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In this paper we have obtained explicit expressions for single and product moments of upper record values from Rayleigh Lomax distribution. Also, recursive relationships between moments of upper records are developed. The characterization theorems based on recurrence relations and conditional expectation are also presented.

Keywords: Rayleigh Lomax distribution, upper record values, single moments, product moments, recurrence relations and characterization.

1. Introduction

Chandler (1952) introduced the record values and documented many of the basic properties of records. An observation is called a record if its value is greater than (or less than) all the previous observations. Record values can be seen in many real-life situations as well as in many statistical applications. For example, it may be helpful as a model for successively largest insurance claims in non-life insurance, for highest water-levels or highest temperatures etc. In reliability theory record values play a very important role. Records are very important when observations are difficult to obtain or when observations are being destroyed when subjected to an experimental test. The prediction of a future record value is also an interesting problem with many real-life applications.

In some situations, record values themselves are viewed as 'outlier' and hence second or third largest values are of special interest, then the model of k^{th} record values is necessary. For more detail survey on this topic, one may refer to the books by Arnold, Balakrishnan, and Nagaraja (1998), Ahsanullah (1995), Kamps (1995) and Ahsanullah and Nevzorov (2015).

Let $\{X_n, n \geq 1\}$ be a sequence of identically and independently distributed (iid) random variables with distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$. For a fixed positive integer k, define the sequences $\{U_n^{(k)}, n \geq 1\}$ of k^{th} upper record times for the sequence $\{X_n, n \geq 1\}$ as follows:

$$U_1^{(k)} = 1$$

$$U_{n+1}^{(k)} = \min\{j > U_n^{(k)} : X_{j:j+k-1} > X_{U_n^{(k)}:U_n^{(k)}+k-1}\}.$$

The sequence $\{R_n^{(k)}, n \geq 1\}$, where $R_n^{(k)} = X_{U_n^{(k)}}$ is called the sequence of k^{th} upper record values of $\{X_n, n \geq 1\}$. Note that for $k = 1$, we have $R_n^{(1)} = X_{U_n}, n \geq 1$, which are the record values of $\{X_n, n \geq 1\}$ (Ahsanullah, 1995). Moreover, we see that $R_1^{(k)} = \min(X_1, X_2, ..., X_n) = X_{1:k}$.

The pdf of $R_n^{(k)}$ is given as

$$f_{R_n^{(k)}}(x) = \frac{k^n}{\Gamma(n)}[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1} f(x), n \geq 1, \tag{1}$$

and the joint pdf of $R_m^{(k)}$ and $R_n^{(k)}$ are given by

$$f_{R_m^{(k)}, R_n^{(k)}}(x, y) = \frac{k^n}{\Gamma(m)\Gamma(n-m)}[-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)}$$

$$\times[\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1} f(y), x < y, 1 \leq m < n, n \geq 2, \tag{2}$$

where

$$\bar{F}(x) = 1 - F(x).$$

The Rayleigh distribution was introduced by Rayleigh (1880) and Lomax distribution was given by Lomax (1954). These are two important distributions in statistical science and widely used in the field of medicine, engineering, business, and actuarial sciences. Fatima, Jan, and Ahmad (2018) introduced a new probability model called Rayleigh Lomax distribution as a mixture of the above two models and studied its distribution properties. This Rayleigh Lomax distribution is more flexible as compared to its base distributions.

A random variable X is said to have Rayleigh Lomax distribution (Fatima *et al*., 2018) if its *pdf* is of the form

$$f(x) = \frac{\beta\lambda}{\theta}\left(\frac{\theta}{\theta + x}\right)^{-2\lambda+1} e^{-\frac{\beta}{2}\left(\frac{\theta}{\theta+x}\right)^{-2\lambda}} ; x \geq -\theta \text{ and } \theta, \lambda, \beta > 0 \tag{3}$$

and the corresponding *df* is

$$F(x) = 1 - e^{-\frac{\beta}{2}\left(\frac{\theta}{\theta+x}\right)^{-2\lambda}} \; ; \; x \geq -\theta \text{ and } \theta, \lambda, \beta > 0 \tag{4}$$

Without loss of generality, throughout the discussion we shall consider $\lambda = 0.5$.

