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Mohd Irfan
Department of Mathematics, National Institute of Technology Raipur, India,
irfan.maya786@gmail.com

A. K. Sharma*
Department of Mathematics, National Institute of Technology Raipur, India,
aksharma.ism@gmail.com

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Mohd Irfan
Department of Mathematics,
National Institute of Technology
Raipur, India

A. K. Sharma
Department of Mathematics,
National Institute of
Technology Raipur, India

This article discusses the problem of estimating the unknown model parameters as well as prediction of future observations from inverse power Maxwell distribution. The maximum likelihood method is applied for estimating the model parameters using Newton-Rapson iterative procedures. The existence and uniqueness of maximum likelihood estimates are established using Cauchy-Schwartz inequality. Approximate confidence intervals are constructed using Fisher information matrix. Using independent gamma informative priors, the Bayes estimates of unknown model parameters are obtained under squared error and Linex loss functions. Two approximation techniques namely: Lindley's approximation and Metropolis-Hastings within Gibbs sampler algorithm have been employed to derive the Bayes estimators and also to construct the associate highest posterior density credible intervals. Based on the informative (observed) sample, Bayesian prediction, predictive density, and predictive intervals are derived for future observation and decision. The performance of proposed methods are evaluated though a Monte Carlo simulation experiment. Two real-life datasets related to tax revenue and heath are incorporated to show the practical utility of proposed methodology in real phenomenon.

Keywords: Inverse power Maxwell distribution; Bayesian estimation; Lindely's approximation; Metropolis-Hasting algorithm; Bayes prediction; Coverage probability; Goodness-of-fit.

1. Introduction

The Maxwell distribution was first investigated by James Clerk Maxwell (1860) for describing the speed distribution of gas particles in a gas or gas mixture at a specific temperature. This distribution plays a fundamental role in understanding the kinetic theory of gases. The Maxwell distribution is an essential concept in statistical physics, as it forms the basis for understanding gas properties, including the distribution of particle velocities, kinetic energy, and pressure in a gas. While the primary application of the Maxwell distribution is in the physics of gases, it has also found uses in statistical modeling outside of physics. For example, it can be used as a model for certain types of random variables where the spread of data has a bell-shaped curve. However, its broader application in survival analysis contexts is less common compared to other probability distributions like the normal, gamma and Weibull distribution.

The Inverse Maxwell distribution is a probability distribution used in statistics, particularly in modeling extreme value data. It is a specialized distribution that arises from the Maxwell distribution. The Inverse Maxwell distribution is valuable in situations where we want to understand extreme values and rare events. It is particularly useful when dealing with data that exhibits a propensity for extreme values or outliers. It allows statisticians and data analysts to model and understand the behavior of these extreme values in a probabilistic manner. Statistically, the distribution is relevant in various applications where the focus is on the tails of distributions and understanding rare events. This distribution is part of the larger family of extreme value distributions and plays a crucial role in extreme value theory, which is concerned with the statistical analysis of extreme events.

The power transform is a very flexible and widely used method for improving the goodness of fit. It is specially used to stabilize the variance and reduce the skewness and kurtosis of the parent distribution. Recently, Al-Kzzaz and EL-Monsef [4] (2021) initiated the inverse power Maxwell distribution (IPMD) with some statistical characteristics and goodness of fit. The following are the main features of IPMD.

- It exhibits an alternative life time sub-models.
- It indicates an upside-down bathtub-shaped hazard rate, a phenomenon that is common to most real-life systems and is highly helpful in survival analysis.
- It can be applied in a variety of domains, including survival analysis and biomedical investiga- tions, and is thought to have a smooth growing failure rate. It is particularly ideal for fitting positive data with a longer right tail.

The cumulative distribution function (cdf) and probability distribution function (pdf) of two parameter inverse power Maxwell distribution (IPMD) are respectively given by

$$F(x,\eta,\theta) = \frac{2}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2},\theta x^{-2\eta}\right); x > 0, \eta, \theta > 0 \tag{1}$$

and

$$f(x,\eta,\theta) = \frac{4}{\sqrt{\pi}}\;\eta\theta^{\frac{3}{2}}x^{-(3\eta+1)}\;exp(-\theta x^{-2\eta}); x > 0, \eta, \theta > 0. \tag{2}$$

where θ is scale parameter and η is shape parameter of distribution and $\Gamma(a,b) = \int_b^\infty x^{a-1} exp(-x)\, dx$ is the incomplete gamma function in upper. The hazard rate function of IPMD is given by

$$h(x,\eta,\theta) = \frac{2\eta\theta^{\frac{3}{2}}\ x^{-3\eta-1}\ exp(-\theta x^{-2\eta})}{\gamma\left(\frac{3}{2},\theta x^{-2\eta}\right)}; x>0, \eta, \theta > 0, \tag{3}$$

where $\gamma\left(\frac{3}{2},\theta x^{-2\eta}\right)$ represents the lower incomplete gamma function. The plot of pdf and hazard rate function of IPMD are depicted in the figures 1 and 2 respectively. It is observed that the IPMD has an upside-down bathtub-shaped hazard rate function that occurs in the majority of real-world scenarios and is extremely helpful in reliability theory. As a result, it is appropriate for fitting the positively skewed data and applicable to numerous study areas, such as biomedical engineering and survival analysis.

Estimation of parameters is very challenging problems in statistical inference. There are two primary approaches to the estimation of parameters: classical (frequentist) and Bayesian. These approaches differ in their underlying philosophies and methodologies for parameter estimation. In classical statistics, maximum likelihood estimation (MLE) is a commonly and wildly used method for estimating parameters. MLE seeks to find the parameter values that maximize the likelihood function, which measures the goodness of fit of a statistical model to observed data. MLE is unbiased and asymptotically efficient, making it a popular choice for point estimation. On the other hand, Bayesian parameter estimation is a modern approach to statistical inference that combines prior knowledge with observed data to make informed decisions about model parameters. Unlike traditional frequentist methods that rely solely on data, Bayesian estimation incorporates prior beliefs or information into the analysis, allowing for a more robust and flexible modeling framework. At its core, Bayesian parameter estimation operates on the principle of conditional probability. It calculates the probability of model parameters given both prior knowledge and observed data. The central idea is to update our beliefs about parameters as we collect more information, allowing for dynamic and continuous learning.

Although a lot of literature is available on the Bayes estimation for different lifetime models including [10], [30] and [5] obtained the Bayes estimates of Binomial parameters using asymmetric loss function. Tyagi and Bhattacharya (1989) [26] first studied the estimation of Maxwell's velocity distribution function under the Bayes approach. Bekker and Roux (2005) [6] discussed the reliability characteristics of Maxwell distribution under the Bayesian approach. Under various loss functions. Ahamad and Fahad (2008) [2] discussed the Bayes estimates of Rayleigh distribution using record value. Dey and Maiti (2010) [8] proposed Bayesian estimation for the parameters of the Maxwell distribution. Dey et al. (2013) [7] discussed Maxwell distribution in Bayesian context under conjugate prior. Sultan and Ahmad (2015) [24] estimated the parameters of Topp-Leone distribution using Bayesian approach.

Ratogi and Meroci (2018) [20] obtained the Bayes estimators of three parameters Weibull-Rayleigh distribution. Sindhu et al. (2019) [22] have estimated the parameters and reliability of the inverted Maxwell mixture model using Bayesian approach. Tomaer and Panwar (2020) [25] reviewed the inverse Maxwell distribution and obtained the Bayes estimates under Jeffery and conjugate prior. Rao and Pandey (2021) [20] investigate the Bayes estimator for exponentialted transmuted Rayleigh distribution. Fuad S. Al-Duais [11] (2022) introduced a new weighted general loss function and derived the Bayes estimators for Weibull model under complete sample. Yilmaz and Kara [29] (2022) derived Bayes estimators of parameters and reliability characteristics of inverse Weibull distribution under different loss functions. Nasir Abbas (2023) [1] discussed the Bayesian and non Bayesian estimation for Bivariate geometric distribution.

Statistical prediction for future observation based on some known prior information is treated as a generic problem in various branches including bio-medical, economic data, finance and industrial experiment, etc. Many authors have discussed the Bayesian prediction under the complete sampling, including Upadhyay and Pandey (1989) [27] for the exponential distribution, Pradhan and Khundu (2012) [18], Dey and Dey (2012) [9] for Rayleigh distribution.

In this study our primary aim is to focused on the parametric inference of the IPMD using both classical and Bayesian techniques under complete sample. First, the maximum likelihood estimation method is applied to obtain the unknwon model parameters using Newton-Rapson iterative procedures. Using the asymptotic normality criteria of the MLEs, approximate confidence intervals (ACIs) have been constructed. Bayes estimator have been derived under Lindley's and Markov Chain Monte Carlo techniques. In addition, highest posterior density (HPD) credible intervals are obtained using MCMC samples. Based on the informative (observed) sample, Bayesian prediction, predictive density and predictive interval are derived for future observation and decision. Further, a Mote Carlo simulation experiment has been carried out to judge the impact of different estimation procedures based on mean square error (MSE).

