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Marshall-Olkin Alpha Power Inverse Lindley Distribution and its Applications

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A new model called as Marshal-Olkin Alpha Power Inverse Lindley (MOAP-IL) distribution has been proposed. The proposed model is more tractable since the density function has complex shapes. Properties and Estimation procedure has been discussed. The usefulness of the model has been verified through real life and simulated data.

Keywords: Marshal-Olkin Alpha Power Inverse Lindley (MOAP-IL), shapes, Stress strength reliability (SSR).

1. Introduction

Developing new class of continuous distributions from classical ones always remain a central focus of researchers. The extended class offers a more flexibility in complex data analysis. In many realistic fields like economics, finance and insurance, survival analysis, these extended distributions turn out to be very useful for the new emerging domains According to Lee et al. (2003), the classical distribution were extended by the method of differential equation, quantile function or transformation techniques prior to 1980 and post 1980, the new class of distributions were mainly obtained as a result of combining classical or adding parameter to classical distributions. These methods were named as method of combination.

Marshal-Olkin (MO) class was pioneered by Marshal-Olkin (1997) by using the method of combination having cumulative distribution function(cdf) and probability density function(pdf) given by (1) and (2) respectively:

$$G(z) = \frac{F(z)}{\gamma + (1-\gamma)G(z)} \tag{1}$$

$$g(z) = \frac{\gamma\, f(z)}{\left[\gamma + (1-\gamma)F(z)\right]^2} \qquad ; z > 0, \gamma > 0 \tag{2}$$

Where F(z) and f(z) are the base model cdf and pdf.

Mahadev and Kundu (2017) recently offered a new family known as Alpha Power Transformation (APT) family. This family was also obtained by including extra shape parameter which incorporates skewness to the base model. The APT family have the following cdf and pdf:

$$G(z)=\begin{cases}\dfrac{\alpha^{F(z)}-1}{\alpha-1} & if\ \alpha>0,\alpha\neq 1\\ F(z) & if\ \alpha=1\end{cases} \tag{3}$$

$$g(z)=\begin{cases}\dfrac{\log(\alpha)\, f(z)\, \alpha^{F(z)}}{\alpha-1} & if\ \alpha>0,\alpha\neq 1\\ f(z) & if\ \alpha=1\end{cases} \tag{4}$$

Nassar et al. (2019) proposed new extension of MO family by incorporating (3) as baseline cdf in (1) and obtained a new family known as Marshal-Olkin Alpha Power (MOAP) with (5) as the cdf:

$$G_{MOAP}(z)=\begin{cases}\dfrac{\alpha^{F(z)}-1}{(\alpha-1)[\gamma+(1-\gamma)(\alpha-1)^{-1}(\alpha^{F(z)}-1)]} & if\ \alpha>0,\alpha\neq 1\\ F(z) & if\ \alpha=1\end{cases} \tag{5}$$

The corresponding pdf is:

$$g_{MOAp}(z)=\begin{cases}\dfrac{\gamma\log(\alpha)\alpha^{F(z)} f(z)}{(\alpha-1)[\gamma+(1-\gamma)(\alpha-1)^{-1}(\alpha^{F(z)}-1)]^2} & if\ \alpha>0,\alpha\neq 1\\ f(z) & if\ \alpha=1\end{cases} \tag{6}$$

This class of distribution possess increasing, decreasing, non-montone hazard rate and can be applied to skewed data thereby finding its applications in various fields such as finance, medical science, insurance etc. Thus this class is very well accommodating and can be effectually employed for complex data analysis.