Therefore (3) and (4) becomes

$$f(x) = \frac{\beta}{2\theta} e^{-\frac{\beta}{2\theta}(\theta+x)} \; ; \; x \geq -\theta \text{ and } \theta, \beta > 0 \tag{5}$$

and

$$\bar{F}(x) = e^{-\frac{\beta}{2\theta}(\theta+x)} \; ; \; x \geq -\theta \text{ and } \theta, \beta > 0 \,. \tag{6}$$

Also,

$$-\ln \bar{F}(x) = \frac{\beta}{2\theta}(\theta + x) \,. \tag{7}$$

It can be seen that

$$\bar{F}(x) = \frac{2\theta}{\beta} f(x) \tag{8}$$

There are numerous works on moments, the relation between moments and characterization using record values by several authors in literature. For detailed survey one may see Nagaraja (1977, 1978, 1988), Grudzien and Szynal (1983), Balakrishnan, Chan, and Ahsanullah (1993), Balakrishnan and Ahsanullah (1994a,b; 1995), Franco and Ruiz (1996, 1997), Pawlas and Szynal (1998, 1999, 2000), Beniek and Szynal (2002), Athar, Yaqub, and Islam (2003), Sultan (2007), Khan, Kulshrestha, and Khan (2015), Khan and Khan (2016) and Khan, Khan, and Khan (2017), and references therein.

In this paper, we have obtained explicit expressions for single and product moments of upper record values from Rayleigh Lomax distribution. Also, recursive relationships between moments of upper records are developed. The characterization theorems based on recurrence relations and conditional expectation are also presented.

2. Single Moments

Theorem 1. For distribution given in (6). Fix a positive integer $k \geq 1$, for $n \geq 1$ and $j = 0,1,...$

$$E(R_n^{(k)})^j = \frac{\theta^j}{(n-1)!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{2}{\beta}\right)^{j-l} \frac{\Gamma(j-l+n)}{k^{j-l}} \tag{9}$$

Proof. In view of (1), we have

$$E(R_n^{(k)})^j = \frac{k^n}{\Gamma(n)} \int_{-\theta}^{\infty} x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx \tag{10}$$

Now using (5), (6) and (7) in (10), we get

$$= \frac{k^n}{\Gamma(n)} \int_{-\theta}^{\infty} x^j \left[(\beta/2)[(\theta+x)/\theta]\right]^{n-1} \left[e^{-(\beta/2)[(\theta+x)/\theta]}\right]^{k-1}$$

$$\times \frac{\beta}{2\theta} e^{-(\beta/2)[(\theta+x)/\theta]} dx \tag{11}$$

After substituting $\frac{\beta}{2}\left(\frac{\theta+x}{\theta}\right) = t$ in (11), we have

$$E(R_n^{(k)})^j = \frac{k^n \theta^j}{\Gamma(n)} \int_0^{\infty} \left(\frac{2t}{\beta} - 1\right)^j t^{n-1} e^{-tk} dt$$

$$= \frac{k^n \theta^j}{\Gamma(n)} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{2}{\beta}\right)^{j-l} \int_0^{\infty} t^{(j-l+n)-1} e^{-tk} dt$$

$$= \frac{\theta^j}{(n-1)!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{2}{\beta}\right)^{j-l} \frac{\Gamma(j-l+n)}{k^{j-l}}.$$

Remark 1. At $k=1$ in (9), we get single moments of upper records from Rayleigh Lomax distribution

$$E(X_{U_n})^j = \frac{\theta^j}{(n-1)!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left(\frac{2}{\beta}\right)^{j-l} \Gamma(j-l+n). \tag{12}$$