The rest of the this paper is arrange as follows. In Section 2, the Maximum likelihood estimators (MLEs) of the unknown parameters including approximate confidence interval and Bayesian estimation together with highest posterior density (HPD) credible interval are derived. Section 3 devoted to point prediction and interval prediction for future failure based on observed sample. To judge the efficiency of the estimation techniques, a simulation study is performed in section 4. In section 5, two real data sets are analysed and the results are compared with inverse power Maxwell distribution (IPMD). At last, the paper is concluded in section 6.

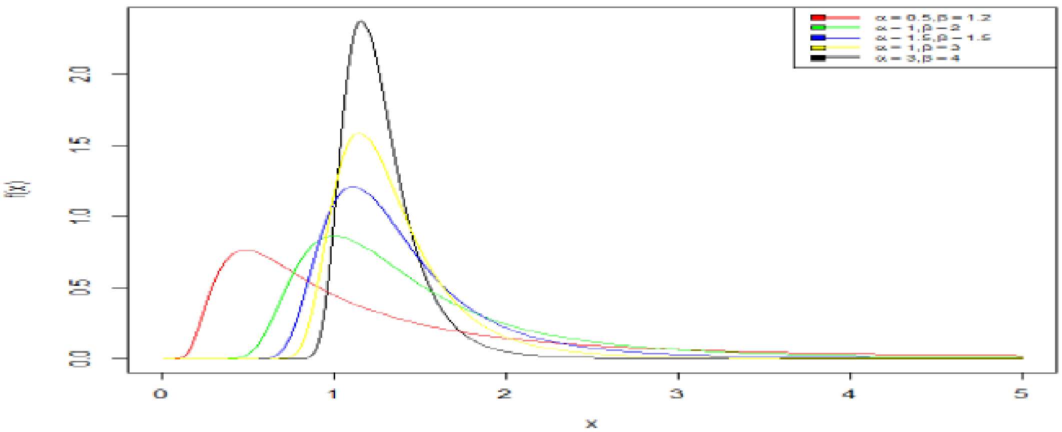

Figure 1: Probability density function of IPMD with different combinations of parameters

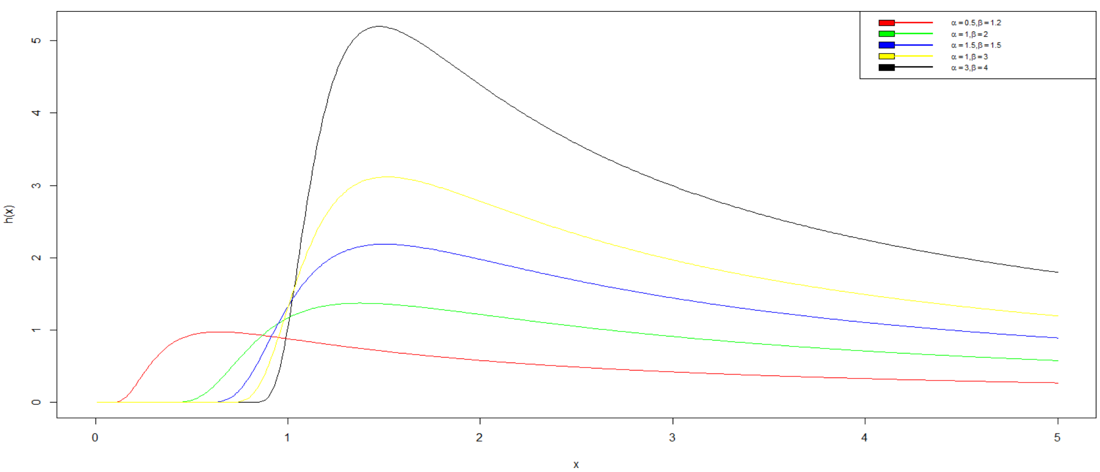

Figure 2: Hazard rate function of IPMD with different combinations of parameters

2. Estimation of the model parameters

In this section, the unknown model parameters and associated confidence intervals of IPMD are obtained using MLE method and Fisher information matrix respectively.

2.1 Maximum likelihood estimation

Let $x_1, x_2, \ldots, x_n$ be the random sample of size n drawn from the inverse power Maxwell distribution (IPMD) with probability density function (2) then the likelihood function, $L(\eta, \theta|x)$ is the joint density of the random sample $x_1, x_2, ..., x_n$ is given by

$$L(\eta,\theta|x) = \prod_{i=1}^{n} \left[\frac{4}{\sqrt{\pi}}\ \eta\theta^{\frac{3}{2}}x^{-(3\eta+1)}\ exp(-\theta x^{-2\eta})\right]$$
$$= \left(\frac{4}{\sqrt{\pi}}\right)^{n} \eta^{n}\theta^{\frac{3n}{2}} \prod_{i=1}^{n} x_i^{(-3\eta-1)} exp(-\theta\sum_{i=1}^{n} x_i^{-2\eta}). \quad (4)$$

Then the log likelihood function of L say *l* can be expressed as follows

$$l(\eta,\theta) = \frac{n}{2}log\left(\frac{16}{\pi}\right) + nlog(\eta) + \frac{3n}{2}log(\theta) - (3\eta+1)\sum_{i=1}^{n} log(x_i) - \theta\sum_{i=1}^{n} x_i^{-2\eta}. \quad (5)$$

The existence and uniqueness of MLEs of the parameters η and θ are provided by the subsequent theorem.

Theorem 2.1. Suppose that observation xi, ($1 \leq i \leq n$) comes from IPMD (2). Then the MLE of θ given η is obtained as

$$\hat{\theta} = \frac{3n}{2\sum_{i=1}^{n} x_i^{-2\eta}}. \quad (6)$$

Proof. Taking the derivative of $l(\eta, \theta)$ with respect to θ and equating to zero, we have

$$\frac{\partial l}{\partial\theta} = \frac{3n}{2\theta} - \sum_{i=1}^{n} x_i^{-2\eta} = 0. \quad (7)$$

On solving for θ we get

$$\hat{\theta} = \frac{3n}{2\sum_{i=1}^{n} x_i^{-2\eta}}. \quad (8)$$

Moreover, since $\frac{\partial^2 l(\eta,\theta)}{\partial\theta^2} = -\frac{3n}{2\theta^2} < 0$, which implies that θˆ is the local maximum of $l(\eta, \theta)$ for given η. Since there is no singular point of $l(\eta, \theta)$ and it consists of single critical point therefore, θ is the maximum of $l(\eta, \theta)$. Hence the MLE of θ given η is exists and unique.

Furthermore, plugging $\hat{\theta} = \hat{\theta}(\eta)$ into equation (5) one has a profile log-likelihood function of η i,e. $l(\eta)$ as

$$l(\eta) = \frac{n}{2}log\left(\frac{16}{\pi}\right) + nlog(\eta) + \frac{3n}{2}log\left(\frac{3n}{2\sum_{i=1}^{n} x_i^{-2\eta}}\right) - (3\eta+1)\sum_{i=1}^{n} log(x_i) - \frac{3n}{2\sum_{i=1}^{n} x_i^{-2\eta}}\sum_{i=1}^{n} x_i^{-2\eta}. \quad (9)$$

The MLE for the parameters η is established from the theorem below

Theorem 2.2. Suppose that observation xi, (1 ≤ i ≤ n) comes from IPMD (2). Then the MLE of η obtained from equation (9) uniquely exists which is derived from the following equation:

$$\frac{1}{\eta} - \frac{3}{n}\sum_{i=1}^{n} log(x_i) - \frac{3}{2}\cdot\frac{A'(\eta)}{A(\eta)} = 0, \tag{10}$$

Where $A(\eta) = \sum_{i=1}^{n} x_i^{-2\eta}\ and\ A'(\eta) = -2\sum_{i=1}^{n}\left[x_i^{-2\eta}\ log(x_i)\right].$

Proof. On differentiating equation (9) and equating to zero we have the likelihood equation as follows

$$\frac{\partial l(\eta,\theta)}{\partial\eta} = \frac{n}{\eta} - 3\sum_{i=1}^{n} log(x_i) + \frac{3n}{\sum_{i=1}^{n} x_i^{-2\eta}}\sum_{i=1}^{n}[x_i^{-2\eta}log(x_i)] = 0. \tag{11}$$

On simplifying we get

$$\frac{1}{\eta} - \frac{3}{n}\sum_{i=1}^{n} log(x_i) + \frac{3}{\sum_{i=1}^{n} x_i^{-2\eta}}\sum_{i=1}^{n}[x_i^{-2\eta}log(x_i)] = 0. \tag{12}$$

In order to prove the existence of the MLE of η we must show the equation (12) has unique with respect to η.