Lindley (1958) proposed Lindley distribution (LD) which has gained focus from researchers. This distribution turns out to a good alternative for exponential distribution in many other cases. LD has been progressively extended and practiced in various realistic situations. Sharma et al. (2015) proposed Inverse Lindley Distribution (ILD) as a stress- strength reliability model and observed its basic properties. The cdf and pdf of ILD is given by:

$$F(z)=\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}} \quad ;\ z>0,\delta>0 \tag{7}$$

$$f(z)=\frac{\delta^2}{(1+\delta)}\left(\frac{1+z}{z^3}\right)e^{\frac{-\delta}{z}} \quad ; z>0; \delta>0 \tag{8}$$

Various authors including Sharma et al. (2016), Eltehiwy (2018), Dey et al (2018), Jan et al. (2019) have extended ILD using different approaches. In this article, a new distribution is obtained by substituting (7) and (8) in (5) and (6) respectively. The newly obtained distribution is named as Marshal-Olkin Alpha Power Inverse Lindley (MOAP-IL) distribution. The proposed model combine the features of MO family and ILD and is therefore proficient in modeling skewed data with increasing, reverse J, symmetrical, unimodal density function. The proposed model competes well with other models hence increasing its flexibility in modeling real life data sets. Some reliability measures and statistical characteristics of the model have been derived. The parameters have been estimated using MLE. The effectiveness of the model has been illustrated through simulated and two real life data sets. Lastly the conclusion has been stated.

2. The Proposed Model

Definition.

A random variable Z follows MOAP-IL if its cdf is:

$$G(z)=\begin{cases}\dfrac{\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}-1}{(\alpha-1)\left[\gamma+(1-\gamma)(\alpha-1)^{-1}\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}-1\right)\right]} & if\ \alpha>0,\alpha\neq 1 \\ \left(1+\dfrac{\delta}{1+\delta}\dfrac{1}{z}\right)e^{\frac{-\delta}{z}} & if\ \alpha=1\end{cases} \quad \text{`} \tag{9}$$

and the corresponding pdf is :

Where $0<z<\infty$, $\delta,\gamma>0$.

Special Cases:

a. If $\alpha\to 1$, MOAP-IL tends to Marshall Olkin-ILD.
b. If $\alpha\to 1$ and $\gamma=1$, the model reduces to ILD.
c. If $\gamma=1$, the model reduces Alpha power ILD.

2.1 Reliability Measures

The Reverse hazard rate $\lambda(z)$, survival function S(z) and hazard rate h(z) of *Z* are respectively obtained as:

$$\lambda(z)=\begin{cases}\dfrac{\gamma\delta^2\log\alpha(1+z)e^{\frac{-\delta}{z}}\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}}{\left[\gamma+(1-\gamma)(\alpha-1)^{-1}\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}-1\right)\right](1+\delta)z^3\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}-1\right)} & \text{if } \alpha>0,\alpha\neq1\\[2ex] \dfrac{\delta^2(1+z)}{z^2(\delta+z(1+\delta))} & \text{if } \alpha=1\end{cases}$$

$$S(z)=\begin{cases}1-\dfrac{\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}-1}{(\alpha-1)\left[\gamma+(1-\gamma)(\alpha-1)^{-1}\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}-1\right)\right]} & \text{if } \alpha>0,\alpha\neq1\\[2ex] 1-\left(1+\dfrac{\delta}{1+\delta}\dfrac{1}{z}\right)e^{\frac{-\delta}{z}} & \text{if } \alpha=1\end{cases}$$

$$h(z)=\begin{cases}\dfrac{\gamma\delta^2\log\alpha(1+z)e^{\frac{-\delta}{z}}\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}}{(1+\delta)\,z^3\left[\gamma+(1-\gamma)(\alpha-1)^{-1}\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}-1\right)\right]}\\ \times\dfrac{1}{\left\{(\alpha-1)\left[\gamma+(1-\gamma)(\alpha-1)^{-1}\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}-1\right)\right]-\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}-1\right)\right\}} & \text{if } \alpha>0,\alpha\neq1\\[2ex] \dfrac{\delta^2(1+z)}{z^2(\delta+z(1+\delta)(e^{\frac{-\delta}{z}}-1)} & \text{if } \alpha=1\end{cases}$$

The behavior of the density function, h(z), S(z) and are displayed in Figure 1. Figure 1 (a) displays the different shapes of density function.