Theorem 2. For distribution given in (6). Fix a positive integer $k \ge 1$, for $n \ge 1$ and $j = 0,1,...$

$$E(R_n^{(k)})^j - E(R_{n-1}^{(k)})^j = \frac{2\theta j}{\beta k} E(R_n^{(k)})^{j-1}. \tag{13}$$

Proof. From Khan *et al.* (2017), we have

$$E(R_n^{(k)})^j - E(R_{n-1}^{(k)})^j = \frac{jk^{n-1}}{(n-1)!} \int_{\alpha}^{\beta} x^{j-1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx \tag{14}$$

Now using (8) in (14), we get

$$E(R_n^{(k)})^j - E(R_{n-1}^{(k)})^j = \frac{2\theta j}{\beta k} \frac{k^n}{\Gamma(n)} \int_{-\theta}^{\infty} x^{j-1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx.$$

Hence the (13).

Remark 2. At $k=1$ in (13), we get the recurrence relations between moments of upper records from Rayleigh Lomax distribution.

$$E(X_{U_n})^j - E(X_{U_{n-1}})^j = \frac{2\theta j}{\beta k} E(X_{U_n})^{j-1}. \tag{15}$$

The distributional properties of k^{th} upper record values in terms of mean, variance, skewness and kurtosis with arbitrarily chosen parameters of Rayleigh Lomax distribution and different values of *k* are tabulated below in Table 1 and 2.

Table 1 $(k=1)$

n	$\theta=.25,\beta=2$				$\theta=.5,\beta=4$			
	Mean	**Variance**	**Skewness**	**Kurtosis**	**Mean**	**Variance**	**Skewness**	**Kurtosis**
1	0.0000	0.0625	2.0032	8.9856	-0.2500	0.0625	2.0000	9.0000
2	0.2500	0.1250	1.4148	5.9964	0.0000	0.1250	1.4142	6.0000
3	0.5000	0.1875	1.1547	4.9991	0.2500	0.1875	1.1547	5.0000
4	0.7500	0.2500	1.0002	4.4987	0.5000	0.2500	1.0000	4.5000
5	1.0000	0.3125	0.8947	4.1974	0.7500	0.3125	0.8944	4.2000
6	1.2500	0.3750	0.8166	3.9989	1.0000	0.3750	0.8165	4.0000
7	1.5000	0.4375	0.7559	3.8570	1.2500	0.4375	0.7559	3.8571
8	1.7500	0.5000	0.7072	3.7493	1.5000	0.5000	0.7071	3.7500
9	2.0000	0.5625	0.6668	3.6652	1.7500	0.5625	0.6667	3.6667
10	2.2500	0.6250	0.6325	3.5993	2.0000	0.6250	0.6325	3.6000
n	$\theta=.75,\beta=6$				$\theta=1.0,\beta=8$			
	Mean	**Variance**	**Skewness**	**Kurtosis**	**Mean**	**Variance**	**Skewness**	**Kurtosis**
1	-0.5000	0.0625	2.0000	8.9984	-0.7500	0.0625	2.0032	9.0512
2	-0.2500	0.1250	1.4136	5.9964	-0.5000	0.1250	1.4142	6.0032
3	0.0000	0.1875	1.1553	5.0005	-0.2500	0.1875	1.1547	4.9986
4	0.2500	0.2500	0.9998	4.5003	0.0000	0.2500	1.0000	4.4992
5	0.5000	0.3125	0.8944	4.1999	0.2500	0.3125	0.8941	4.2005
6	0.7500	0.3750	0.8164	4.0003	0.5000	0.3750	0.8165	4.0000
7	1.0000	0.4375	0.7561	3.8562	0.7500	0.4375	0.7559	3.8571
8	1.2500	0.5000	0.7070	3.7505	1.0000	0.5000	0.7071	3.7500
9	1.5000	0.5625	0.6667	3.6666	1.2500	0.5625	0.6668	3.6657
10	1.7500	0.6250	0.6324	3.6004	1.5000	0.6250	0.6325	3.5999