Let

$$\phi_1(\eta) = \frac{1}{\eta} \ \text{ and } \ \phi_2(\eta) = \frac{3}{2}\cdot\frac{A'(\eta)}{A(\eta)} + \frac{3}{n}\sum_{i=1}^{n} log(x_i) \tag{13}$$

From the equation (13) it is observed that ϕ1(η) is decreasing function in η and $\lim_{\eta\to 0}\phi_1(\eta) = +\infty$ and $\lim_{\eta\to\infty}\phi_1(\eta) = 0$. On the other hand for the function ϕ2(η) using Cauchy-Schwartz inequality we have

$$\frac{d\phi_2(\eta)}{d\eta} = \frac{3}{2}\cdot\frac{A(\eta)A''(\eta) - [A'(\eta)]^2}{[A(\eta)]^2} \geq 0, \tag{14}$$

where $A(\eta) = \sum_{i=1}^{n} x_i^{-2\eta}$, $A'(\eta) = -2\sum_{i=1}^{n}\left[x_i^{-2\eta}\ log(x_i)\right]$ and $A''(\eta) = 4\sum_{i=1}^{n}\left[x_i^{-2\eta}\ log^2(x_i)\right]$,

which implies that ϕ2(η) increasing function of η. Moreover, since

$$\lim_{\eta\to 0}\phi_2(\eta) = 0 \tag{15}$$

and

$$\lim_{\eta \to \infty} \phi_2(\eta) = \frac{3}{n} \sum_{i=1}^{n} log(x_i) - 3\ log(x_1) \tag{16}$$

From equations (13), (14), (15) and (16) it is clear that the functions $\phi_1(\eta)$, and $\phi_2(\eta)$ have a unique intersection point; therefore, the MLE of η, as the root of the equation $\phi_1(\eta) = \phi_2(\eta)$, exists and unique.

2.2 Approximate confidence interval (ACI)

This subsection focused on the construction of frequentest confidence intervals. The confidence interval 100(1 - τ)% can be derived for unknown parameters η and θ based on the asymptotic behavior of the MLEs η̂ and θ̂. The MLEs (η̂, θ̂) asymptotic normal distribution with mean (η, θ) and variance-covariance matrix I^{-1}(η̂, θ̂) under the certain regularity conditions, that is

$$(\hat{\eta}, \hat{\theta}) \sim N((\eta, \theta), \hat{I}^{-1}(\hat{\eta}, \hat{\theta})),$$

where $\hat{I}^{-1}(\hat{\eta}, \hat{\theta})$ is the observed information matrix and defined as

$$I(\eta, \theta) = \left[-\frac{\partial^2 l(\eta, \theta)}{\partial \eta_j \partial \theta_k} \right]_{(\eta,\theta)=(\hat{\eta},\hat{\theta})} ; 1 \leq j, k \leq 2 \tag{17}$$

and the second derivative of l(η, θ) can be obtained from the equation (5) as

$$\frac{\partial^2 l}{\partial \eta^2} = -\frac{n}{\eta^2} - 4\theta \sum_{i=1}^{n} \left[x_i^{-2\eta} (log x_i)^2 \right], \quad \frac{\partial^2 l}{\partial \theta^2} = -\frac{3n}{2\theta^2},$$

$$\frac{\partial^2 l}{\partial \eta \partial \theta} = \frac{\partial^2 l}{\partial \theta \partial \eta} = 2 \sum_{i=1}^{n} \left[x_i^{-2\eta} log(x_i) \right].$$

The expression of the equation (17) is known as variance and covariance matrix whose diagonal ele- ments represents the variance of η̂ and θ̂ while non-diagonal elements represent the covariance between the η̂ and θ̂. Thus, the 100(1 - τ)% ACI of η and θ are obtained as $\left(\hat{\eta} - z_{\tau/2}\sqrt{var(\hat{\eta})}, \hat{\eta} + z_{\tau/2}\sqrt{var(\hat{\eta})}\right)$ and $\left(\hat{\theta} - z_{\tau/2}\sqrt{var(\hat{\theta})}, \hat{\theta} + z_{\tau/2}\sqrt{var(\hat{\theta})}\right)$,

where $z_{\frac{\tau}{2}}$ represents the upper $\frac{\tau}{2}$-th quantile for standard normal distribution.

3. Bayesian estimation

In this section, Bayesian approach is used to estimate the unknown parameters of IPMD using two different approximation techniques namely Lindley's and Metropolis-Hastings algorithm under squared error loss functions (SELF) and linex loss functions (LLF).

3.1 Loss functions

In Bayesian estimation, loss function plays a crucial role in decision-making and model selection. It quantifies the cost or loss associated with the choice of different models, parameter values, or decision actions. The goal is to select models or parameters that minimize the expected loss, making the decision process more rational and well-informed. The two main loss functions viz. squared error loss function (SELF) and linex loss function (LLF) functions are considered. The SELF is symmetric and LLF is asymmetric. If the $\hat{\zeta}$ is the estimator of the parameters ζ then the SELF is given by

$$L_{SELF}(\zeta,\hat{\zeta}) = (\zeta-\hat{\zeta})^2, \tag{18}$$

which provide equal penalty for overestimation as well as under estimation. This loss function may not be the best tool to utilise in certain situations. For instance, in stock market underestimation of market risk is worse than overestimation for investors. Furthermore, overestimation is more harmful than underestimation when evaluating the cure rate based on a particular treatment. In these situations the linex loss function (see, Varian (1975) [28]) is useful and given by

$$L_{LLF} = e^{(\zeta-\hat{\zeta})} - h(\zeta-\hat{\zeta}) - 1; h \neq 0. \tag{19}$$

Under the loss functions (18) and (19) the Bayes estimates can be expressed as follows

$$\hat{\zeta}_{SELF} = E_{\zeta}(\zeta|X) \tag{20}$$

and

$$\hat{\zeta}_{LLF} = -\frac{1}{h}\, log[E_{\zeta}(e^{-h\zeta}|x)], h \neq 0. \tag{21}$$

3.2 Prior information

In Bayesian estimation, prior distribution play a dominant role. Note that if a suitable prior available regarding the unknown parameters, the informative priors are an appropriate way of incorporating the information into the model. In this study, the model parameters are not known, and also joint conjugate prior does not exist. Therefore independent gamma prior is suitable for this study because the gamma distribution is log concave function in the interval (0, ∞). Let the η and θ have two independent gamma prior distribution which are as follows

$$u_1(\eta) = \frac{b_1^{a_1}}{\Gamma a_1}\eta^{a_1-1} exp\left(-b_1\eta\right); \eta \in (0,\infty) \tag{22}$$

and

$$u_2(\theta) = \frac{b_2^{a_2}}{\Gamma a_2}\theta^{a_2-1} exp\left(-b_2\theta\right); \theta \in (0, \infty), \tag{23}$$

where a_1, b_1, a_2, $b_2 > 0$ are hyper parameters.

The joint prior density of η and θ can be expressed as follows

$$u(\eta, \theta) \propto \eta^{a_1-1}\theta^{a2-1} exp(-b_1\eta - b_2\theta). \tag{24}$$

3.3 Posterior analysis

The posterior distribution is obtained by combining (4) and (24), it is possible to express the joint posterior density function of η anf θ as

$$\pi(\eta, \theta|x) = \frac{L(x|\eta, \theta)\pi(\eta, \theta)}{\int_\eta \int_\theta L(x|\eta, \theta)\pi(\eta, \theta)\ d\eta d\theta}$$
$$\propto \eta^{n+a_1-1}\theta^{\frac{3n}{2}+a_2-1}\prod_{i=1}^{n} x_i^{-3\eta-1} exp(-\eta b_1) exp\left(-\theta(b_2 + \sum_{i=1}^{n} x_i^{-2\eta})\right). \tag{25}$$

Bayes estimators of η and θ under square error loss function (SELF) and LINEX loss function are given as follows

$$\hat{\eta}_{BS} = E(\eta|x) = \int_0^\infty \int_0^\infty \eta\ \pi(\eta, \theta|x) d\eta d\theta, \tag{26}$$

$$\hat{\theta}_{BS} = E(\theta|x) = \int_0^\infty \int_0^\infty \theta\ \pi(\eta, \theta|x) d\eta d\theta, \tag{27}$$

$$\hat{\eta}_{BL} = -\frac{1}{h}\ ln\ E(exp(-h\eta)|x));\ \ h \neq 0 \tag{28}$$

and

$$\hat{\theta}_{BL} = -\frac{1}{h}\ ln\ E(exp(-h\theta)|x));\ \ h \neq 0. \tag{29}$$

Since the aforementioned equations cannot be analytically solved, therefore, Lindley's approximation is employed for further analysis.