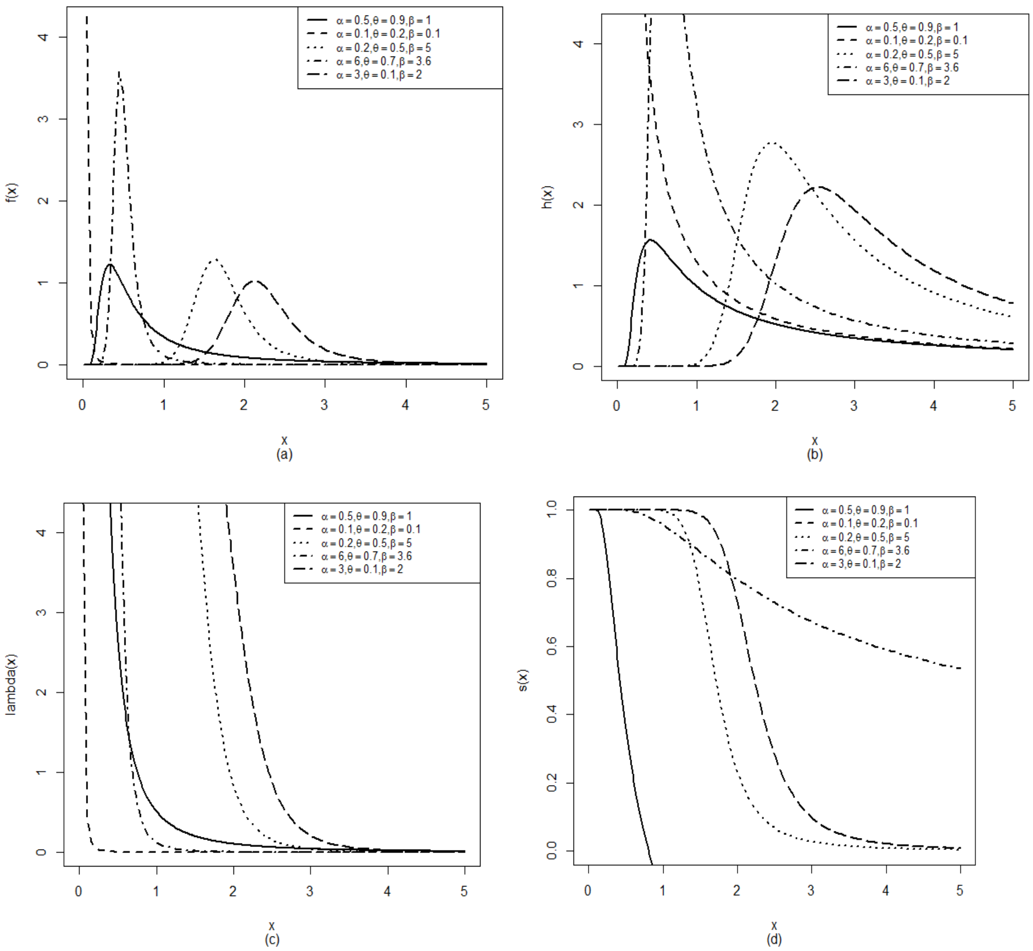

Figure 1. (a) Pdf plot (b) hazard rate (c) reverse hazard rate (d) survival plot of MOAP-IL for different parameter combinations.

2.2 Structural Properties.

To explore the statistical features of the proposed model, the mixture representation of MOAP family has been used. The pdf of Z represented as a linear combination of exponentiated ILD (EILD) is given as:

$$g(z) = \sum_{s=0}^{\infty} v_s h_{s+1}(z) \tag{11}$$

Where

$$v_s = \begin{cases} \sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^j \binom{k}{j}(k+1)(1-\gamma)^k \dfrac{\log(\alpha)^{s+1}(j+1)^s}{(\gamma(\alpha-1))^{k+1}(s+1)!} & ;\ 0<\gamma<1 \\ \sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^j \binom{k}{j}(k+1)(1-\gamma^{-1})^k \dfrac{\log(\alpha)^{s+1}(k+1-j)^s}{\gamma(\alpha-1)^{k+1}(s+1)!} & ;\ \gamma>1 \end{cases} \tag{12}$$