Table 2 $(k=3)$

n	$\theta=.25,\beta=2$				$\theta=.5,\beta=4$			
	Mean	**Variance**	**Skewness**	**Kurtosis**	**Mean**	**Variance**	**Skewness**	**Kurtosis**
1	-0.1667	0.0069	2.0689	9.7995	-0.4167	0.0070	1.9928	8.7993
2	-0.0833	0.0139	1.4351	6.0120	-0.3333	0.0139	1.4235	6.2057
3	0.0000	0.0208	1.1667	5.0851	-0.2500	0.0208	1.1417	4.8973
4	0.0833	0.0278	1.0128	4.4574	-0.1667	0.0278	1.0006	4.5060
5	0.1667	0.0347	0.8982	4.2056	-0.0833	0.0348	0.8895	4.1590
6	0.2500	0.0417	0.8103	4.0004	0.0000	0.0417	0.8103	3.9680
7	0.3333	0.0486	0.7621	3.8378	0.0833	0.0487	0.7510	3.8599
8	0.4167	0.0556	0.7015	3.7738	0.1667	0.0555	0.7116	3.7536
9	0.5000	0.0625	0.6688	3.6480	0.2500	0.0625	0.6656	3.6752

10	0.5833	0.0695	0.6330	3.5853	0.3333	0.0695	0.6313	3.5810
n	$\theta=.75, \beta=6$				$\theta=1.0, \beta=8$			
	Mean	**Variance**	**Skewness**	**Kurtosis**	**Mean**	**Variance**	**Skewness**	**Kurtosis**
1	-0.6667	0.0069	2.0254	10.4974	-0.9167	0.0069	1.7726	4.8972
2	-0.5833	0.0140	1.4500	6.2468	-0.8333	0.0139	1.4083	6.0335
3	-0.5000	0.0208	1.1334	4.8539	-0.7500	0.0208	1.1251	4.6661
4	-0.4167	0.0278	1.0074	4.5330	-0.6667	0.0277	0.9898	4.4300
5	-0.3333	0.0347	0.8866	4.1226	-0.5833	0.0348	0.8934	4.1245
6	-0.2500	0.0417	0.8220	4.0004	-0.5000	0.0417	0.8162	3.9968
7	-0.1667	0.0486	0.7598	3.8731	-0.4167	0.0486	0.7533	3.8564
8	-0.0833	0.0556	0.7073	3.7657	-0.3333	0.0556	0.7116	3.7617
9	0.0000	0.0625	0.6656	3.6608	-0.2500	0.0625	0.6656	3.6496
10	0.0833	0.0695	0.6317	3.6078	-0.1667	0.0694	0.6313	3.6017

3. Product Moments

Theorem 3. For the distribution given in (6). Fix a positive integer $k \geq 1$ and for $1 \leq m < n$, $n \geq 2$, $i, j = 0, 1, \ldots$

$$E[(R_m^{(k)})^i (R_n^{(k)})^j] = \frac{\theta^{i+j}}{\Gamma(m)\Gamma(n-m)} \sum_{l=0}^{j} \sum_{p=0}^{i+j-l} (-1)^p \binom{j}{l} \binom{i+j-l}{p} \left(\frac{2}{\beta}\right)^{i+j-p} \times \frac{\Gamma(l+n-m)\Gamma(i+j-l-p+m)}{k^{i+j-p}} \quad (16)$$

Proof. In view of (2), we have

$$E[(R_m^{(k)})^i (R_n^{(k)})^j] = \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_{-\theta}^{\infty} \int_{x}^{\infty} x^i y^j [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)}$$

$$\times [-\ln \bar{F}(y) + \ln \bar{F}(x)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dy dx.$$