3.4 Lindely approximation

The Lindley's approximation was first introduced by Lindley's (1980) [13], which plays a significant role in Bayesian estimation. The beauty of this approximation is that it computes the Bayes estimate quite accurately without doing any numerical integration.

Let v(η, θ) be the function of η and θ, then using the equation (12) the average value of v(η, θ) is given by

$$\hat{v} = E(v(\eta,\theta)|x) = \frac{\int_\eta \int_\theta v(\eta,\theta) e^{l(\eta,\theta|x)+\rho(\eta,\theta)} d\eta d\theta}{\int_\eta \int_\theta e^{l(\eta,\theta|x)+\rho(\eta,\theta)} d\eta d\theta}. \tag{30}$$

The Bayes estimator v(η, θ) is the solution of the above equation. Unfortunately, the Bayes estimator can not be obtained analytically because it involves the ratio of two integrals. In order to go through these challenges it is used Lindley's approximation then E(u(θ, λ x)) can be approximated by vˆ(η, θ) (say) as

$$\begin{aligned} \hat{v}(\eta,\theta) \;=\;& v(\hat{\eta},\hat{\theta}) + 0.5(\hat{v}_{\eta\eta}\hat{\sigma}_{\eta\eta} + \hat{v}_{\theta\theta}\hat{\sigma}_{\theta\theta}) + \hat{v}_{\eta\theta}\hat{\sigma}_{\eta\theta} + \hat{v}_{\eta}(\hat{\sigma}_{\eta\eta}\hat{\rho}_{\eta} + \hat{\sigma}_{\theta\eta}\hat{\rho}_{\theta}) + \\ & \hat{v}_{\theta}(\hat{\sigma}_{\eta\theta}\hat{\rho}_{\eta} + \hat{\sigma}_{\theta\theta}\hat{\rho}_{\theta}) + 0.5\hat{L}_{\eta\eta\eta}(\hat{v}_{\eta}\hat{\sigma}^2_{\eta\eta} + \hat{v}_{\theta}\hat{\sigma}_{\eta\eta}\hat{\sigma}_{\eta\theta}) + 0.5\hat{L}_{\eta\eta\theta} \\ & \left(3\hat{v}_{\eta}\hat{\sigma}_{\eta\eta}\hat{\sigma}_{\eta\theta} + \hat{v}_{\theta}(\hat{\sigma}_{\eta\eta}\hat{\sigma}_{\theta\theta} + 2\hat{\sigma}^2_{\eta\theta})\right) + 0.5\hat{L}_{\eta\theta\theta}\left(\hat{v}_{\eta}(\hat{\sigma}_{\eta\eta}\hat{\sigma}_{\theta\theta} + 2\hat{\sigma}^2_{\eta\theta})\right. \\ & \left.+3\hat{v}_{\theta}\hat{\sigma}_{\eta\theta}\hat{\sigma}_{\theta\theta}\right) + 0.5\hat{L}_{\theta\theta\theta}(\hat{v}_{\eta}\hat{\sigma}_{\eta\theta}\hat{\sigma}_{\eta\eta} + \hat{u}_{\theta}\hat{\sigma}^2_{\theta\theta}). \end{aligned} \tag{31}$$

where ρ(η, θ) is the logarithm of joint prior density, vηη is the second derivative of the function v(η, θ) with respect to η, which vˆηη is the same expression as evaluated for v(ηˆ, θˆ). Other notations are defined in similar way as

$$\hat{v}_\eta = \frac{\partial v}{\partial \eta}, \quad \hat{v}_\theta = \frac{\partial u}{\partial \theta}, \quad \hat{v}_{\theta\eta} = \hat{v}_{\eta\theta} = \frac{\partial^2 v}{\partial\eta\partial\eta}, \quad \hat{\rho}_\eta = \frac{\partial log\ u(\eta,\theta)}{\partial\eta}, \quad \hat{\rho}_\theta = \frac{\partial log\ u(\eta,\theta)}{\partial\theta}$$

$$\hat{l}_\theta = \frac{\partial l}{\partial\theta}, \hat{l}_\eta = \frac{\partial l}{\partial\eta}, \hat{l}_{\eta\theta} = \hat{l}_{\eta\theta} = \frac{\partial^2 l}{\partial\eta\theta}, \hat{l}_{\theta\theta} = \frac{\partial^2 l}{\partial\theta^2}, \hat{l}_{\eta\eta} = \frac{\partial^2 l}{\partial\eta^2}, \hat{l}_{\theta\theta\theta} = \frac{\partial^3 l}{\partial\theta^3}, \hat{l}_{\eta\eta\eta} = \frac{\partial^3 l}{\partial\eta^3},$$

$$\hat{l}_{\theta\theta\eta} = \frac{\partial^3 l}{\partial\theta^2\partial\eta}, \hat{l}_{\theta\eta\eta} = \frac{\partial^3 l}{\partial\theta\eta^2},$$

where σ_{ij} is the (i, j)th element of matrix $[-\hat{l}_{ij}]^{-1}; i,j = 1,2.$

The Bayes estimates under square error loss function (SELF) are obtained as If v(η, θ) = η, then $v_\eta = 1$, $v_{\theta\theta} = v_\theta = v_{\eta\eta} = v_{\theta\eta} = v_{\eta\theta} = 0$. Thus, the Bayes estimator of η is expressed as follows

$$\begin{aligned} \hat{\eta}_{BS} \;=\;& \hat{\eta} + (\hat{\rho}_\eta\hat{\sigma}_{\eta\eta} + \hat{\rho}_\theta\hat{\sigma}_{\theta\eta}) + 0.5\left[\hat{L}_{\eta\eta\eta}\hat{\sigma}^2_{\eta\eta} + 3\hat{L}_{\eta\eta\theta}\hat{\sigma}_{\eta\theta}\hat{\sigma}_{\eta\eta} + \hat{L}_{\eta\eta\theta}(\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\eta\eta} + 2\hat{\sigma}^2_{\eta\theta})\right. \\ & \left.+\hat{L}_{\theta\theta\theta}\hat{\sigma}_{\eta\theta}\hat{\sigma}_{\theta\theta}\right]. \end{aligned} \tag{32}$$

If v(η, θ) = θ, then $v_\theta = 1$, $v_{\theta\theta} = v_\theta = v_{\eta\eta} = u_{\theta\eta} = u_{\eta\theta} = 0$. So, the Bayes estimator of θ is found as follows:

$$\begin{aligned} \hat{\theta}_{BS} \;=\;& \hat{\theta} + (\hat{\rho}_\theta\hat{\sigma}_{\theta\theta} + \hat{\rho}_\eta\hat{\sigma}_{\eta\theta}) + 0.5\left[\hat{L}_{\eta\eta\eta}\hat{\sigma}_{\eta\eta}\hat{\sigma}_{\eta\theta} + 3\hat{L}_{\eta\theta\theta}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\eta\theta} + \hat{L}_{\eta\eta\theta}(\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\eta\eta} + 2\hat{\sigma}^2_{\eta\theta})\right. \\ & \left.+\hat{L}_{\theta\theta\theta}\hat{\sigma}^2_{\theta\theta}\right]. \end{aligned} \tag{33}$$

If u(η, θ) = exp(−hη), $v_\eta = -h\exp(-h\eta)$, $u_{\eta\eta} = h2\exp(-h\eta)$, $u_\theta = v_{\theta\theta} = v_{\eta\theta} = v_{\theta\eta} = 0$.

