Journal of Modern Applied Statistical Methods

Volume 23 | Issue 1 Article 9

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Recommended Citation

Etebong P. Clement, Emmanuel J. Ekpenyong, Matthew J. Iseh (2024). Modified Classes of Stratified Exponential Ratio Estimators of Population Mean with Equal Optimal Efficiency. Journal of Modern Applied Statistical Methods, 23(1), https://doi.org/10.56801/Jmasm.V23.i1.9

Journal of Modern Applied Statistical Methods
2024, Vol. 23, No. 1,
Doi: 10.56801/Jmasm.V23.i1.9

ISSN 1538 – 9472

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This paper introduces new classes of exponential ratio estimators for population mean and derives expressions for their biases and mean square errors (MSEs) under the large sample approximation. Asymptotic optimum estimator and its approximate MSE are derived for each class of estimators. Their properties are studied and some special members of their families are identified. Analytically, the proposed classes of estimators have been shown to have equal optimal efficiency under certain prescribed conditions. It is observed that the proposed classes of estimators fare better than the regression estimator and all the modified existing estimators under review with appreciable efficiency gains.

Keywords: asymptotic optimum estimator, auxiliary variable, large sample approximation, optimality conditions, stratified random sampling.

1. Introduction

In sample surveys the scientific technique for selecting a sample is that of selecting a probability sample that is usually based upon a stratification of the population. It is well known that stratification is one of the design tools that give increased precision. It is noted that the regression estimator of mean is the most efficient estimator. The ratio (and product) estimator of mean is equally good if the regression line is a straight line and passes through the origin. However in most practical situations the regression line does not pass through the origin.

To address this problem, most survey statisticians have carried out researches towards the modification of the existing ratio and product estimators to provide better alternatives and improve their precision. In the progression for improving the performance of the ratio (and product) estimators, authors have of recent proposed various improved ratio and product estimators in sample surveys. Notably among them include: Khoshnevisan *et al.* [21]; Singh and Vishwakarma [26], Chaudhary *et al.* [2], Koyuncu and Kadilar [22], Vishwakarma *et al*. [31], Tailor *et al.* [30], Malik and Singh [23], Tailor [29], Khare and Sinha [19], Sharma *et al.* [24], Khare *et al.* [20], Singh and Audu [27], Clement and Enang [8], [9],, Clement [3], [4], [5], [6], [7], Clement and Inyang [10], [11], Clement *et al* [12], Enang and Clement [15], Clement and Etukudoh [13], and Inyang and Clement [17].

Keeping this in view, this paper introduces new classes of exponential ratio estimators for population mean that have equal optimal efficiency under certain prescribed conditions and as well fare better than the regression estimator.

2. Definitions and Basic Notations

Consider a finite population $U = \left(U_{1,} U_2, \dots, U_N\right)$ of size (N). Let (X) and (Υ) denote the auxiliary and study variables taking values $X_i \ and \ Y_i$ respectively on the i-th unit $U_i(i = 1, 2, \dots, N)$ of the population.

Let the population be divided into H strata with N_h units in the hth stratum from which a simple random sample of size n_h is taken without replacement. The total population size be $N = \sum_{h=1}^{H} N_h$ and the sample size $n = \sum_{h=1}^{H} n_h$, respectively. Associated with the ith element of the hth stratum are y_{hi} and x_{hi} with $x_{hi} > 0$ being the covariate; where y_{hi} is the y value of the ith element in stratum h, and x_{hi} is the x value of the ith element in stratum h, $h = 1,2, \dots, H$ and $i = 1,2, \dots, N_h$. For the hth stratum, let $W_h = N_h/N$ be the stratum weights and $f_h = n_h/N_h$, the sample fraction.

Let the hth sample stratum means of the study variable Y and auxiliary variable X

$\left(\bar{y}_h = \sum_{i=1}^{n_h} y_{hi}/n_h \, ; \, \bar{x}_h = \sum_{i=1}^{n_h} x_{hi}/n_h\right)$ be the unbiased estimators of the population stratum means $\left(\bar{Y}_h = \sum_{i=1}^{N_h} y_{hi}/N_h \, ; \, \bar{X}_h = \sum_{i=1}^{N_h} x_{hi}/N_h\right)$ of Y and X respectively, based on n_h observations.