And $h_{s+1}(z)=(s+1)\left[\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}\right]^{s}\frac{\delta^2}{(1+\delta)}\left(\frac{1+z}{z^3}\right)e^{\frac{-\delta}{z}}$ is the pdf of EILD with‘s’ as the parameter of exponentiation. The cdf of Z can be signified as:

$$G(z)=\sum_{s=0}^{\infty}\nu_s H_{s+1}(z) \tag{13}$$

Where $H_{s+1}(z)$ is the cdf of EILD with power parameter (s+1).

2.3 Moments

The key features of a probability distribution including skewness, peakedness can be investigated using moments. The μ_r' of Z has been calculated using (11) as:

$$\mu_r'=\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\nu_s(s+1)\binom{s}{m}\frac{\delta^{m+2}}{(1+\delta)^{m+1}}\frac{\Gamma(m-r+1)}{(\delta(1+s))^{m-r+1}}\left(\frac{m-r+1}{\delta(1+s)}+1\right)$$

$$=\sum_{s=0}^{\infty}\nu_s E(X_r')$$

Where Z ~ EILD(s, δ)

Also the MGF of Z is:

$$M_Z(t)=\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\nu_s(s+1)\binom{s}{m}\frac{\delta^{m+2}}{(1+\delta)^{m+1}}\left(\frac{t^n}{n!}\right)\frac{\Gamma(m-n+1)}{(\delta(1+s))^{m-n+1}}\left(\frac{m-n+1}{\delta(1+s)}+1\right)$$

2.4 Incomplete Moments

Incomplete moment (IC) possess applicability in many applied areas and are used in estimating Bonferroni and Lorenz curves. The n^{th} IC of Z is:

$$m_n'(t)=\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\nu_s(s+1)\binom{s}{m}\frac{\delta^{m+2}}{(1+\delta)^{m+1}}\left[(\delta(1+s))^{n-m-2}\Gamma(m-n+2,t)+(\delta(1+s))^{n-m-1}\Gamma(m-n+1,t)\right]$$

2.5 Order Statistics

The pdf of *Z(r:n)* is:

$$g_r(z)=\frac{n!}{(r-1)(n-r)!}\sum_{s=0}^{\infty}\sum_{p=0}^{n-r}\sum_{q=0}^{p}(-1)^p\binom{n-r}{p}\binom{s}{q}(s+1)\frac{\delta^{q+2}}{(1+\delta)^{q+1}}\,\nu_s\,(G(z))^{p+r-1}\left(\frac{1+z}{z^{q+3}}\right)e^{\frac{-\delta(1+s)}{z}} \tag{14}$$

Where $G(z)$ for $\alpha>1,\alpha\neq 1$ and $\alpha=1$ is given by (9).

Substituting the value of r=1 and r=n in (14), the pdf of minimum and maximum order statistics can be obtained respectively. Furthermore, the cdf of *Z(r:n)* is:

$$G_r(z)=\frac{n!}{(r-1)(n-r)!}\sum_{p=0}^{n-r}\frac{(-1)^r}{(r+p)}\binom{n-r}{p}(G(z))^{r+p} \tag{15}$$

2.6 Stress-Strength Reliability

Let Z (strength) ~MOAP-IL $(\alpha_{1,}\gamma_{1,}\delta)$ and Y (stress) ~MOAP-IL $(\alpha_{2,}\gamma_{2,}\delta)$ are i.i.d random variables. Then, the stress-strength reliability is:

$$R=\sum_{s',s''=0}^{\infty}\sum_{m=0}^{\infty}\nu_{s'}\nu_{s''}\binom{s'+s''}{m}(s'+1)\frac{\delta^{m+2}}{(1+\delta)^{m+1}}\frac{\Gamma(m+1)}{(\delta(s'+s''+1))^{m+1}}\left[\frac{m+1}{\delta(s'+s''+1)}+1\right]$$

Where $\nu_{s'},\nu_{s''}$ are given by equation (12) for Z and Y respectively.