Hence, the Bayes estimator of θ under LINEX loss function is obtained as

$$\hat{\eta}_{BL} = -\frac{1}{h}\ ln\ E(exp(-h\eta)|x));\ \ h \neq 0, \tag{34}$$

where

$$\begin{aligned} E(exp(-h\eta)|x) \ &= \ exp(-h\hat{\eta}) + 0.5\hat{v}_{\eta\eta}\hat{\sigma}_{\eta\eta} + \hat{v}_{\eta}(\hat{\sigma}_{\eta\eta}\hat{\rho}_{\eta} + \hat{\sigma}_{\theta\eta}\hat{\rho}_{\theta}) + \\ &0.5\hat{L}_{\eta\eta\eta}\hat{u}_{\eta}\hat{\sigma}^2_{\eta\eta} + 1.5\hat{L}_{\eta\eta\theta}\hat{v}_{\eta}\hat{\sigma}_{\eta\eta}\hat{\sigma}_{\eta\theta} + 0.5\hat{L}_{\eta\theta\theta}\left(\hat{v}_{\eta}(\hat{\sigma}_{\eta\eta}\hat{\sigma}_{\theta\theta} + 2\hat{\sigma}^2_{\eta\theta})\right) \\ &+0.5\hat{L}_{\theta\theta\theta}\hat{v}_{\eta}\hat{\sigma}_{\eta\theta}\hat{\sigma}_{\eta\eta}. \end{aligned}$$

If $v(\eta, \theta) = exp(-h\theta)$, then $u_\theta = -h\ exp(-h\theta)$, $v_{\theta\theta} = h^2\ exp(-h\theta)$, $v_\eta = v_{\eta\eta} = v_{\eta\theta} = u_{\theta\eta} = 0$. Thus,

$$\hat{\theta}_{BL} = -\frac{1}{h}\ ln\ E(exp(-h\theta)|x);\ \ h \neq 0, \tag{35}$$

where

$$\begin{aligned} E(exp(-h\theta)|x) \ &= \ exp(-h\hat{\theta}_{ML}) + 0.5\hat{v}_{\theta\theta}\hat{\sigma}_{\theta\theta} + \hat{v}_{\theta}(\hat{\sigma}_{\theta\theta}\hat{\rho}_{\theta} + \hat{\sigma}_{\eta\theta}\hat{\theta}_{\eta}) + \\ &+0.5\hat{L}_{\eta\eta\eta}\hat{v}_{\theta}\hat{\sigma}_{\rho\theta}\hat{\sigma}_{\theta\theta} + 1.5\hat{L}_{\eta\theta\theta}\hat{v}_{\theta}\hat{\sigma}_{\eta\eta}\hat{\sigma}_{\eta\theta} + 0.5\hat{L}_{\eta\eta\theta}\left(\hat{v}_{\theta}(\hat{\sigma}_{\eta\eta}\hat{\sigma}_{\theta\theta} + 2\hat{\sigma}^2_{\eta\theta})\right) \\ &+0.5\hat{L}_{\theta\theta\theta}\hat{v}_{\theta}\hat{\sigma}^2_{\theta\theta}. \end{aligned}$$

It is not possible to determine the credible interval using Lindley's approximation. Therefore it is introduced the Metropolis-Hasting algorithm to acquired the credible interval in the next subsection.

3.5 Metropolis-Hastings algorithm

The Metropolis-Hastings within Gibbs (MH-within-Gibbs) sampler is a powerful Markov chain Monte Carlo (MCMC) algorithm used for drawing samples from complex probability distributions, especially in Bayesian statistics and statistical modeling. It combines elements of two important MCMC techniques: the Metropolis-Hastings algorithm and the Gibbs sampler. For more detail see Metropolis et al. (1953) [17], and Smith and Robert (1993) [23].

From equation (25), up-to normalising constant, the marginal posterior density of η given θ and data can be obtained as

$$\pi(\eta|\theta, x) \propto \eta^{n+a_1-1} exp\left[-\left\{\eta b_1 + \theta\sum_{i=1}^{n} x_i^{-2\eta} + 3\eta\sum_{i=1}^{n} log(x_i)\right\}\right] \tag{36}$$

Similarly, up-to normalising constant, the full conditional distribution for θ given η and data is given by

$$\pi(\theta|\eta, x) \propto \theta^{\frac{3n}{2}+a_2-1} exp\left[-\theta\left(b_2 + \sum_{i=1}^{n} x_i^{-2\eta}\right)\right], \tag{37}$$

It can be observed that, equation (37) follows a gamma density with shape parameter $(\frac{3n}{2}+a_2)$ and scale parameter $(\sum_{i=1}^{n} x_i^{-2\eta}+b_2)$. Therefore, it is simple to create the sample of θ by gamma generating technique. Furthermore, the full conditional posterior distribution for η does not follows any well known distribution and it is difficult to sample directly by standard methods. Therefore, to compute Bayes estimates and associated HPD credible intervals, the M-H within Gibss sampler algorithm is used for this purpose. The following steps are considered for generating the sample from unknown posterior distribution based on the Metropolis-Hasting within Gibbs sampler approach

Step 1: Set j = 1 and assume that $\eta_0 = \hat{\eta}$ and $\theta_0 = \hat{\theta}$.

Step 2 Generate θ_j from $\pi(\theta|\eta, x)$ using Gibbs sampler algorithm.

Step 3: Using the M-H algorithm, generate a posterior sample for η_j and from the conditional distributions $\pi(\eta_{j-1}|\theta_j, x)$ with normal proposal distribution $N(\eta_j, var(\eta))$.

(i) Generate a candidate η^* from $N(\eta_j, var(\eta))$.

(ii) Compute $\nu_\eta = min\left(\frac{\pi(\eta^*|\theta_{(j-1)},x)}{\pi(\eta_{(j-1)}|\theta_{(j-1)},x)}, 1\right)$.

(iii) Generate u from uniform distribution $U(0,1)$.

(iv) If $u \leq \nu_\eta$, set $\eta_j = \eta^*$, else set $\eta^j = \eta_{j-1}$.

Step 4: Set j = j+1.

Step 5: Repeat the steps 2-4, N times to extract samples $(\eta_1, \theta_1), (\eta_2, \theta_2), \ldots, (\eta_N, \theta_N)$.

Step 6: The Bayes estimator of the parameters η and θ under square error loss function (SELF) and LINEX loss function can be obtained form the following formula as

$$\hat{\eta}_{BS} = \frac{1}{N-M}\sum_{j=M+1}^{N} \eta_j, \tag{38}$$

$$\hat{\theta}_{BS} = \frac{1}{N-M}\sum_{j=M+1}^{N} \theta_j, \tag{39}$$

$$\hat{\eta}_{BL} = -\frac{1}{h}\, ln\left(\frac{1}{N-M}\sum_{j=M+1}^{N} exp(-h\eta_j)\right); \quad h \neq 0 \tag{40}$$

and

$$\hat{\theta}_{BL} = -\frac{1}{h}\, ln \left(\frac{1}{N-M} \sum_{j=M+1}^{N} exp(-h\theta_j) \right); \quad h \neq 0, \tag{41}$$

where M is the burn-in period.

Step 7: In order to build the HPD credible interval of η, first arrange $\eta_1, \eta_2, ..., \eta_N$ in increasing order, as $\eta_{(1)}, \eta_{(2)}, ..., \eta_{(N)}$, then for arbitrary $0 < \tau < 1$, the 100(1 τ)% credible interval of η can be obtained as $(\hat{\eta}^{[k]}, \hat{\eta}^{[k+N-(\tau N+1)]})$ where $k = 1, 2, \ldots, [N\tau]$ and [z] represents the greatest integer less or equal to z. Hence, the credible interval is computed such that

$$\left(\hat{\eta}^{[k^*+N-(\tau N+1)]} - \hat{\eta}^{[k^*]]}\right) = min_{k=1}^{N\tau} \left(\hat{\eta}^{[k+N-(\eta N+1)]} - \hat{\eta}^{[k]]}\right).$$

Similarly, the HPD credible interval of θ can be also derived.

4. Prediction of future observation

In this section, it is addressed point as well as interval prediction of future observation from IPMD under Bayesian paradigm. The Bayes prediction of unknown observable belongs to informative sample. Let z be the future observation independent of the given data x1, x2, ..., xn. Then the posterior predictive density of z for the given observed data, is defined as

$$\pi(z|x) = \int_0^\infty \int_0^\infty f(z|\eta,\theta)\pi(\eta,\theta|x) d\eta\ d\theta. \tag{42}$$

Let us assume a future sample $z_1, z_2, ..., z_s$ with size s, independent of the observed sample $x_1, x_2, ..., x_n$ and let $z_{(1)} < z_{(2)}, ..., < z_{(r)} <, ..., z_{(s)}$ be the sample order statistics. Suppose it is interested in the predictive density of the future order statistic z(r) given the observed sample $x_1, x_2, ..., x_n$. Then the probability density function of the rth order statistic in the future sample is represented by $y_{(r)}(z|\eta, \theta)$ and expressed as

$$y_{(r)}(z|\eta,\theta) = r\binom{s}{r}[F(z|\eta,\theta)]^{r-1}[1-F(z|\eta,\theta)]^{s-r} f(z|\eta,\theta), \tag{43}$$

where $F(z|\eta,\theta) = \frac{2}{\sqrt{\pi}}\, \Gamma\left(\frac{3}{2}, \theta z^{-2\eta}\right)$ and $f(z|\eta,\theta) = \frac{4}{\sqrt{\pi}}\, \eta\theta^{\frac{3}{2}} z^{-(3\eta+1)}\, exp(-\theta z^{-2\eta})$.