Let $\bar{y}_h = \bar{Y}_h \left(1 + e_{hy}\right)$, $\bar{x}_h = \bar{X}_h \left(1 + e_{hx}\right)$ so that $E(e_{hx}) = E\left(e_{hy}\right) = 0$, $E(e_{hx}^2) = \gamma_h C_{hx}^2$, $E\left(e_{hy}^2\right) = \gamma_h C_{hy}^2$, $E\left(e_{hx} e_{hy}\right) = \gamma_h \rho_{hxy} C_{hx} C_{hy}$

where

$$\gamma_h = \left(\frac{1 - f_h}{n_h}\right) = \left(\frac{1}{n_h} - \frac{1}{N_h}\right) ; C_{hx}^2 = \frac{S_{hx}^2}{\bar{X}_h^2} \, ; \, C_{hy}^2 = \frac{S_{hy}^2}{\bar{Y}_h^2} \, ; \, K_h = \rho_{hxy} \frac{C_{hy}}{C_{hx}} \, ;$$

$\rho_{hxy} = \frac{S_{hxy}}{S_{hx} S_{hy}}$; $S_{hx}^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (x_{hi} - \bar{X}_h)^2$; $S_{hy}^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (y_{hi} - \bar{Y}_h)^2$ and

$$S_{hxy} = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}(x_{hi} - \bar{X}_h)\ (y_{hi} - \bar{Y}_h)$$

3. Proposed Estimator 1

The first suggested estimator is defined as:

$$\tau_1 = \sum_{h=1}^{H} w_h \{\lambda_h \bar{y}_h + \alpha_h(\bar{X}_h - \bar{x}_h) exp[\varphi_h(\bar{X}_h - \bar{x}_h)/(\bar{X}_h + \bar{x}_h)]\} \quad (1)$$

Where $\varphi_h, \lambda_h, \alpha_h,$ are suitably chosen scalars such that $\lambda_h > 0\ ; -\infty \leq \varphi_h \leq \infty$ and $-\infty \leq \alpha_h \leq \infty$

Expressing (1) in terms of the e's gives

$$\tau_1 = \sum_{h=1}^{H} w_h \bar{Y}_h \left(1 + e_{hy}\right) - \sum_{h=1}^{H} w_h \bar{X}_h\, \alpha_h exp\left[-\frac{\varphi_h e_{hx}}{2}\left(1 + \frac{e_{hx}}{2}\right)^{-1}\right]$$

Now, it is assumed that $|e_{hy}| < 1; |e_{hx}| < 1$ so that expanding $\left(1 + \frac{e_{hx}}{2}\right)^{-1}$ and $exp\frac{\varphi_h e_{hx}}{2}\left(1 + \frac{1}{2}e_{hx}\right)^{-1}$ as a series in power of e_{hx}, multiplying out and retaining terms of the e's to the second degree, gives

$$\tau_1 = \sum_{h=1}^{H} w_h \bar{Y}_h\, \lambda_h\left(1 + e_{hy}\right) - \sum_{h=1}^{H} w_h \bar{X}_h\, \alpha_h \left[1 - \frac{\varphi_h e_{hx}}{2}\left(1 - \frac{e_{hx}}{2} + \frac{e_{hx}^2}{4} + \cdots\right) + \frac{\varphi_h^2 e_{hx}^2}{8}\right]$$

So that to first order approximation

$$(\tau_1 - \bar{Y}) = \sum_{h=1}^{H} w_h \bar{Y}_h \left\{\lambda_h\left(1 + e_{hy}\right) - \frac{1}{2}\alpha_h \psi(2e_{hx} - \varphi_h e_{hx}^2) - 1\right\} \quad (2)$$

where $\bar{Y} = \sum_{h=1}^{H} w_h \bar{Y}_h$, $\bar{X} = \sum_{h=1}^{H} w_h \bar{X}_h$ and $\psi = \frac{\sum_{h=1}^{H} w_h \bar{X}_h}{\sum_{h=1}^{H} w_h \bar{Y}_h}$

The Bias of τ_1 is obtained from (2) as follows:

$$B(\tau_1) = E(\tau_1 - \bar{Y}) = \sum_{h=1}^{H} w_h \bar{Y}_h \left\{(\lambda_h - 1) + \frac{1}{2}\alpha_h \psi \varphi_h C_{hx}^2\right\} \quad (3)$$

