} \hat{\lambda}_{hiau} \right], \]
\[ = E_1 \left[ \sum_{h=1}^{L} W_h \frac{1}{n_h} \sum_{i=1}^{\bar{N}_h} M_{hi} \hat{\lambda}_{hia} \right], \]
\[ = \sum_{h=1}^{L} W_h \frac{1}{n_h} \sum_{i=1}^{\bar{N}_h} M_{hi} \hat{\lambda}_{hia} \]
\[ = \sum_{h=1}^{L} W_h \hat{\lambda}_{hia} = \hat{\lambda}_a. \] \hspace{1cm} (4.36)

Thus the theorem is proved.

**Theorem 4.4** The variance of the estimator \( \hat{\lambda}_{asppzwru} \) is given by
\[ V(\hat{\lambda}_{asppzwru}) = \sum_{h=1}^{L} W_h^2 \frac{1}{n_h M_{hi}^2} \left[ \sum_{i=1}^{\bar{N}_h} P_{hi} \left( \frac{M_{hi} \hat{\lambda}_{hia}^2}{p_{hi}} - M_{hi} \hat{\lambda}_{ha}^2 \right) \right] + \sum_{i=1}^{\bar{N}_h} M_{hi}^2 \frac{\Phi_{hi}^{(12)}}{p_{hi} C_{hi} m_{hi}}, \] \hspace{1cm} (4.37)
where
\[ \Phi_{hi}^{(12)} = \left[ \left( d_{h2i}^2 C_{h3i} + C_{h2i}^2 d_{h3i} - 2 C_{h2i} C_{h3i} d_{h2i} d_{h3i} \right) \hat{\lambda}_{hia} \right] + \left( d_{h2i}^2 C_{h2i} + C_{h2i}^2 d_{h2i} - 2 C_{h2i} d_{h2i} \right) \hat{\lambda}_{hiv}. \]

\[ C_{h3i} = U_{hli} + \left( 1 - U_{hli} \right) \left[ P_{hli} \left\{ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right], \]
\[ d_{h3i} = U_{hli} + \left( 1 - U_{hli} \right) \left[ Q_{1hi} \left\{ 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right]. \]

**Proof:** The variance of the estimator \( \hat{\lambda}_{asppzwru} \) is decomposed as
\[ V(\hat{\lambda}_{asppzwru}) = V_1E_2(\hat{\lambda}_{asppzwru}) + E_1V_2(\hat{\lambda}_{asppzwru}). \] \hspace{1cm} (4.38)
The first term of (4.38) is simplified to
\[ V_i E_2 (\hat{\lambda}_{aspcwru}) = V_i E_2 \left[ \sum_{h=1}^{L} W_h \frac{1}{n_h M_{h0}} \sum_{i=1}^{m_h} \frac{M_{hi} \lambda_{hia}}{p_{hi}} \right], \]
\[ = V_i \left[ \sum_{h=1}^{L} W_h \frac{1}{n_h M_{h0}} \sum_{i=1}^{m_h} \frac{M_{hi} \lambda_{hia}}{p_{hi}} \right], \]
\[ = \sum_{h=1}^{L} W_h \frac{1}{n_h M_{h0}} \sum_{i=1}^{m_h} p_{hi} \left( \frac{M_{hi} \lambda_{hia}}{p_{hi}} - M_{h0} \lambda_{ha} \right)^2. \]  
(4.39)

The second term of (4.38) is simplified to
\[ E_i V_2 (\hat{\lambda}_{aspcwru}) = E_i V_2 \left[ \sum_{h=1}^{L} W_h \frac{1}{n_h M_{h0}} \sum_{i=1}^{m_h} \frac{M_{hi} \lambda_{hia}}{p_{hi}} \right], \]
\[ = E_i \left[ \sum_{h=1}^{L} W_h \frac{1}{n_h M_{h0}} \sum_{i=1}^{m_h} \frac{M_{hi}^2}{p_{hi}} V_2 (\lambda_{hia}) \right]. \]  
(4.40)

Now,
\[ V_2 (\hat{\lambda}_{hia}) = V_2 \left[ \frac{1}{C_{h_{hi}} m_{hi}} \left( d_{h_{2i}} \sum_{j=1}^{m_{hi}} y_{hi1j} - C_{h2i} \sum_{j=1}^{m_{hi}} y_{hi2j} \right) \right] \]
\[ = \frac{1}{C_{h_{hi}} m_{hi}^2} \left[ d_{h_{2i}} \sum_{j=1}^{m_{hi}} V_2 (y_{hi1j}) + C_{h_{2i}} \sum_{j=1}^{m_{hi}} V_2 (y_{hi2j}) - 2 d_{h_{2i}} C_{h_{2i}} \sum_{j=1}^{m_{hi}} \text{Cov}(y_{hi1j}, y_{hi2j}) \right], \]
\[ = \frac{1}{C_{h_{hi}} m_{hi}^2} \left[ d_{h_{2i}} \lambda_{h_{hi}} + C_{h_{2i}} \sum_{j=1}^{m_{hi}} \lambda_{h_{i2}} - 2 d_{h_{2i}} C_{h_{2i}} \sum_{j=1}^{m_{hi}} \lambda_{h_{i12}} \right], \]
\[ = \frac{1}{C_{h_{hi}} m_{hi}^2} \left[ d_{h_{2i}} \lambda_{h_{hi}} + C_{h_{2i}} \lambda_{h_{2i}} - 2 d_{h_{2i}} C_{h_{2i}} \lambda_{h_{i12}} \right], \]  
(4.41)