Now in view of (5), (6) and (7), we have

$$E[(R_m^{(k)})^i (R_n^{(k)})^j] = \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_{-\theta}^{\infty} \int_{x}^{\infty} x^i y^j \left[(\beta/2)[(\theta+x)/\theta]\right]^{m-1} \left(\frac{\beta}{2\theta}\right)$$

$$\times \left[(\beta/2)[(\theta+y)/\theta] - (\beta/2)[(\theta+x)/\theta]\right]^{n-m-1} \left[e^{-(\beta/2)[(\theta+y)/\theta]}\right]^{k-1}$$

$$\times \frac{\beta}{2\theta} e^{-(\beta/2)[(\theta+y)/\theta]} dy dx$$

$$= \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_{-\theta}^{\infty} x^i \left(\frac{\beta}{2\theta}\right) \left[(\beta/2)[(\theta+x)/\theta]\right]^{m-1} I(x) dx, \quad (17)$$

where,

$$I(x) = \int_{x}^{\infty} y^j \left[(\beta/2)[(\theta+y)/\theta] - (\beta/2)[(\theta+x)/\theta]\right]^{n-m-1} \left[e^{-(\beta/2)[(\theta+y)/\theta]}\right]^{k-1}$$

$$\times \frac{\beta}{2\theta} e^{-(\beta/2)[(\theta+y)/\theta]} dy \tag{18}$$

Substituting $(\beta/2)[(\theta+y)/\theta]-(\beta/2)[(\theta+x)/\theta]=t$ in (18), we get

$$I(x)=\int_0^\infty \left(\frac{2\theta t}{\beta}+x\right)^j t^{n-m-1} e^{-k\left[t+(\beta/2)[(\theta+x)/\theta]\right]} dt$$

$$=e^{-k\left[(\beta/2)[(\theta+x)/\theta]\right]} \sum_{l=0}^{j} \binom{j}{l} \left(\frac{2\theta}{\beta}\right)^l x^{j-l} \int_0^\infty t^{l+n-m-1} e^{-kt} dt$$

$$=e^{-k\left[(\beta/2)[(\theta+x)/\theta]\right]} \sum_{l=0}^{j} \binom{j}{l} \left(\frac{2\theta}{\beta}\right)^l x^{j-l} \frac{\Gamma(l+n-m)}{k^{l+n-m}}.$$

Now on substituting value of $I(x)$ in (17), we get

$$E[(R_m^{(k)})^i (R_n^{(k)})^j]=\frac{k^n}{\Gamma(m)\Gamma(n-m)} \sum_{l=0}^{j} \binom{j}{l} \left(\frac{2\theta}{\beta}\right)^l x^{j-l} \frac{\Gamma(l+n-m)}{k^{l+n-m}}$$

$$\times \int_{-\theta}^\infty x^{i+j-l} \left(\frac{\beta}{2\theta}\right) \left[(\beta/2)[(\theta+x)/\theta]\right]^{m-1} e^{-k\left[(\beta/2)[(\theta+x)/\theta]\right]} dx. \tag{19}$$

Let $\frac{\beta}{2}\left(\frac{\theta+x}{\theta}\right)=u$, then *RHS* of (19) becomes

$$=\frac{k^n}{\Gamma(m)\Gamma(n-m)} \sum_{l=0}^{j} \binom{j}{l} \left(\frac{2\theta}{\beta}\right)^l x^{j-l} \frac{\Gamma(l+n-m)}{k^{l+n-m}} \int_0^\infty \left(\frac{2\theta u}{\beta}-\theta\right)^{i+j-l} u^{m-1} e^{-ku} du$$

$$=\frac{\theta^{i+j}}{\Gamma(m)\Gamma(n-m)} \sum_{l=0}^{j} \sum_{p=0}^{i+j-l} (-1)^p \binom{j}{l} \binom{i+j-l}{p} \left(\frac{2}{\beta}\right)^{i+j-p} \int_0^\infty u^{(i+j-l-p+m)-1} e^{-ku} du ,$$

which yields (16).

Remark 3. If we put $j=0$ in (16), we get moment of single record as obtained in (9).