Its Mean Square Error is obtained by Taylor's series approximation as:

$$MSE(\tau_1) = E(\tau_1 - \bar{Y})^2 = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2\, E\left[\lambda_h\left(1 + e_{hy}\right) - \frac{1}{2}\alpha_h \psi(2e_{hx} - \varphi_h e_{hx}^2) - 1\right]^2$$

$$MSE(\tau_1) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \, E\{(\lambda_h - 1)^2 + \lambda_h^2 e_{hy}^2 - 2\lambda_h \alpha_h \psi e_{hy} e_{hx} + \alpha_h \psi e_{hx}^2 [(\lambda_h - 1)\varphi_h + \alpha_h \psi]\}$$

$$MSE(\tau_1) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \, \{(\lambda_h - 1)^2 + \lambda_h^2 \gamma_h C_{hy}^2 - 2\lambda_h \alpha_h \psi \gamma_h \rho_{hxy} C_{hy} C_{hx} + [(\lambda_h - 1)\varphi_h + \alpha_h \psi] \alpha_h \psi \gamma_h C_{hx}^2\} \quad (4)$$

3.1 Conditions for optimal efficiency of estimator 1

This section establishes the optimality conditions under which the suggested estimator 1 attains its optimal efficiency. Let set $\frac{\partial MSE(\tau_1)}{\partial \varphi_h} = \frac{\partial MSE(\tau_1)}{\partial \alpha_h} = \frac{\partial MSE(\tau_1)}{\partial \lambda_h} = 0$; so that

$$\lambda_h \alpha_h \psi \gamma_h C_{hx}^2 = \alpha_h \psi \gamma_h C_{hx}^2 \quad (5)$$

$$\alpha_h \psi \gamma_h C_{hx}^2 = \lambda_h \psi^2 \gamma_h \rho_{hxy} C_{hy} C_{hx} \quad (6)$$

$$\alpha_h \varphi_h \psi \gamma_h C_{hx}^2 = 2\left[1 - \lambda_h \left(1 + \gamma_h C_{hy}^2\right) + \alpha_h \psi \gamma_h \rho_{hxy} C_{hy} C_{hx}\right] \quad (7)$$

Following from [(5), (6), (7)], the suggested estimator 1 attains its optimal efficiency when the optimal values of λ_h , α_h and φ_h are respectively:

$$\lambda_{h,opt} = 1 \quad (8)$$

$$\alpha_{h,opt} = \psi K_h \quad (9)$$

$$\varphi_{h,opt} = \frac{-2C_{hy}^2\left(1 - \rho_{hxy}^2\right)}{\rho_{hxy} C_{hy} C_{hx}} \quad (10)$$

Substituting the values of $\lambda_{h,opt}$, $\alpha_{h,opt}$ and $\varphi_{h,opt}$ in [(8), (9), (10)] for λ_h , α_h and $\varphi_h$in (1); an asymptotically optimum exponential ratio estimator (AOE) for population mean ($\bar{Y}$) in stratified random sampling is obtained as:

$$\tau_{1,opt} = \sum_{h=1}^{H} w_h \left\{\lambda_{h,opt} \bar{y}_h + \alpha_{h,opt}(\bar{X}_h - \bar{x}_h) exp\left[\varphi_{h,opt}(\bar{X}_h - \bar{x}_h)/(\bar{X}_h + \bar{x}_h)\right]\right\} \quad (11)$$

Similarly, substituting the values of $\lambda_{h,opt}$, $\alpha_{h,opt}$ and $\varphi_{h,opt}$ in [(8), (9), (10)] for λ_h , α_h and $\varphi_h$in (4); gives the MSE of the asymptotically optimum exponential ratio estimator (AOE) $\tau_{1,opt}$ (or minimum MSE of τ_1) as:

$$MSE(\tau_{1,opt}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 \left(1 - \rho_{hxy}^2\right) = min.\, MSE(\tau_1) \quad (12)$$

Following from the above, the following theorem is established.

Theorem 1

Given

$$\tau_1 = \sum_{h=1}^{H} w_h \{\lambda_h \bar{y}_h + \alpha_h(\bar{X}_h - \bar{x}_h)exp[\varphi_h(\bar{X}_h - \bar{x}_h)/(\bar{X}_h + \bar{x}_h)]\}$$

such that $\lambda_h > 0$; $-\infty \leq \varphi_h \leq \infty$ and ; $-\infty \leq \alpha_h \leq \infty$.