2.7 Renyi Entropy

If Z has the pdf given by (10), Renyi entropy denoted by $I_{\varsigma}(z)$ is:

$$I_{\varsigma}(z)=\frac{1}{1-\varsigma}\log\sum_{s=0}^{\infty}\left(\nu_s(s+1)\frac{\delta^2}{1+\delta}\right)^{\varsigma}\int_0^{\infty}\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)^{s\varsigma}\left(\frac{1+z}{z^3}\right)^{\varsigma}e^{\frac{-\delta\varsigma(1+s)}{z}}dz \qquad (16)$$

After simplification, (16) reduces to

$$I_{\varsigma}(z)=\frac{1}{1-\varsigma}\log\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\nu_s(s+1)\right)^{\varsigma}\binom{s\varsigma}{m}\binom{\varsigma}{n}\left(\frac{\delta^{m+2\varsigma}}{(1+\delta)^{m+\varsigma}}\right)\frac{\Gamma(3\varsigma+m-n-1)}{(\delta\varsigma(s+1))^{3\varsigma+m-n-1}}$$

2.8 Stochastic Ordering

The MOAP-IL distribution follows likelihood ratio ordering and from the following result (Shaked and Shanthikumar, 1994)

$$\begin{array}{l} Z\leq_{lr}Y\Rightarrow Z\leq_{hr}Y\Rightarrow Z\leq_{mlr}Y \\ \qquad\quad\Downarrow \\ \qquad\quad Z\leq_{st}Y \end{array} \qquad (17)$$

it follows that MOAP-IL is stochastically ordered. The result is confirmed in the theorem.

Theorem: Suppose X ~MOAP-IL $(\alpha_{1,}\gamma_{1,}\delta_1)$ and Y ~MOAP-IL $(\alpha_{2,}\gamma_{2,}\delta_2)$. If $\gamma_1=\gamma_2=\gamma$ and $\alpha_1<\alpha_2$ (or $\alpha_1=\alpha_2=\alpha$ and $\gamma_1<\gamma_2$) then $Z\leq_{lr}Y\ \forall z$.

Proof: The likelihood ratio i.e

$$\frac{f_Z(z)}{f_Y(z)}=\frac{\delta^2}{(1+\delta)}\left(\frac{1+z}{z^3}\right)e^{\frac{-\delta}{z}}\left[(\log\alpha_1-\log\alpha_2)+\frac{2(1-\gamma_2)(\alpha_2-1)^{-1}\alpha_2^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}\log\alpha_2}{\gamma_2+(1-\gamma_2)(\alpha_2-1)^{-1}\left(\alpha_2^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}-1\right)}-\frac{2(1-\gamma_1)(\alpha_1-1)^{-1}\alpha_1^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}\log\alpha_1}{\gamma_1+(1-\gamma_1)(\alpha_1-1)^{-1}\left(\alpha_1^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}-1\right)}\right]$$

decreases in x for $\gamma_1=\gamma_2=\gamma$ and $\alpha_1<\alpha_2$ (or $\alpha_1=\alpha_2=\alpha$ and $\gamma_1<\gamma_2$), therefore $Z\leq_{lr}Y\ \forall z$ and hence $Z\leq_{st}Y$.

2.9 Estimation

The parameters of MOAP-IL model are estimated using MLE. The log likelihood function from a sample of ‘n’ size taken from (10) is:

$$\log L=n\log\gamma+n\log(\log\alpha)-n\log(\alpha-1)+2n\log\delta-n\log(1+\delta)+\sum_{i=1}^{n}\log\left(\frac{1+z_i}{z_i^{3}}\right)-\sum_{i=1}^{n}\frac{\delta}{z_i}$$
$$\times\sum_{i=1}^{n}\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)e^{\frac{-\delta}{z_i}}\log\alpha\quad-2\log\sum_{i=1}^{n}\left[\gamma+(1-\gamma)(\alpha-1)^{-1}\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)e^{\frac{-\delta}{z_i}}}-1\right)\right]\qquad(18)$$