where \( \lambda_{h_{i12}} = \text{Cov}(y_{hi1j}, y_{hi2j}) = E(y_{hi1j} y_{hi2j}) - E(y_{hi1j})E(y_{hi2j}) \)
\[ = \left\{ U_{hi1} + (1 - U_{hi1}) \left[ P_{hi1} \left\{ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} + P_{3hi}^2 \frac{k_{hi}}{(k_{hi} - 1)} \right] \right\} \]
\[ \times \left\{ U_{h_{2i}} + (1 - U_{h_{2i}}) \left[ Q_{1hi} \left\{ 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} + Q_{3hi}^2 \frac{k_{hi}}{(k_{hi} - 1)} \right] \right\} \lambda_{h_{hi}}, \]
\[ + \left\{ (1 - U_{hi1}) P_{2hi} \left\{ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} \right\} \left\{ (1 - U_{h_{2i}}) Q_{2hi} \left\{ 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} \right\} \lambda_{h_{hiy}}, \]
and \( (\lambda_{h_{hi}}, \lambda_{h_{hiy}}) \) are same as define earlier.

Putting the values from (4.31) in (4.40) and then substituting from (4.39) and (4.40) in (4.48), we obtained the expression of the variance of the estimator \( \hat{\lambda}_{aspcwru} \) as given in (4.37).
PARTIAL RANDOMIZED RESPONSE MODEL FOR ESTIMATING A RARE SENSITIVE ATTRIBUTE IN PROBABILITY PROPORTIONAL TO SIZE MEASURES USING POISSON DISTRIBUTION

The estimator of an unbiased estimator of the variance of $\hat{\lambda}_{\text{appswru}}$ is given by

$$\hat{V}(\hat{\lambda}_{\text{appswru}}) = \sum_{h=1}^{H} W_h^2 \frac{1}{n_h(n_h - 1)M_{h0}^2} \sum_{i=1}^{N_h} \left( \frac{M_{hi} \hat{\lambda}_{\text{hiau}} - \hat{\lambda}_{\text{hau}}}{p_{hi}} \right)^2. \quad (4.42)$$

Estimation when the first-stage sample is selected using SRSWR scheme

Let the first-stage sample units are drawn with equal probability and with replacement in two-stage stratified sampling, the probabilities of selecting the $i$th cluster from the $h$th stratum is $p_{hi} = \frac{1}{N_h}$. Then the unbiased estimator for the parameter $\lambda_a$ is defined by

$$\hat{\lambda}_{\text{aswru}} = \sum_{h=1}^{H} W_h \frac{N_h}{n_h M_{h0}} \sum_{i=1}^{N_h} M_{hi} \hat{\lambda}_{\text{hiau}}. \quad (4.43)$$

The variance of the estimator $\hat{\lambda}_{\text{aswru}}$ is

$$V(\hat{\lambda}_{\text{aswru}}) = \sum_{h=1}^{H} W_h^2 \frac{N_h}{n_h M_{h0}^2} \left[ \frac{N_h}{(N_h - 1)} \sum_{i=1}^{N_h} \left( M_{hi} \lambda_{\text{hia}} - \bar{M}_h \lambda_{\text{ha}} \right)^2 + \sum_{i=1}^{N_h} \frac{M_{hi}^2}{m_{hi} C_{hi}^2} \Phi_{hi}^{(12)} \right] \quad (4.44)$$

and its estimate is given by

$$\hat{V}(\hat{\lambda}_{\text{aswru}}) = \sum_{h=1}^{H} W_h^2 \frac{1}{n_h(n_h - 1)M_{h0}^2} \sum_{i=1}^{N_h} \left( N_h M_{hi} \hat{\lambda}_{\text{hiau}} - \bar{\hat{\lambda}}_{\text{hau}} \right)^2. \quad (4.45)$$

Estimation when the first-stage sample is selected using PPS sampling.