Let the predictive density of $z_{(r)}$ denoted as $y^*_{(r)}(z|\eta, \theta)$, then

$$y^*_{(r)}(|z|\eta,\theta) = \int_0^\infty \int_0^\infty y_{(r)}(z|\eta,\theta)\ \pi(\eta,\theta|x) d\eta\ d\theta. \tag{44}$$

Using the equation (43) we get

$$y^*_{(r)}(z|\eta,\theta) = \int_0^\infty \int_0^\infty r\binom{s}{r}\left[\frac{2}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2},\theta z^{-2\eta}\right)\right]^r \left[\frac{2}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2},\theta z^{-2\eta}\right)\right]^{s-r} \frac{4}{\sqrt{\pi}}\,\eta\theta^{\frac{3}{2}} z^{-(3\eta+1)}\,exp(-\theta z^{-2\eta})\pi(\eta,\theta|x)\,d\eta\,d\theta. \quad (45)$$

On simplifying

$$y^*_{(r)}(z|\eta,\theta) = \frac{4}{\sqrt{\pi}} r\binom{s}{r}\int_0^\infty \int_0^\infty \eta\theta^{\frac{3}{2}} z^{-(3\eta+1)}\,exp(-\theta z^{-2\eta})\pi(\eta,\theta|x) \left[\frac{2}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2},\theta z^{-2\eta}\right)\right]^r \left[\frac{2}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2},\theta z^{-2\eta}\right)\right]^{s-r} d\eta\,d\theta. \quad (46)$$

Since the analytical solution of (46) is not possible. Therefore it is again used the Metropolis-Hasting within Gibbs sampler algorithm to approximate the solution. Suppose that $\{(\eta_i, \theta_i), i = 1, 2, ..., N\}$ is an MCMC sample obtained from $\pi(\eta_i, \theta_i|x)$ using M-H algorithm within Gibbs sampler algorithm described in the subsection 3.5 then a simulation consistent estimator of $y^*_{(r)}(z|x)$ can be found as

$$y^*_{(r)}(z|\eta,\theta) = \frac{1}{N-M}\sum_{i=M+1}^{N} g_{(r)}(z|\eta_i,\theta_i). \quad (47)$$

Now, it is facile to obtained the predictive mean of the future r − th order lifetime, which is given by

$$\begin{aligned}\hat{z}_{BS} &= E(z|x)\\ &= \int_0^\infty z\; y^*_{(r)}(z|\eta,\theta)\,dz\\ &= \int_0^\infty z\frac{\sum_{i=M+1}^{N} y_{(r)}(z|\eta_i,\theta_i)}{N-M}\,dz\\ &= \frac{\sum_{i=M+1}^{N}\int_0^\infty z\; y_{(r)}(z|\eta_i,\theta_i)}{N-M}\,dz. \end{aligned} \quad (48)$$

Further the predictive density of $z_{(r)}$ is assumed to be represented as $Y^*_{(r)}(z|\eta, \theta)$ and given by

$$y_{(r)}(z|\eta,\theta) = r\binom{s}{r}\int_0^z [F(t|\eta,\theta)]^{r-1}\,[1-F(t|\eta,\theta)]^{s-r} f(t|\eta,\theta)dt]. \quad (49)$$

The simulation consistent estimator of $Y_r^*(.|x)$ is found as follows

$$Y^*_{(r)}(z|x) = \frac{1}{N-M}\sum_{i=M+1}^{N} Y_{(r)}(z|\eta_i,\theta_i). \tag{50}$$

It is derived the two sided predictive interval of the r^{th} order statistic Z_r from the future observations $Z_{(1)}, Z_{(1)}, ..., Z_{(s)}$ of size s, independent of the observed sample $\{x_1, x_2, ..., x_n\}$. The two sided 100τ % symmetric predictive interval for $Z_{(r)}$, computed by solving the following two nonlinear equation equations for lower bound l and upper boundu.

$$P\left[Z_{(r)} > l|x\right] = 1 - Y^*_{(r)}(l|x) = \frac{1+\tau}{2}, \tag{51}$$

$$P\left[Z_{(r)} > u|x\right] = 1 - Y^*_{(r)}(u|x) = \frac{1-\tau}{2}. \tag{52}$$

Since, the analytical solution is not possible therefore, it is proposed the following algorithm to obtain the solution to the equations (51) and (52).

Step 1: Set an initial guess of v, say $\tilde{v}$, and set v = $\tilde{v}$.

Step 2: Compute $Y^*_{(r)}(v|x) = \frac{1}{N-M}\sum_{i=M+1}^{N} Y_{(r)}(v|\eta_i,\theta_i)$ using MCMC and sample obtained from the equations (23) and (24).

Step 3: If $Y^*_{(r)}(v|x) < \xi$ then increase the value of v by a fixed small number, say ϵ, otherwise, decrease the value of v by ϵ.

Step 4: Repeat the steps 2 and 3, until $Y^*_{(r)}(v|x) \simeq \xi$,

where v may be l or u, and $\xi = 1 - \frac{\tau}{2}$ or $\frac{\tau}{2}$.

5. Simulation study

In this section we investigate a comprehensive numerical simulation to assess the effectiveness of the proposed estimation procedures using R statistical software. The quality of the different estimators are evaluated based on the following measures.

- Absolute bias (AB): The AB is defined as $\frac{1}{N}\sum_{i=1}^{N}|\zeta_i - \hat{\zeta}_i|$ where, ζ_i denotes parameters (η and θ) and $\hat{\zeta}_i$ denotes their estimates and N is the number of replications. The smaller value of AB indicate that the correlation between experimental data and prediction model is more precise.
- Mean square error (MSE): The MSE is defined as $\frac{1}{N}\sum_{i=1}^{N}(\zeta_i - \hat{\zeta}_i)^2$. The lower value of the MSE indicates the better performance of the estimates.
- Average length (AL): Average length of $100(1 - \tau)$% confidence intervals has been judged. The thinner length of AL represents the high performance of the intervals estimates.

- Coverage probability (CP): The probability of containing the true values of the parameters in between the intervals estimates.

To generate the sample from IPMD we used inverse transformation technique. It is well known that in inverse transformation method, random numbers from a particular distribution are generated by solving the non-linear equation $x = F^{-1}(p)$ where F (x) is the cumulative distribution function of IPMD and p follows the uniform distribution in the interval (0,1) i.e. $p \sim u(0, 1)$. Following the same procedure for generation of random number from IPMD using inverse transformation technique

$$F(x,\eta,\theta) = \frac{2}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2},\theta x^{-2\eta}\right) = p. \tag{53}$$

In both the frequentest and Bayes methods, the sample sizes n = (30, 60, 90, 120) and the true value of shape parameter $\eta = 0.5, 1$ and scale parameter $\theta = 1.2, 2$ are taken. In order to compute Bayes estimates two different loss functions namely; SELF and LLF have been used. Two independent the gamma distributions are considered for the prior distributions of η and θ. Here, the hyper- parameters for the combinations $(\eta, \theta) = (0.5, 1.2)$ and $(\eta, \theta) = (1, 2)$ are (a1, b1, a2, b2) = (1, 2, 6, 5) and (a1, b1, a2, b2) = (2, 2, 10, 5) respectively. To derive Bayes estimates Lindley approximation method and MCMC algorithm are simulated with 1000 replications. The 95% ACIs and HPD credible intervals for unknown parameters are also computed. The numerical results of different point and intervals estimates are presented in Tables 1-4. Further, prediction results of z4, z7 and z9 in future sample with sample size 10 based on the informative samples from IPMD are derived under the similar situation. In addition, the predictive mean, prediction intervals also constructed and tabulated in Table 5. The following interpretation may be read out from these tables:

- As the effective sample size n grows the AB and MSE of all estimates decrease. This trends indicates that the estimated results are more precise when sample size become larger.
- In terms of ABs and MSEs, Bayes estimates have superior performance than the MLEs.
- For the shape parameter η, the performance of LLF (h = 0.5) under Lindely method outperform than the MCMC approach and following mathematical relation can be establish for ABs and MSEs:

$$MLE_{\eta} > MCMC_{\eta} > Lindley_{\eta}.$$

- For the scale parameter θ, the performance of LLF (h = 0.5) under MCMC technique is better than the Lindely method and following mathematical relation can be establish for ABs and MSEs:

$$MLE_\theta > Lindley_\theta > MCMC_\theta.$$

- With the increase of effective sample size n, the AL of the confidence intervals decreases, which suggests that the estimation is more precise. However, no specific trends of CP have been seen throughout these numerical experiment.
- HPD intervals are preferable to ACIs in terms of ALs and CPs.
- From Table 5, it has been investigated that the interval lengths of future observation become wider as the r increases.

Table 1. The average estimates (AEs) (First row), absolute biases (ABs), and mean square errors (MSEs) (third row) of the MLEs and Bayes estimators under SELF and LLF. The true values of η and θ are 0.5 and 1.2.