Remark 4. At $k=1$ in (16), we get product moments of upper records from Rayleigh Lomax distribution as

$$E[(X_{U_m})^i (X_{U_n})^j]=\frac{\theta^{i+j}}{\Gamma(m)\Gamma(n-m)} \sum_{l=0}^{j} \sum_{p=0}^{i+j-l} (-1)^p \binom{j}{l} \binom{i+j-l}{p} \left(\frac{2}{\beta}\right)^{i+j-p}$$

$$\times \Gamma(l+n-m)\Gamma(i+j-l-p+m). \tag{20}$$

Theorem 4. For the distribution given in (6). Fix a positive integer $k \ge 1$ and for $1 \le m < n$, $n \ge 2$, $i, j = 0,1,\ldots$

$$E[(R_m^{(k)})^i (R_n^{(k)})^j] - E[(R_m^{(k)})^i (R_{n-1}^{(k)})^j] = \frac{2\theta j}{\beta k} E[(R_m^{(k)})^i (R_n^{(k)})^{j-1}] . \tag{21}$$

Proof. From Khan *et al*. (2017), we have

$$E[(R_m^{(k)})^i (R_n^{(k)})^j] - E[(R_m^{(k)})^i (R_{n-1}^{(k)})^j] = \frac{jk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_\alpha^\beta \int_x^\beta x^i y^{j-1}$$

$$\times [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^k \, dydx . \tag{22}$$

Now on using relation (8) in (22), we get

$$E[(R_m^{(k)})^i (R_n^{(k)})^j] - E[(R_m^{(k)})^i (R_{n-1}^{(k)})^j] = \frac{2j\theta}{k\beta} \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_{-\theta}^\infty \int_x^\infty x^i y^{j-1}$$

$$\times [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dydx .$$

Hence the relation (21).

Remark 5. Theorem 3.2 can be deduced for relation between single moment of k^{th} upper record values by setting $i = 0$. Further, at $k = 1$ in (3.6), we get relation between product moments of upper records from Rayleigh Lomax distribution.

The product moments of k^{th} upper record values with arbitrary chosen parameters of Rayleigh Lomax distribution and different *k* are tabulated below in Table 3, 4, 5 and 6.

Product Moments: $(m < n)$

Table 3. $(\theta = 0.25, \beta = 2, k = 1)$

n / *m*	2	3	4	5	6	7	8	9	10
1	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625
2		0.2500	0.3125	0.3770	0.4375	0.5000	0.5625	0.6250	0.6875
3			0.5625	0.6875	0.8125	0.9375	1.0625	1.1875	1.3125
4				1.0000	1.1875	1.3750	1.5625	1.7500	1.9375
5					1.5625	1.8125	2.0625	2.3125	2.5625
6						2.2500	2.5625	2.8750	3.1875
7							3.0625	3.4375	3.8125
8								4.0000	4.4375
9									5.0625

Table 4. $\theta = 0.5, \beta = 5, k = 1$

n / *m*	2	3	4	5	6	7	8	9	10
1	0.0700	0.0100	-0.0500	-0.1100	-0.1700	-0.2300	-0.2900	-0.3500	-0.4100
2		0.0700	0.0500	0.0300	0.0100	-0.0100	-0.0300	-0.0500	-0.0700
3			0.1500	0.1700	0.1900	0.2100	0.2300	0.2500	0.2700
4				0.3100	0.3700	0.4300	0.4900	0.5500	0.6100
5					0.5500	0.6500	0.7500	0.8500	0.9500
6						0.8700	1.0100	1.1500	1.2900
7							1.2700	1.4500	1.6300
8								1.7500	1.9700
9									2.3100

Table 5. $\theta = 0.25, \beta = 2, k = 2$

n / *m*	2	3	4	5	6	7	8	9	10
1	0.0039	0.0000	-0.0010	-0.0010	-0.0007	-0.0005	-0.0003	-0.0002	-0.0001
2		0.0039	0.0020	0.0010	0.0005	0.0002	0.0001	0.0001	0.0000
3			0.0049	0.0029	0.0017	0.0010	0.0005	0.0003	0.0002
4				0.0049	0.0029	0.0017	0.0010	0.0005	0.0003
5					0.0042	0.0024	0.0014	0.0008	0.0004
6						0.0032	0.0018	0.0010	0.0006
7							0.0023	0.0013	0.0007
8								0.0015	0.0009
9									0.0010