When $n_h$ clusters are drawn with replacement depending on cluster size, we consider the PPS. Let $p_{hi} = \frac{M_{hi}}{M_{h0}}$. The unbiased estimator for the mean number of persons who possess the rare sensitive attribute $\lambda_a$ is

$$\hat{\lambda}_{\text{appswru}} = \sum_{h=1}^{H} W_h \frac{1}{n_h} \sum_{i=1}^{N_h} \hat{\lambda}_{\text{hiau}}, \quad (4.46)$$

and its variance is obtained as

$$V(\hat{\lambda}_{\text{appswru}}) = \sum_{h=1}^{H} W_h^2 \frac{1}{n_h M_{h0}} \left[ \sum_{i=1}^{N_h} M_{hi} \left( \hat{\lambda}_{\text{hia}} - \lambda_{\text{ha}} \right)^2 + \sum_{i=1}^{N_h} \frac{M_{hi}^2}{m_{hi} C_{hi}^2} \Phi_{hi}^{(12)} \right]. \quad (4.47)$$

Also, the estimator for the variance of $\hat{\lambda}_{\text{appswru}}$ is given by

$$\hat{V}(\hat{\lambda}_{\text{appswru}}) = \sum_{h=1}^{H} W_h^2 \frac{1}{n_h(n_h - 1)M_{h0}^2} \sum_{i=1}^{N_h} \left( \hat{\lambda}_{\text{hiau}} - \bar{\hat{\lambda}}_{\text{hau}} \right)^2. \quad (4.48)$$

Estimation by PPSWR for Stratification
Let \( \delta_{hi} \) be the probability that the \( i^{th} \) unit belongs to the first-stage sample and \( \delta_{hij} \) the probability that both the \( i^{th} \) and \( j^{th} \) units belong to first-stage sample, using probability proportional to size sampling without replacement from the \( h^{th} \) stratum. The unbiased estimator of \( \lambda_d \) is given by

\[
\hat{\lambda}_{\text{asppsworu}} = \sum_{h=1}^{L} W_h \hat{\lambda}_{h}\text{au},
\]

(4.49)

where the estimator \( \hat{\lambda}_{h}\text{au} \) in stratum \( h \) is defined by

\[
\hat{\lambda}_{h}\text{au} = \frac{1}{M_{h0}} \sum_{i=1}^{n_h} M_{hi} \hat{\lambda}_{hiau}.
\]

(4.50)

The variance of estimator \( \hat{\lambda}_{\text{asppsworu}} \) is given by

\[
V(\hat{\lambda}_{\text{asppsworu}}) = \sum_{h=1}^{L} W_h^2 \frac{1}{M_{h0}^2} \left[ \sum_{i=1}^{n_h} \sum_{j>i}^{n_h} \left( \delta_{hi} \delta_{hj} - \delta_{hij} \right) \left( \frac{M_{hi} \hat{\lambda}_{hiau} - M_{hj} \hat{\lambda}_{hja}}{\delta_{hi}} \right)^2 + \sum_{i=1}^{n_h} M_{hi}^2 \Phi_{hi}^{(12)} \right].
\]

(4.51)

Further the estimator for the variance of \( \hat{\lambda}_{\text{asppsworu}} \) is given by

\[
V(\hat{\lambda}_{\text{asppsworu}}) = \sum_{h=1}^{L} W_h^2 \frac{1}{M_{h0}^2} \left[ \sum_{i=1}^{n_h} \sum_{j>i}^{n_h} \left( \delta_{hi} \delta_{hj} - \delta_{hij} \right) \left( \frac{M_{hi} \hat{\lambda}_{hiau} - M_{hj} \hat{\lambda}_{hja}}{\delta_{hi}} \right)^2 + \sum_{i=1}^{n_h} M_{hi}^2 \Phi_{hi}^{(12)} \right],
\]

(4.52)

where

\[
\Phi_{hi}^{(12)} = \left[ d_{h2i}^2 C_{h3i} + C_{h2i}^2 d_{h3i} - 2C_{h2i} C_{h3i} d_{h2i} d_{h3i} \right] \hat{\lambda}_{hiau} + \left[ d_{h2i}^2 C_{h2i} + C_{h2i}^2 d_{h2i} - 2C_{h2i}^2 d_{h2i} \right] \hat{\lambda}_{hia}.
\]

Comparing PPSWR and the equal probability two-stage sampling for stratification

From (4.44) and (4.47) we have

\[
V(\hat{\lambda}_{\text{aswrn}}) - V(\hat{\lambda}_{\text{asppsworu}}) = \sum_{h=1}^{L} W_h^2 \left[ \frac{N_h}{n_h M_{h0}} \left( \frac{N_h}{N_h - 1} \sum_{i=1}^{n_h} (M_{hi} \hat{\lambda}_{hiau} - \overline{M}_h \hat{\lambda}_{ha})^2 + \sum_{i=1}^{n_h} M_{hi}^2 \Phi_{hi}^{(12)} C_{h0}^2 m_{hi} \right) \right] - \frac{1}{n_h M_{h0}} \left[ \sum_{i=1}^{n_h} M_{hi} \left( \hat{\lambda}_{hiau} - \hat{\lambda}_{ha} \right)^2 + \sum_{i=1}^{n_h} M_{hi} \Phi_{hi}^{(12)} \right].
\]