			Lindley			**MCMC**		
	n	**MLE**	**SELF**	**LLF**		**SELF**	**LLF**	
				h = -0.5	h = 0.5		h = -0.5	h = 0.5
η		0.5246	0.4998	0.5009	0.4987	0.5185	0.5192	0.5179
		0.05915	0.04820	0.04829	0.04814	0.05335	0.05359	0.05312
	30	0.006141	0.003892	0.003920	0.003869	0.004948	0.004996	0.004900
θ		1.206	1.303	1.310	1.296	1.200	1.204	1.196
		0.1655	0.1341	0.1409	0.1270	0.1209	0.1210	0.1210
		0.04332	0.02931	0.03210	0.02661	0.02283	0.02297	0.02274
η		0.5097	0.4993	0.4999	0.4987	0.5079	0.5082	0.5076
		0.04183	0.03830	0.03836	0.03826	0.03998	0.04005	0.03991
	60	0.002857	0.002326	0.002336	0.002318	0.002590	0.002602	0.002579
θ		1.204	1.254	1.258	1.249	1.202	1.204	1.199
		0.10883	0.09869	0.10107	0.09634	0.09215	0.09224	0.09211
		0.01826	0.01550	0.01625	0.01478	0.01305	0.01310	0.01301
η		0.5097	0.5068	0.5002	0.5000	0.5056	0.5072	0.5065
		0.03463	0.03248	0.03252	0.03244	0.03350	0.03355	0.03346
	90	0.001894	0.001646	0.001652	0.001641	0.001776	0.001782	0.001771
θ		1.204	1.237	1.240	1.234	1.202	1.204	1.201
		0.09201	0.08703	0.08840	0.08571	0.08210	0.08222	0.08200
		0.01326	0.01183	0.01219	0.01149	0.01056	0.01060	0.01054
η		0.5047	0.4998	0.5001	0.4995	0.5063	0.5065	0.5061
		0.02883	0.02767	0.02769	0.02766	0.02834	0.02838	0.02830
	120	0.001316	0.001191	0.001194	0.001188	0.001277	0.001282	0.001273
θ		1.204	1.229	1.231	1.226	1.195	1.197	1.192
		0.08282	0.07882	0.07961	0.07805	0.07522	0.07496	0.07553
		0.010676	0.009778	0.009986	0.009579	0.008727	0.008683	0.008786

Table 2. The average estimates (AEs) (First row), average Biases (ABs), and mean square errors (MSEs) (third row) of the MLEs and Bayes estimators under SELF and LLF. The True values of η and θ are 1 and 2.

			Lindley			**MCMC**		
	n	**MLE**	**SELF**	**LLF**		**SELF**	**LLF**	
				h = -0.5	h = 0.5		h = -0.5	h = 0.5
η		1.049	1.035	1.041	1.030	1.037	1.040	1.035
		0.1251	0.1159	0.1178	0.1142	0.1169	0.1178	0.1160
	30	0.02666	0.02245	0.02333	0.02167	0.02286	0.02329	0.02246

		2.056	2.021	2.043	1.999	2.023	2.033	2.013
θ		0.2459	0.1797	0.1857	0.1771	0.1933	0.1955	0.1914
		0.10564	0.05210	0.05586	0.05047	0.06203	0.06395	0.06035
		1.028	1.023	1.026	1.020	1.023	1.025	1.022
η		0.08426	0.08140	0.08212	0.08073	0.08153	0.08188	0.08120
	60	0.01191	0.01102	0.01126	0.01080	0.01108	0.01119	0.01097
		2.041	2.028	2.040	2.017	2.026	2.031	2.021
θ		0.1760	0.1534	0.1563	0.1513	0.1556	0.1567	0.1546
		0.05281	0.03915	0.04092	0.03783	0.04040	0.04116	0.03971
		1.013	1.010	1.012	1.009	1.011	1.011	1.010
η		0.06150	0.06025	0.06057	0.05998	0.06040	0.06054	0.06026
	90	0.006234	0.005957	0.006034	0.005886	0.005983	0.006021	0.005948
		2.025	2.019	2.027	2.011	2.016	2.020	2.013
θ		0.1444	0.1325	0.1341	0.1312	0.1331	0.1337	0.1325
		0.03370	0.02813	0.02895	0.02749	0.02830	0.02870	0.02805
		1.011	1.009	1.010	1.008	1.009	1.010	1.008
η		0.05611	0.05534	0.05554	0.05516	0.05539	0.05549	0.05529
	120	0.005272	0.005105	0.005154	0.005059	0.005124	0.005147	0.005101
		2.014	2.010	2.016	2.005	2.006	2.009	2.003
θ		0.1254	0.1178	0.1186	0.1171	0.1180	0.1183	0.1177
		0.02569	0.02254	0.0229	0.02220	00.02256	0.02273	0.02241

Table 3. Average lengths (ALs) and coverage probabilities (CPs) of the 95% approximate confidence interval (ACI) and highest posterior density (HPD) credible interval. The true values of η and θ are 0.5 and 1.2.

Parameters	**n**	**ACI**	**CP**	**HPD**	**CP**
η	30	0.2861	0.9500	0.2740	0.9640
θ		0.7847	0.9400	0.7208	0.9910
η	60	0.1967	0.9420	0.1936	0.9550
θ		0.5531	0.9590	0.5491	0.9840
η	90	0.1593	0.9410	0.1582	0.9420
θ		0.4513	0.9520	0.4611	0.9760
η	120	0.1374	0.9470	0.1553	0.9740
θ		0.3907	0.9400	0.5787	0.9970

Table 4. Average lengths (ALs) and coverage probabilities (CPs) of the 95% approximate confidence interval (ACI) and highest posterior density (HPD) credible interval. The true values of η and θ are 1 and 2.

Parameters	**n**	**ACI**	**Als**	**HPD**	**Als**
η	30	0.5736	0.9520	0.5528	0.9630
θ		1.2150	0.9510	1.0790	0.9790
η	60	0.3962	0.9490	0.3870	0.9500
θ		0.8479	0.9470	0.7966	0.9640

η θ	90	0.3206 0.6825	0.9420 0.9570	0.3150 0.6551	0.9480 0.9660
η θ	120	0.2751 0.5901	0.9410 0.9310	0.2711 0.5723	0.9410 0.9460

Table 5. Point prediction and 95% confidence interval with s = 10.

n	r	Point Prediction	Interval Prediction
30	4	0.959	(0.380, 1.91)
	7	2.172	(0.80, 5.21)
	9	5.734	(1.430, 19.61)
60	4	0.8173	(0.39, 1.57)
	7	1.694	(0.721, 3.75)
	9	3.914	(1.18, 11.97)
90	4	0.813	(0.367, 1.483)
	7	1.714	(0.722, 3.80)
	9	4.036	(1.190, 6.66)
120	4	0.8265	(0.391, 1.57)
	7	1.625	(0.710, 3.15)
	9	3.949	(1.20, 5.90)

6. Study on real data sets

6.1 Data set I (taxes revenue data)

The data set I depicts Egypt's real monthly tax receipts between January 2006 and November 2010 (in 1000 million Egyptian pounds). The information was taken from Nassar and Nada (2011) [14]. IPMD was found to be a better fit to this data set than some other models by Al-Kzzaz and EL-Monsef (2021) [4] after studying this data set. The MLEs, Bayes estimates and associated confidence intervals of the unknown model parameters η and θ are presented in the following Tables 6 and 7. In order to compute Bayes estimates all the hyper-parameters are assumed as 0.001 because no prior information is available. To derive the Bayes estimates an MCMC samples of size 5000 are generated. Further, the prediction results for future observable is also obtained under similar situations.

Table 6. Estimated values of η and θ based on the real data set I.

		Lindley			MCMC		
	MLE	SELF	LLF		SELF	LLF	
			h = -2	h = 2		h = -2	h = 2
η	0.891	0.895	0.896	0.887	0.892	0.892	0.8867
θ	83.7	82.03	86.22	81.15	84.41	130	53.42

Table 7. The 95% confidence interval and HPD credible interval of the parameters θ and λ based on the real data set I.

Parameters	**ACI**	**HPD**
θ	(0.72, 1.06)	(0.842, 0.939)
λ	(21.6, 145.8)	(61.8, 109.7)

Table 8. Point prediction and 95% confidence interval for s = 10 based on the real data set I.

r	**Point Prediction**	**Interval Prediction**
4	9.34	(6.54, 13.4)
7	13.8	(9.0, 21.9)
9	19.4	(8.95, 32.7)

The following observation can point point form the above tables:

- The MLE and Bayes estimates are quite near to each other for the shape parameters η. However, the HPD credible intervals is narrower than the ACI.
- For the scale parameters θ, the MLE is differ form the Bayes estimates using MCMC approach. However, the Bayes estimates using Lindley method provides similar results. The HPD credible intervals is more faithful than the ACI as expected.
- The length of prediction intervals becomes wider as the size of r increases.