The normal equations obtained after differentiating (18) w.r.t α,γ,δ are:

$$\frac{\partial\log L}{\partial\alpha}=\frac{n}{\log\alpha}-\frac{n}{\alpha-1}+\frac{\sum_{i=1}^{n}\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)e^{\frac{-\delta}{z_i}}}{\alpha}-2\sum_{i=1}^{n}\frac{1}{\left[\gamma+(1-\gamma)(\alpha-1)^{-1}\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)e^{\frac{-\delta}{z_i}}}-1\right)\right]}(1-\gamma)$$
$$\left[(\alpha-1)^{-1}\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)e^{\frac{-\delta}{z_i}}-1}\right)\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)e^{\frac{-\delta}{z_i}}-(\alpha-1)^{-2}\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)e^{\frac{-\delta}{z_i}}}-1\right)\right]$$

$$\frac{\partial \log L}{\partial \gamma}=\frac{n}{\gamma}-2\sum_{i=1}^{n}\left[\frac{1-(\alpha-1)^{-1}\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)e^{\frac{-\delta}{z_i}}}-1\right)}{\gamma+(1-\gamma)(\alpha-1)^{-1}\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)e^{\frac{-\delta}{z_i}}}-1\right)}\right]$$

$$\frac{\partial \log L}{\partial \delta}=\frac{2n}{\delta}-\frac{n}{1+\delta}-\sum_{i=1}^{n}\frac{1}{z_i}+\log\alpha\sum_{i=1}^{n}\left[\frac{e^{\frac{-\delta}{z_i}}}{z_i}\left(\frac{1}{(1+\delta)^2}-\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)\right)\right]-2\sum_{i=1}^{n}(1-\gamma)(\alpha-1)^{-1}(\log\alpha)e^{\frac{-\delta}{z_i}}\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)e^{\frac{-\delta}{z_i}}}\left[\frac{\frac{e^{\frac{-\delta}{z_i}}}{z_i}\left(\frac{1}{(1+\delta)^2}-\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)\right)}{\gamma+(1-\gamma)(\alpha-1)^{-1}\left(\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z_i}\right)e^{\frac{-\delta}{z_i}}}-1\right)}\right]$$

Since the above system of equations are complicated mathematical expressions and are very difficult to solve for a particular estimate, therefore the estimates of parameters are obtained using R software.

2.10 Applications

This section is assigned to prove the real life application of the proposed model. The proposed model has been compared with other existing lifetime models such as Generalised Inverse Weibull Distribution (GIWD), Alpha Power ILD (APILD), Logarithmic ILD (LILD), Generalized ILD (GILD) and ILD using two real life datasets. The comparison has been made on the basis of *AIC, BIC, -logL*. Among all compared models, the one with lowest value of *AIC, BIC, -logL* is considered a better model. Cramer-von Mises statistic (W*), AD statistics (A*), K-S statistics has been employed for the model fitting. The distributions that has been taken in consideration for comparison have the following pdfs:

GIWD: $g(z)=\alpha\gamma\delta^{\gamma}z^{-\gamma-1}e^{-\alpha\left(\frac{\delta}{z}\right)^{\gamma}}$; $z>0$

APILD: $g(z)=\frac{\log\alpha}{\alpha-1}\frac{\delta^2}{(1+\delta)}\left(\frac{1+z}{z^3}\right)e^{\frac{-\delta}{z}}\alpha^{\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}}$; $z>0$

$$LILD: \quad g(z)=\frac{\delta^2(\gamma-1)(1+z)e^{\frac{-\delta}{z}}}{(1+\delta)(\ln\gamma)z^3\left(1-(1-\gamma)\left(1-\left(1+\frac{\delta}{1+\delta}\frac{1}{z}\right)e^{\frac{-\delta}{z}}\right)\right)} \qquad ; \quad z>0$$

$$GILD: \quad g(z)=\frac{\alpha\delta^2}{(1+\delta)}\left(\frac{1+z^\alpha}{z^{2\alpha+1}}\right)e^{\frac{-\delta}{z^\alpha}} \qquad ; \quad z>0$$

$$ILD: \quad g(z)=\frac{\delta^2}{(1+\delta)}\left(\frac{1+z}{z^3}\right)e^{\frac{-\delta}{z}} \qquad ; z>0$$