(4.53)

Under the assumption \( (N_h - 1) \equiv N_h \), and \( \overline{M}_h = \frac{M_{h0}}{N_h} \), expression (4.53) reduces to

\[
V(\hat{\lambda}_{\text{aswrn}}) - V(\hat{\lambda}_{\text{asppsworu}}) = \sum_{h=1}^{L} W_h^2 \frac{1}{n_h M_{h0} M_{h}} \left[ \sum_{i=1}^{n_h} (M_{hi} - \overline{M}_h)^2 \hat{\lambda}_{hiau}^2 + \overline{M}_h \sum_{i=1}^{n_h} (M_{hi} - \overline{M}_h) \left( \hat{\lambda}_{hiau} - \hat{\lambda}_{ha} \right)^2 + \sum_{i=1}^{n_h} (M_{hi} - \overline{M}_h)^2 \Phi_{hi}^{(12)} \right].
\]

(4.54)

It is observed from (4.54) that if \( M_{hi} = \overline{M}_h \), then \( V(\hat{\lambda}_{\text{aswrn}}) = V(\hat{\lambda}_{\text{asppsworu}}) \). Further
putting \( M_{hi} = \frac{M_i}{N} = \frac{M_{i0}}{N_h} \), the probability \( p_{hi} = \frac{M_{hi}}{M_{i0}} \) in PPS is equal to \( \frac{1}{N_h} \), which is the probability in the stratified PPS with the equal Probability two-stage sampling.

**Numerical Study**

We have evaluated and compared the suggested estimation methods with those of Singh and Suman (2019) estimators.

**Comparison of the suggested procedure with Singh and Suman (2019) estimators**

In this section we present Singh and Suman (2019) estimators under two-stage sampling and stratified two-stage sampling schemes, where first-stage samples are selected from the clustered population using probability proportional to size with replacement (PPSWR) sampling scheme.

(a) When the proportion of persons having the unrelated rare attribute is known under two-stage sampling scheme:

\[
\hat{\lambda}_{appswr}^{(1)} = \frac{1}{nM_0} \sum_{i=1}^{n} \frac{M_i}{P_i} \hat{\lambda}_{ia}^{(1)},
\]

where

\[
\hat{\lambda}_{ia}^{(1)} = \frac{1}{J_i} \left[ \left( 1/m_i \right) \sum_{j=1}^{m_i} y_{ij} - (1-U_i) P_{2i} \left( 1 + P_{3i} \frac{k_i}{(k_i-1)} \right) \right] \lambda_{iy}
\]

\[
J_i = \left[ U_i + (1-U_i) P_{2i} \left( 1 + P_{3i} \frac{k_i}{(k_i-1)} \right) \right] \text{ and } P_i = \frac{M_i}{M_0}.
\]

Thus

\[
\hat{\lambda}_{appswr}^{(1)} = \frac{1}{n} \sum_{i=1}^{n} \hat{\lambda}_{ia}^{(1)}
\]

The Variance of \( \hat{\lambda}_{appswr}^{(1)} \) is given by

\[
V(\hat{\lambda}_{appswr}^{(1)}) = \frac{1}{nM_0} \left[ \sum_{i=1}^{N} M_i (\hat{\lambda}_{ia} - \hat{\lambda}_{a})^2 + \sum_{i=1}^{N} M_i \eta_i \right],
\]

where

\[
\eta_i = \frac{\lambda_{ia}}{J_i} + \frac{(1-U_i) P_{2i} \left( 1 + P_{3i} \frac{k_i}{(k_i-1)} \right) \lambda_{iy}}{J_i^2}
\]

From (3.12) and (5.3) we have
When the proportion of persons having the unrelated rare attribute is unknown under two-stage sampling design:

\[ \hat{\lambda}_{appswru}^{(1)} = \frac{1}{nM_0} \sum_{i=1}^{n} \frac{M_i}{m_i} \left[ \sum_{j=1}^{m_i} y_{ij} - C_{2i} \sum_{j=1}^{m_i} y_{ij2} \right], \]  

\[ J_{li} = \left[ U_{li} \left(1 - U_{li}\right) Q_{2i} \left\{1 + Q_{3i} \frac{k_i}{(k_i - 1)}\right\} - U_{li} \left(1 - U_{li}\right) P_{2i} \left\{1 + P_{3i} \frac{k_i}{(k_i - 1)}\right\} \right] \]

\[ + \left(1 - U_{li}\right) \left(1 - U_{li}\right) \left(P_{li} Q_{2i} - P_{2i} Q_{li}\right) \left\{1 + P_{3i} \frac{k_i}{(k_i - 1)}\right\} \left\{1 + Q_{3i} \frac{k_i}{(k_i - 1)}\right\} \]

\[ d_{2i} = (1 - U_{li}) Q_{2i} \left\{1 + Q_{3i} \frac{k_i}{(k_i - 1)}\right\}, \]

\[ C_{2i} = (1 - U_{li}) P_{2i} \left\{1 + P_{3i} \frac{k_i}{(k_i - 1)}\right\}. \]