6.2 Data set II (health data set)

This data set describes the lifetime of relief (in minutes) of 20 patients receiving an analgesic, presented by Gross and Clark (1975) [12].

Table 9. The goodness-of-fit measures for the data set II.

Model	**-2 log(l)**	**AIC**	**BIC**	**HQIC**	**AICC**
IPMD	**30.837**	**34.837**	**36.828**	**35.226**	**35.543**
MD	40.355	42.355	43.351	42.550	42.577
IMD	36.466	38.466	39.462	38.661	38.688
ED	65.674	67.674	68.670	67.868	67.896
WD	41.173	45.173	47.164	45.562	45.879
IGD	32.783	36.783	38.774	37.172	37.489

In order to examine the data set II, the MLEs of unknown model parameters of IPMD are obtained together with different goodness-of-fit criteria such as Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Akaike information criterion with correction (AICC). In addition, empirical and theoretical density, Q-Q plot, empirical and theoretical CDFs and P-P plot are also displayed in the Figure 3. For comparison purpose, some other lifetime models have been considered, including Maxwell distribution (MD), inverse Maxwell distribution (IMD), exponential distribution (ED), Weibull distribution (WD) and inverse Gompertz distribution (IGD). Their probability distribution function are given in the equations (6.1), (6.2), (6.3), (6.4) and (6.5) respectively. For goodness-of-fit test statistics the Kolmogrov-Smirnov (K-S) distance is computed for each model together with p-values. From all of these, it has been observed that the IPMD fits more accurately than the other model.

Table 10. The goodness-of-fit test statistics for the data set II.

Model	η	θ	**K-S**	**p-value**
IPMD	1.5785	6.6342	0.0929	0.9952
MD	-	0.3675	0.1720	0.5947
IMD	-	4.1411	0.1921	0.4516
ED	-	0.5263	0.43951	0.00089
WD	2.12998	2.78703	0.18497	0.5006
IGD	0.1103	6.1456	1.000	$< 2.2e^{-16}$

$$f_{MD}(x|\theta) = \frac{4}{\pi}\theta^{\frac{3}{2}}x^2exp(-\theta x^2); x > 0, \theta > 0, \tag{54}$$

$$f_{IMD}(x|\theta) = \frac{4}{\pi}\theta^{\frac{3}{2}}\ x^{-4}exp(-\theta x^{-2}); x > 0, \theta > 0, \tag{55}$$

$$f_{ED}(x|\theta) = \theta\ exp(-\theta x), x > 0, \theta > 0, \tag{56}$$

$$f_{WD}(x|\eta,\theta) = \frac{\eta}{\theta}\left(\frac{x}{\lambda}\right)^{\eta-1} e^{(-\frac{x}{\theta})^{\eta}}; x > 0, \eta, \theta > 0 \tag{57}$$

and

$$f_{IGD}(x|\eta,\theta) = \frac{\eta}{x^2}exp\left(-\frac{\eta}{\theta}\left(exp\left(\frac{\eta}{x}\right) - 1\right) + \frac{\eta}{x}\right); x > 0, \eta, \theta > 0. \tag{58}$$

The maximum likelihood estimation (MLEs) and Bayes estimates of the parameters θ and λ are presented in the following table.

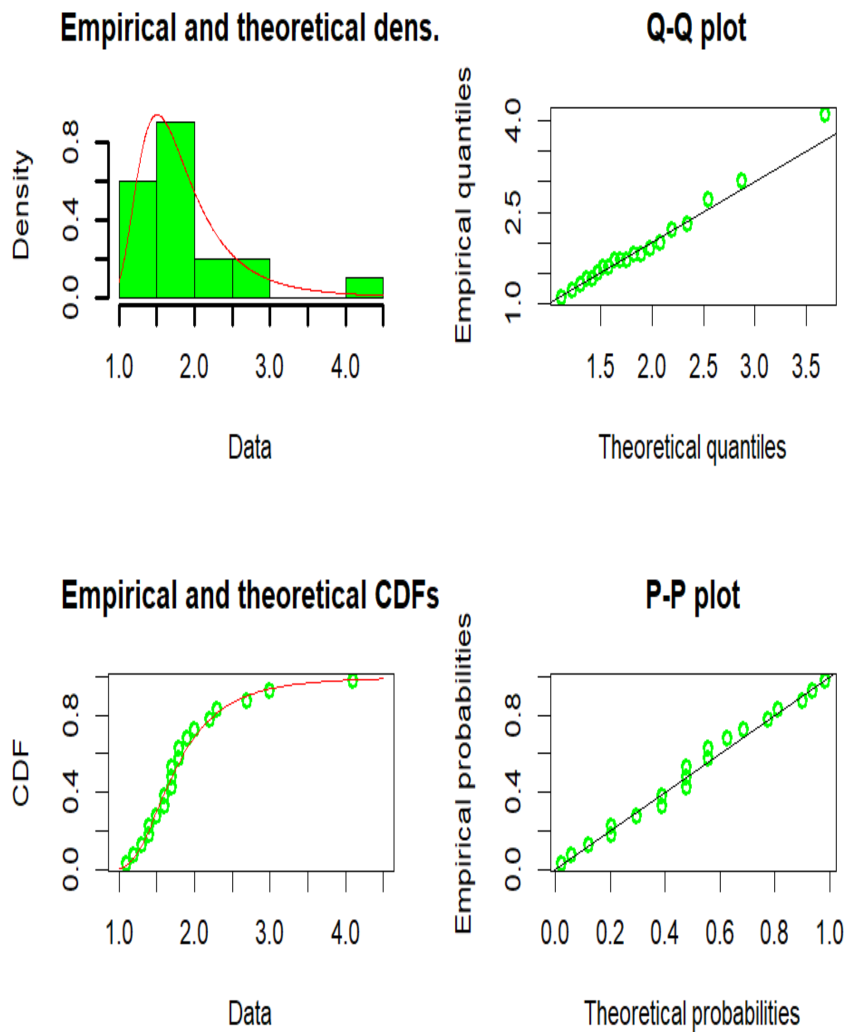

Figure 3: The estimated density and distribution plots of IPMD for the data set II.

Table 11. Estimated values of η and θ based on the real data set II.

		Lindley			**MCMC**		
	MLE	**SELF**	**LLF**		**SELF**	**LLF**	
			h = -2	**h = 2**		**h = -2**	**h= 2**
η	1.579	1.580	1.611	1.464	1.560	1.594	1.528
θ	6.634	6.552	7.298	5.929	6.753	10.467	5.214

Table 12. The 95% confidence interval and HPD credible interval of the parameters θ and λ based on the real data set II.

Parameters	**ACI**	**HPD**
θ	(1.049, 2.109)	(1.205, 1.889)
λ	(3.070, 10.199)	(4.060, 9.819)

Table 13. Point prediction and 95% confidence interval with s = 10 based on the real data set II.

r	**Point Prediction**	**Interval Prediction**
4	1.579	(1.26, 1.93)
7	1.98	(1.53, 2.56)
9	2.51	(1.77, 3.68)

The following interpretations may be read out from tables 11-13

- The MLE and Bayes estimates are close to each other for the shape parameters η. However, the HPD credible intervals is narrower than the ACI.
- For the scale parameters θ, the MLE is slightly differ form the Bayes estimates using MCMC approach. However, the Bayes estimates using Lindley method provides similar results. The HPD credible intervals is more faithful than the ACI as expected.
- As the size of r grows, the length of the prediction intervals expands.

7. Conclusions

In this article, statistical inference for a IPMD based on the complete sample has been investi- gated. Point and intervals estimates are proposed based on frequentist and Bayesian approaches. Since the MLEs of the unknown model parameters cannot be deduced explicitly, this problem has been solved using Newton's iterative technique. The existence and uniqueness of unknown parameters are also obtained. Similarly, due to complicated form of Bayes estimates, two approxi- mation techniques namely; Lindley and MCMC algorithm are implemented. To judge the quality of various estimation approaches, a Monte Carlo simulation experiment has been conducted. It has been explored that the Bayesian approach provides superior results to the classical approach. Moreover, in case of intervals estimations, HPD credible intervals shows the better results than the ACIs in terms of their average length. Two real-life data set have been analyzed to show the practical utility of proposed estimation techniques. It is anticipated that economic profession- als and healthcare data analysts will benefit from the findings and approaches presented in this study.

7.1 Conflicts of interest

There are no conflicts of interest regarding the publication of this paper.

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