Table 6. $\theta = 0.75, \beta = 6, k = 2$

n / *m*	2	3	4	5	6	7	8	9	10
1	0.0820	0.0312	0.0107	0.0029	0.0002	-0.0005	-0.0005	-0.0004	-0.0003
2		0.0273	0.0098	0.0029	0.0005	-0.0002	-0.0004	-0.0003	-0.0002
3			0.0088	0.0029	0.0007	0.0000	-0.0002	-0.0002	-0.0001
4				0.0029	0.0010	0.0002	0.0000	-0.0001	-0.0001
5					0.0012	0.0005	0.0002	0.0001	0.0000
6						0.0007	0.0004	0.0002	0.0001
7							0.0005	0.0003	0.0002
8								0.0004	0.0002
9									0.0003

4. Characterizations

Theorem 5. Fix a positive integer $k \geq 1$ and let j be a non-negative integer. Then for $n \geq k$, the random variable X holds the relation

$$E(R_n^{(k)})^j - E(R_{n-1}^{(k)})^j = \frac{2\theta j}{\beta k} E(R_n^{(k)})^{j-1} \tag{23}$$

if and only if

$$\bar{F}(x) = e^{-\frac{\beta}{2\theta}(\theta+x)}; \ x \ge -\theta \ \text{ and } \ \theta, \beta > 0\,, \tag{24}$$

with $R_n^{(k)} = X_{U_n^{(k)}}$.

Proof. The necessary part follows immediately from (13). Now suppose the relation (23) is satisfied, then on using Khan, Khan, and Khan (2017), we have

$$\frac{jk^{n-1}}{\Gamma(n)} \int_{-\theta}^{\infty} x^{j-1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k \, dx$$

$$= \frac{2\theta\, j}{\beta k} \frac{k^n}{\Gamma(n)} \int_{-\theta}^{\infty} x^{j-1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx\,,$$

which implies

$$\int_{-\theta}^{\infty} x^{j-1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} \{\bar{F}(x) - (2\theta / \beta) f(x)\} dx = 0 \tag{25}$$

Now applying a generalization of the M*ü*ntz-Sz*á*sz Theorem (see for example Hwang and Lin (1984)) in (25), we obtain

$$\bar{F}(x) - \frac{2\theta}{\beta} f(x) = 0$$

or

$$\frac{f(x)}{\bar{F}(x)} = \frac{\beta}{2\theta}\,. \tag{26}$$

Now integrating both the sides of (26) *w.r.t* x between $(-\theta, y)$, the sufficiency part is proved.

The Theorem 5 can be used to characterize the Rayleigh Lomax distribution in terms of moments of minimal order statistics.

Corollary 1. Under the condition as stated in Theorem 5 with $n = 1$, the following relation holds

$$E(X_{1:k}^{j}) = \frac{2\theta\, j}{\beta k} E(X_{1:k}^{j-1}) - \theta \tag{27}$$

and subsequently

$$E(X) = \frac{2\theta}{\beta} - \theta \tag{28}$$

if and only if

$$\bar{F}(x) = e^{-\frac{\beta}{2\theta}(\theta+x)}; \ x \ge -\theta \ \text{ and } \ \theta, \beta > 0\,, \tag{29}$$

with $R_0^{(k)} = -\theta$.

Theorem 6. Fix a positive integer $k \geq 1$ and for $1 \leq m < n$, $n \geq 2$, $i, j > 0$. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (5) is that

$$E[(R_m^{(k)})^i (R_n^{(k)})^j] - E[(R_m^{(k)})^i (R_{n-1}^{(k)})^j] = \frac{2j\theta}{k\beta} E[(R_m^{(k)})^i (R_n^{(k)})^{j-1}] \tag{30}$$

with $R_n^{(k)} = X_{U_n^{(k)}}$.