The variance of \( \hat{\lambda}_{appswru}^{(1)} \) is given by

\[ V\left(\hat{\lambda}_{appswru}^{(1)}\right) = \frac{1}{nM_0} \sum_{i=1}^{N} \frac{M_i}{m_i} \left(\hat{\lambda}_{ia} - \lambda_{a}\right)^2 + \sum_{i=1}^{N} M_i \eta_{i}^{(12)}, \]  

\[ \eta_{i}^{(12)} = \left[d_{2i}^2 C_{3i} + C_{2i}^2 d_{3i}^2 - 2C_{2i} C_{3i} d_{2i} d_{3i}\right] \hat{\lambda}_{ia}^{(1)} + \left[d_{2i}^2 C_{3i} + C_{2i}^2 d_{2i}^2 - 2C_{2i}^2 d_{2i}^2\right] \hat{\lambda}_{iy}^{(1)}, \]  

\[ d_{3i}^{(1)} = U_{li} + (1 - U_{li}) Q_{li} \left\{1 + Q_{3i} \frac{k_i}{(k_i - 1)}\right\}, \]

\[ C_{3i}^{(1)} = U_{li} + (1 - U_{li}) P_{li} \left\{1 + P_{3i} \frac{k_i}{(k_i - 1)}\right\}. \]

When the proportion of persons having the unrelated rare attribute is known under two-stage sampling design:

\[ \hat{\lambda}_{appswr}^{(1)} = \sum_{h=1}^{k} W_h \frac{1}{n_k M_{h0}} \sum_{i=1}^{n_k} M_{hi} \hat{\lambda}_{hia}^{(1)}, \]

where \( \hat{\lambda}_{hia}^{(1)} = \frac{1}{J_{hi}} \left[\left(1/m_{hi}\right) \sum_{j=1}^{m_{hi}} y_{hij} - (1 - U_{hi}) P_{2hi} \left\{1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)}\right\}\hat{\lambda}_{hiy}\right]. \]
\[ J_{hi} = \left[ U_{hi} + (1 - U_{hi}) P_{1hi} \left\{ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} \right] \text{ and } P_{hi} = \frac{M_{hi}}{M_{h0}}. \]

The Variance of \( \hat{\lambda}_{\text{asppsru}}^{(1)} \) is given by
\[
V(\hat{\lambda}_{\text{asppsru}}^{(1)}) = \sum_{h=1}^{L} W_{h}^{2} \frac{1}{n_{h} M_{h0}} \left[ \sum_{i=1}^{N_{h}} M_{hi} \left( \lambda_{hia} - \lambda_{ha} \right)^{2} + \sum_{i=1}^{N_{h}} M_{hi} \eta_{hi} \right],
\]
where \( \eta_{hi} = \frac{\lambda_{hia}}{J_{hi}} + \frac{(1 - U_{hi}) P_{2hi} \left\{ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} \lambda_{hidy}}{J_{hi}^{2}} \).

When the proportion of persons having the unrelated rare attribute is unknown under stratified two-stage sampling design:
\[
\hat{\lambda}_{\text{asppsru}}^{(1)} = \sum_{h=1}^{L} W_{h} \frac{1}{n_{h} M_{h0}} \sum_{i=1}^{N_{h}} M_{hi} \hat{\lambda}_{hiau}, \quad (5.8)
\]
where
\[
\hat{\lambda}_{hiau} = \frac{1}{J_{hi} m_{hi}} \left[ (1 - U_{h2i}) Q_{2hi} \left\{ 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} \sum_{j=1}^{m_{hi}} y_{h1j} \right], \quad (5.9)
\]
\[
J_{hi} = \left[ U_{h1i} (1 - U_{h2i}) Q_{2hi} \left\{ 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} - U_{h2i} (1 - U_{h1i}) P_{2hi} \left\{ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} \right]
\]
\[
+ (1 - U_{h1i}) (1 - U_{h2i}) \left( P_{1hi} Q_{2hi} - P_{2hi} Q_{1hi} \right) \left\{ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\} \right] \left\{ 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\},
\]
\[
P_{hi} = \frac{M_{hi}}{M_{h0}}.
\]