The two data sets have been taken from Maguire et al. (1952) Ghitany et al. (2008) respectively. The ML estimates together with *–logL*, *AIC*, *BIC* values are presented in Table 1, 2 respectively. Table 3, 4 displays **W*, A*** and **K-S** of the two data sets respectively. From these tables, it is apparent that the proposed model competes well relative to other models.

Table 1. ML Estimates and Information Measures for Coal mining data

Model	Estimates α	δ	γ	-logL	AIC	BIC
MOAP-IL	15.45(63.35)	2.70(2.90)	21.22(36.35)	710.27	1426.54	1434.62
APIL	514.09(607.35)	11.77(2.38)	-	721.13	1446.26	1451.65
LIL	-	13.54(2.73)	39.49(17.48)	724.28	1452.56	1457.94
GIL	0.64(0.04)	14.25(2.19)	-	726.51	1457.01	1462.39
IL	-	35.54(3.32)	-	761.86	1525.72	1528.41
GIW	4.97(1.23e+3)	0.64(4.06e-2)	4.73(1.84e+3)	726.30	1458.61	1466.67

Table 2. ML Estimates and Information Measures for Bank data

Model	Estimates α	δ	γ	-logL	AIC	BIC
MOAP-IL	481.11(466.21)	5.06(1.19)	0.07(0.03)	326.10	658.20	666.02
APIL	0.22(0.177)	8.43(1.35)	-	335.25	674.52	679.73
LIL	-	6.84(1.10)	0.63(0.36)	336.29	676.58	681.79
GIL	1.15(0.08	7.23(0.90)	-	334.78	673.56	678.77
IL	-	6.10(0.54)	-	336.62	675.24	677.85
GIW	3.09(420.32)	1.90(221.79)	1.16(0.07)	334.38	674.76	682.58

The pp and cdf plots for the two data sets are presented in Figure 2 and 3 respectively. The asymptotic 95% CI for the two data sets are respectively given by (19) and (20):

$\alpha \in (7.20, 124.89)$
$\delta \in (0.98, 8.37)$
$\gamma \in (4.34, 98.36)$
and
$\alpha \in (4.31, 1392.65)$
$\delta \in (0.72, 7.41)$
$\gamma \in (0.001, 0.15)$
respectively.

Table 3. Goodness of fit statistics for Coal mining data

Model	W*	A*	K-S
MOAP-IL	0.32	2.25	0.13
APIL	0.82	4.33	0.16
LIL	1.08	5.81	0.17
GIL	0.75	4.41	0.14
IL	4.63	23.80	0.33
GIW	0.71	4.40	0.14

Table 4. Goodness of fit statistics for Bank data

Model	W*	A*	K-S
MOAP-IL	0.09	1.07	0.07
APIL	0.37	2.66	0.11
LIL	0.49	3.48	0.15
GIL	0.43	2.93	0.11
IL	0.57	3.91	0.16
GIW	0.42	2.89	0.11

3. Simulated Data Analysis.

The proposed distribution has been compared with other lifetime models using simulated data. The data has been generated using inverse cdf technique for n=500, the results are presented in Table 5. It is evident from the table also that MOAP-IL competes well for generated data.