Proof. The necessary part follows immediately from (21). Now we shall proof sufficiency part.

In view of Khan, Khan, and Khan (2017), the (30) can be expressed as

$$\frac{jk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_\alpha^\beta \int_x^\beta x^i y^{j-1} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^k \, dydx$$

$$= \frac{2j\theta}{k\beta} \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_{-\theta}^\infty \int_x^\infty x^i y^{j-1}$$

$$\times [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y) dydx,$$

which implies that

$$\int_{-\theta}^\infty \int_x^\infty x^i y^{j-1} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1}$$

$$\times [\bar{F}(y)]^{k-1} \left[\bar{F}(y) - (2\theta / \beta) f(y) \right] dydx = 0. \tag{31}$$

Now applying a generalization of the Müntz-Szász Theorem (see for example Hwang and Lin (1984)) in (31), we get

$$\bar{F}(y) = \frac{2\theta}{\beta} f(y)$$

Hence the sufficiency part.

Theorem 7. Let $R_i = X_{U(i)}, i = 1, 2, \ldots$ be the i^{th} upper record value from a continuous population with df $F(x)$ and pdf $f(x)$. Then for $1 \leq l < j < n$

$$E\left[R_j \mid R_l = x, R_n = y \right] = \frac{(n-j)x + (j-l)y}{(n-l)}, \; l = m, m+1 \tag{32}$$

and consequently

$$E\left[R_{m+1} \mid R_m = x, R_{m+2} = y\right] = \frac{x+y}{2} \tag{33}$$

if and only if

$$\bar{F}(x) = e^{-\frac{\beta}{2\theta}(\theta+x)};\ x \geq -\theta \text{ and } \theta, \beta > 0\,, \tag{34}$$

Proof. Khan, Akhter, Wahid, and Kumar (2016) shown that for $1 \leq l < j < s$ and $l = r, r+1$

$$E\left[\xi(R_j) \mid R_l = x, R_s = y\right] = \frac{(s-j)\xi(x) + (j-l)\xi(y)}{(s-l)} \tag{35}$$

if and only if

$$\bar{F}(x) = e^{-[a\xi(x)+b]},\ x \in (\alpha, \beta) \tag{36}$$

On comparing (36) with (34), we have

$$a = \frac{\beta}{2\theta}, \xi(x) = x, b = \frac{\beta}{2} \text{ and } x \in (-\theta, \infty)\,.$$

Thus, the theorem can be proved in view of (35).

Theorem 8. Under the condition as stated in Theorem 7.

$$E\left[R_n \mid R_m = x\right] = x - \theta + (n-m)\frac{2\theta}{\beta} \tag{37}$$

and

$$E\left[R_{m+1} \mid R_m = x\right] = x - \theta + \frac{2\theta}{\beta} \tag{38}$$

if and only if

$$\bar{F}(x) = e^{-\frac{\beta}{2\theta}(\theta+x)};\ x \geq -\theta \text{ and } \theta, \beta > 0\,, \tag{39}$$

Proof: Khan, Faizan, and Haque (2010) shown that

$$E\left[h(R_n) \mid R_m = x\right] = (n-m)c + h(x) \tag{40}$$

if and only if

$$\bar{F}(x) = e^{-\frac{1}{c}h(x)};\ x \in (\alpha, \beta)\,. \tag{41}$$

Therefore, theorem can be proved in view of (40) and (41).

5. Conclusion

The above studies demonstrate the explicit expression for the moments of generalized upper record values (k^{th} upper records) from Rayleigh Lomax distribution. The recurrence relations between moments are also established. These relations can be used to reduce the amount of direct computation and moments of any order can be calculated easily. The characterization results can be used to know if the designed models fit the requirements probability distribution under study or not. Due to complexity in the expression of single as well as product moments, we fixed the value of $\lambda = 0.5$. So problem is still open for the readers for varying λ.

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