The Variance of \( \hat{\lambda}_{\text{asppsru}}^{(1)} \) is given by
\[
V(\hat{\lambda}_{\text{asppsru}}^{(1)}) = \sum_{h=1}^{L} W_{h}^{2} \frac{1}{n_{h} M_{h0}} \left[ \sum_{i=1}^{N_{h}} M_{hi} \left( \lambda_{hia} - \lambda_{ha} \right)^{2} + \sum_{i=1}^{N_{h}} M_{hi} \eta_{hi} \right],
\]
where
\[
\eta_{hi} = \left[ \left\{ d_{h2i}^{2} C_{h3i}^{(1)} + C_{h2i}^{2} d_{h3i}^{(1)} \right\} \lambda_{hia} + \left\{ d_{h2i}^{2} C_{h3i}^{(1)} + C_{h2i}^{2} d_{h3i}^{(1)} \right\} \lambda_{hia} \right], \quad (5.7)
\]
Two clusters \( n=2 \) are selected, [see Nargis and Shabbir (2019)],

\[
C_{hi} = (1 - U_{hi}) P_{hi} \left\{ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\},
\]

\[
d_{hi} = (1 - U_{hi}) Q_{hi} \left\{ 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\},
\]

\[
C^{(1)}_{hi} = U_{hi} + (1 - U_{hi}) P_{hi} \left\{ 1 + P_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\},
\]

\[
d^{(1)}_{hi} = U_{hi} + (1 - U_{hi}) Q_{hi} \left\{ 1 + Q_{3hi} \frac{k_{hi}}{(k_{hi} - 1)} \right\}.
\]

The percent relative efficiencies (PRE’s) of the proposed estimators \( \hat{\lambda}_{appswr}, \hat{\lambda}_{appswru}, \)

\( \hat{\lambda}_{appswr}, \) \( \hat{\lambda}_{appswru}, \) and \( \hat{\lambda}_{appswru}^{(1)} \) with respect to Singh and Suman (2019) estimators \( \hat{\lambda}_{appswr}^{(1)}, \)

\( \hat{\lambda}_{appswr}^{(1)}, \) \( \hat{\lambda}_{appswr}, \) and \( \hat{\lambda}_{appswru}^{(1)} \) are respectively defined by

\[
e_1 = \text{PRE}\left(\hat{\lambda}_{appswr}, \hat{\lambda}_{appswr}^{(1)}\right) = \frac{V(\hat{\lambda}_{appswr})}{V(\hat{\lambda}_{appswr}^{(1)})} * 100, \quad (5.10)
\]

\[
e_2 = \text{PRE}\left(\hat{\lambda}_{appswr}, \hat{\lambda}_{appswru}^{(1)}\right) = \frac{V(\hat{\lambda}_{appswr})}{V(\hat{\lambda}_{appswru})} * 100, \quad (5.11)
\]

\[
e_3 = \text{PRE}\left(\hat{\lambda}_{appswr}, \hat{\lambda}_{appswru}^{(1)}\right) = \frac{V(\hat{\lambda}_{appswr})}{V(\hat{\lambda}_{appswru})} * 100, \quad (5.12)
\]

\[
e_4 = \text{PRE}\left(\hat{\lambda}_{appswr}, \hat{\lambda}_{appswru}^{(1)}\right) = \frac{V(\hat{\lambda}_{appswr})}{V(\hat{\lambda}_{appswru})} * 100. \quad (5.13)
\]

To carry out the numerical comparison under two stage sampling design, we consider a population of five clusters \( N=5 \) with sizes \( M_i = \{1000, 2000, 2000, 3000, 4000\} \) for \( i = 1, 2, 3, 4, 5 \). Two clusters \( n=2 \) are selected using the PPSWR sampling scheme depending on the cluster sizes as \( p_i = \frac{M_i}{M_0} \),

where \( M_0 = \sum_{i=1}^{5} M_i = 12000 \) [see, Lee et al (2013)]. It is assumed that

(a) \( \lambda_{1y} = \lambda_{2y} = \lambda_{3y} = \lambda_{4y} = \lambda_{5y} = 1 \) for the rare unrelated attribute,

(b) the values of \( P_{li}, P_{2i} \) and \( U_i \) are same in all clusters; \( P_{li} = P_1, \quad P_{2i} = P_2 = \frac{(1-P_1)}{3} \)

, \( P_{3i} = (1-P_{li} - P_{2i}) = P_3 = (1-P_1 - P_2) = \frac{2(1-P_1)}{3} \), [see Nargis and Shabbir (2019)],

\( U_i = U \) for \( i = 1, 2, 3, 4, 5 \).

(c) the total number of cards, \( k_i = k = 100, i = 1, 2, 3, 4, 5 \). in a deck for each cluster.
In the stratified PPS, a population is stratified into two strata (h=2), and there are two clusters $N_1 = 2$ with sizes $M_{1i} = (1000,2000)$ for $i = 1,2$ in stratum 1, and three clusters $N_2 = 3$ with sizes $M_{2i} = (2000,3000,4000)$ for $i = 1,2,3$ in stratum 2. We select a cluster from each stratum ($n_1 = n_2 = 1$). In both procedures, we suppose that the samples in each cluster are drawn with 10% and the parameters for the rare unrelated attribute which was assumed to be known are equal to 1.