Table 5. ML Estimates and Information Measures for Simulated Data

Model	Estimates			-logL	AIC	BIC
	α	δ	γ			
MOAP-IL	1.00(1.68)	1.68(0.14)	1.22(1.05)	1226.95	2458.89	2468.54
APIL	1.49(0.55)	1.68(0.14)	-	1227.9	2461.95	2469.38
LIL	-	1.68(0.13)	1.49(0.5)	1228.92	2460.83	2469.26
GIL	0.95(0.03)	1.83(0.06)	-	1227.56	2459.12	2471.55
IL	-	1.82(0.06)	-	1228.6	2463.26	2472.48
GIW	1.82(43.76)	0.74(17.71)	1.00(0.03)	1228.03	2462.05	2474.69

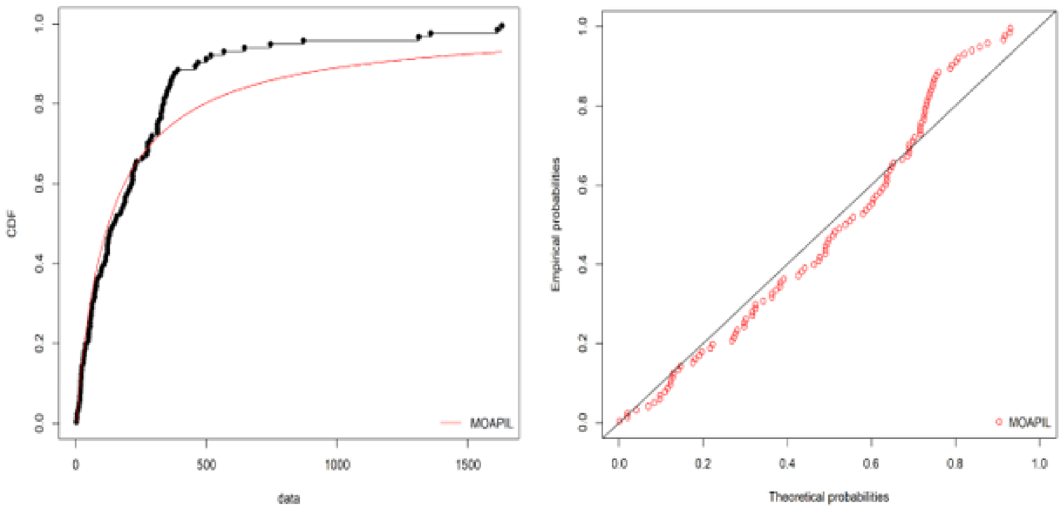

Figure 2. Cdf plot and PP plot for the coal mining data.

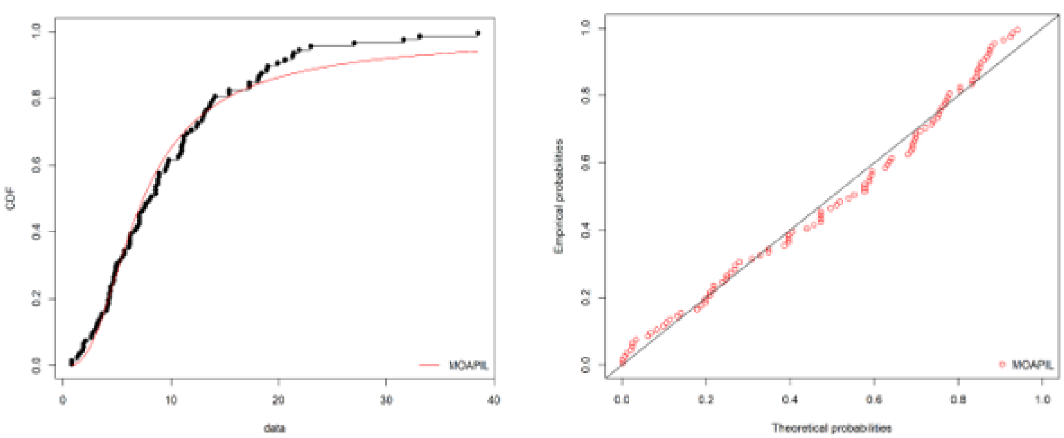

Figure 3. Cdf and pp plots for the Bank data set.

4. Concluding Remarks

A new extension of ILD has been proposed. This model can serve as better alternative for other existing lifetime models. This model exhibits various complex shapes of density function. Some fundamental characteristics along with the parameter estimates have been obtained. The model conformity has been done using two real life and generated data. The model is expected to stand more flexible than the compared models and can be used in various applied areas.

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