We have taken $M_0 = 12000$ and $P_{1hi}, P_{2hi}, P_{3hi}, U_{1hi}, \text{and } U_{2hi}$ are equal for all clusters and strata

$$P_{111} = P_{121} = P_{1}, P_{211} = P_{221} = P_{2}, P_{311} = P_{321} = P_{3},$$

$$Q_{111} = Q_{221} = Q_{1}, Q_{221} = Q_{2}, Q_{311} = Q_{321} = Q_{3},$$

clusters and strata

$$U_{111} = U_{112} = U_{211} = U_{212} = U_{213} = U_{1},$$

$$U_{121} = U_{122} = U_{221} = U_{222} = U_{223} = U_{2},$$

$$W_i = \frac{N_i}{N}, W_2 = \frac{N_2}{N}, k_{hi} = k = 100, \lambda_i = 1 ; i = 1 \text{to } 5$$

Findings are shown in Tables 5.1 to 5.4.

**Table 5.1-** PRE’s of the estimator $\lambda_{appwr}$ with respect to the estimator $\lambda_{appwr}^{(1)}$ for $\lambda_i = 1; i = 1,2,3,4,5$ and $k = 100$.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
<th>$\lambda_5$</th>
<th>$\lambda_a$</th>
<th>PRE ($= \epsilon_i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.1</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P_2$</td>
<td>0.3</td>
<td>0.267</td>
<td>0.2</td>
<td>0.133</td>
<td>0.067</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P_3$</td>
<td>0.6</td>
<td>0.533</td>
<td>0.4</td>
<td>0.267</td>
<td>0.133</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>
Table 5.2- PRE’s of the estimator $\hat{\lambda}_{\text{app-wru}}$ with respect to the estimator $\hat{\lambda}_{\text{app-wru}}^{(0)}$ for $\lambda_{xy} = 1; i = 1,2,3,4,5$ and $k = 100$

<table>
<thead>
<tr>
<th>$\lambda_{xy}$</th>
<th>$\lambda_{xy}$</th>
<th>$\lambda_{xy}$</th>
<th>$\lambda_{xy}$</th>
<th>$\lambda_{xy}$</th>
<th>$\lambda_{xy}$</th>
<th>PRE ($e_2$)</th>
</tr>
</thead>
<tbody>
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<td>0.1</td>
<td>0.1</td>
<td>0.2</td>
<td>0.2</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3</td>
<td>0.3</td>
<td>0.267</td>
<td>0.267</td>
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<td>0.6</td>
<td>0.533</td>
<td>0.533</td>
<td>0.4</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.1</td>
<td>0.3</td>
<td>0.1</td>
<td>0.3</td>
<td>0.1</td>
<td>0.3</td>
</tr>
<tr>
<td>0.8</td>
<td>0.3</td>
<td>0.233</td>
<td>0.3</td>
<td>0.233</td>
<td>0.3</td>
<td>0.233</td>
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<tr>
<td>1</td>
<td>0.6</td>
<td>0.466</td>
<td>0.6</td>
<td>0.466</td>
<td>0.6</td>
<td>0.466</td>
</tr>
</tbody>
</table>

Mean total number of persons possessing the rare sensitive attribute $A$

$U_1$, $U_2$
PARTIAL RANDOMIZED RESPONSE MODEL FOR ESTIMATING A RARE SENSITIVE ATTRIBUTE IN PROBABILITY PROPORTIONAL TO SIZE MEASURES USING POISSON DISTRIBUTION

| 1 1 1 2 1 | 102.93 100.46 116.99 100.85 100.77 124.12 |
| 1 1 2 1 1 | 104.14 100.68 121.43 101.25 101.09 129.08 |
| 1 2 1 1 1 | 104.14 100.68 121.43 101.25 101.09 129.08 |
| 2 1 1 1 1 | 107.04 101.29 129.02 102.31 101.84 136.62 |
| 0.7 1 1 1 1 1 | 121.07 186.13 114.46 130.07 106.17 110.59 |
| 0.8 1 1 1 2 1 | 101.4 153.89 100.62 103.89 100.17 100.37 |
| 1 2 1 1 1 | 102.04 162.17 100.91 105.5 100.25 100.54 |
| 1 1 1 1 1 | 102.04 162.17 100.91 105.5 100.25 100.54 |
| 2 1 1 1 1 | 103.73 173.48 101.72 109.39 100.48 101.03 |

Table 5.3- PRE’s of the estimator $\hat{\lambda}_{appwr}$ with respect to the estimator $\hat{\lambda}_{appwr}^{(i)}$ for $\lambda_{xy} = i; i = 1, 2, 3, 4, 5$ and $k = 100$.

<table>
<thead>
<tr>
<th>Mean total number of persons possessing the rare sensitive attribute A</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.3</td>
<td>0.6</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.267</td>
<td>0.533</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.2</td>
<td>0.4</td>
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</tr>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>0.267</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>U</th>
<th>$\lambda_{1a}$</th>
<th>$\lambda_{2a}$</th>
<th>$\lambda_{3a}$</th>
<th>$\lambda_{4a}$</th>
<th>$\lambda_{5a}$</th>
<th>$\lambda_{a}$</th>
<th>PRE ( = $e_1$)</th>
</tr>
</thead>
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<tr>
<td>0.01</td>
<td>1 1 1 1 1</td>
<td>134.7 108.47 101.45 100.33 100.05</td>
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</tr>
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</tr>
<tr>
<td>1 2 1 1 1</td>
<td>139.08 109.62 101.67 100.38 100.06</td>
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</tr>
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</tr>
<tr>
<td>1 2 2 1 1</td>
<td>126.5 107.78 101.49 100.35 100.06</td>
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</tr>
<tr>
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<td>368.79 211.1 131.53 109.58 101.88</td>
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<td>111.73 104.03 100.85 100.21 100.03</td>
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<td>122.44 107.76 101.66 100.41 100.07</td>
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<td>132.63 111.44 102.48 100.62 100.1</td>
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<td>104.17 102.11 100.63 100.18 100.03</td>
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</tr>
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<td>1 1 2 1 1</td>
<td>106.12 103.1 100.93 100.27 100.05</td>
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<td>104.77 104.22 100.73 100.21 100.04</td>
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<td></td>
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</tbody>
</table>

38
Table 5.4: PRE’s of the estimator $\hat{\lambda}_{\text{appsera}}$ with respect to the estimator $\hat{\lambda}_{\text{appsera}}^{(i)}$ for $\lambda_{iy} = 1; i = 1, 2, 3, 4, 5$ and $k = 100$.

<table>
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<tr>
<th>Mean total number of persons possessing the rare sensitive attribute A</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$Q_1$</th>
<th>$Q_2$</th>
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<tbody>
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<td>$U_1$</td>
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<td>0.1</td>
<td>0.2</td>
<td>0.2</td>
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<td>2</td>
<td>100.61</td>
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<td>1</td>
<td>102.39</td>
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</table>
Tables 5.1 and 5.3 demonstrate the PREs of the estimators $\hat{\lambda}_{appswr}$ and $\hat{\lambda}_{appswru}$ under two-stage sampling scheme for the known and unknown unrelated rare attribute. Tables 5.2 and 5.4 demonstrate the PREs of the estimators $\hat{\lambda}_{asppswr}$ and $\hat{\lambda}_{asppswru}$ under two-stage stratified sampling scheme for the known and unknown unrelated rare attribute.

It is observed from Tables 5.1-5.4 that the PRE’s exceed 100% for all chosen parametric values which indicates that proposed model and estimation procedures perform better than the Singh and Suman (2019) RRT model.

It is further observed from Tables 5.1 and 5.3 that the value of PREs increase for decreasing value of $U$ with considerable gain in efficiency for smaller values of $U$.

From Tables 5.1 and 5.3, it is clearly seen that there is higher PRE’s for smaller values of $P_i'$s for different sets of mean total number of persons possessing the rare sensitive attribute $A$ i.e. $\lambda_{ia}$.

Tables 5.2 and 5.4 exhibit that there is gain in PREs for smaller values of both $P_{hi}'s$ and $Q_{hi}'s$. It is also observed that we get higher PREs when the value of $P_{hi}'s$ is smaller than $Q_{hi}'s$. Thus if $P_{hi}'s < Q_{hi}'s$, the proposed estimators $\hat{\lambda}_{asppswru}^{(1)}$ and $\hat{\lambda}_{asppswru}^{(1)}$ perform better than the $\hat{\lambda}_{appswr}$ and $\hat{\lambda}_{appswru}$ respectively.

6. Conclusion

In this paper we have proposed alternative estimation procedures to estimate the mean number of persons having a rare sensitive attribute under a Poisson distribution using two-stage and stratified two-stage sampling schemes. Probability proportional to size sampling scheme has been used to drawn the first-stage units and simple random sampling with replacement is used to select the second-stage units. We have obtained the variances of the suggested estimators and their estimates, if the parameter of the unrelated non-sensitive attribute is either known or unknown. It is proved that the estimators using Narjis and Shabbir (2020) randomized response model under two stage and stratified two stage sampling schemes are advantageous in terms of PRE’s when compared with Singh and Suman (2019). With theoretical and empirical results associated with proposed estimators we also observed that in proposed estimation procedures design parameters play important roles in increasing or decreasing the efficiency. Thus, the envisaged two-stage unrelated question RRT is therefore recommended to the survey practitioners for its use in practice.
References


PARTIAL RANDOMIZED RESPONSE MODEL FOR ESTIMATING A RARE SENSITIVE ATTRIBUTE IN PROBABILITY PROPORTIONAL TO SIZE MEASURES USING POISSON DISTRIBUTION