Then, to the first degree of approximation

$MSE(\tau_{1,opt}) \leq MSE(\tau_1)$

with strict equality holding if $\varphi_h = [-2C_{hy}^2(1-\rho_{hxy}^2)/\rho_{hxy}C_{hy}C_{hx}]$, $\lambda_h = 1$, $\alpha_h = \psi K_h$

where $K_h = \rho_{hxy}C_{hy}/C_{hx}$; $MSE(\tau_{1,opt}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2(1-\rho_{hxy}^2)$,

$MSE(\tau_1) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \{(\lambda_h - 1)^2 + \lambda_h^2 \gamma_h C_{hy}^2 - 2\lambda_h \alpha_h \psi \gamma_h \rho_{hxy} C_{hy} C_{hx} + [(\lambda_h - 1)\varphi_h + \alpha_h \psi]\alpha_h \psi \gamma_h C_{hx}^2\}$ and $\psi = [\sum_{h=1}^{H} w_h \bar{X}_h / \sum_{h=1}^{H} w_h \bar{Y}_h]$.

3.2 Properties of estimator 1

This section studies the properties of the proposed estimator 1 and identifies some special members of its family and derives their mean square errors (MSEs) under certain prescribed conditions.

(i) Property 1

When $\lambda_h = 1$, $\alpha_h = 1$ and $\varphi_h = 1$; then the proposed estimator 1 reduces to

$$\tau_{1,1} = \sum_{h=1}^{H} w_h \{\bar{y}_h + (\bar{X}_h - \bar{x}_h)exp[(\bar{X}_h - \bar{x}_h)/(\bar{X}_h + \bar{x}_h)]\} \quad (13)$$

with MSE given as

$$MSE(\tau_{1,1}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h [C_{hy}^2 - \psi C_{hx}^2(2K_h - \psi)] \quad (14)$$

(ii) Property 2

When $\lambda_h = 1$, $\alpha_h = 1$ and $\varphi_h = 2$; then the proposed estimator 1 reduces to

$$\tau_{1,2} = \sum_{h=1}^{H} w_h \{\bar{y}_h + (\bar{X}_h - \bar{x}_h)exp[2(\bar{X}_h - \bar{x}_h)/(\bar{X}_h + \bar{x}_h)]\} \quad (15)$$

with MSE given as

$$MSE(\tau_{1,2}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h [C_{hy}^2 - \psi C_{hx}^2(2K_h - \psi)] \quad (16)$$

(iii) Property 3

When $\lambda_h = 1$, $\alpha_h = 1$ and $\varphi_h = \varphi_h$; then the proposed estimator 1 reduces to

$$\tau_{1,3} = \sum_{h=1}^{H} w_h \{\bar{y}_h + (\bar{X}_h - \bar{x}_h)exp[\varphi_h(\bar{X}_h - \bar{x}_h)/(\bar{X}_h + \bar{x}_h)]\} \quad (17)$$

with MSE given as

$$MSE(\tau_{1,3}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h [C_{hy}^2 - \psi C_{hx}^2(2K_h - \psi)] \quad (18)$$

Remark 1: Following from properties [1-3], it is deduced that irrespective of the values of φ_h, whenever $\lambda_h = \alpha_h = 1$; the resulting family members of the proposed estimator 1 have the same MSE given by $MSE(\tau_{1,*}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left[C_{hy}^2 - \psi C_{hx}^2 (2K_h - \psi)\right]$.

(iv). Property 4

When $\lambda_h = 1, \alpha_h = 0$ and $\varphi_h = 1$; the proposed estimator 1 reduces to the usual unbiased stratified random sampling estimator given as:

$$\tau_{1,4} = \sum_{h=1}^{H} w_h \bar{y}_h \tag{19}$$

with MSE given as

$$MSE(\tau_{1,4}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 \tag{20}$$

Remark 2: It is observed from investigation that irrespective of the values of φ_h, whenever $\lambda_h = 1, \alpha_h = 0$; the resulting family member of the proposed estimator 1 is always the unbiased stratified random sampling estimator of population mean given in (19).

(v) Property 5

When $\lambda_h = \lambda_h, \alpha_h = \alpha_h$ and $\varphi_h = 0$. The proposed estimator 1 reduces to

$$\tau_{1,5} = \lambda_h \bar{y}_{st} + \alpha_h (\bar{X} - \bar{x}_{st}) \tag{21}$$

where $\bar{y}_{st} = \sum_{h=1}^{H} w_h \bar{y}_h$; $\bar{x}_{st} = \sum_{h=1}^{H} w_h \bar{x}_h$; $\bar{X} = \sum_{h=1}^{H} w_h \bar{X}_h$

with MSE given as

$$MSE(\tau_{1,5}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \left\{(\lambda_h - 1)^2 + \lambda_h^2 \gamma_h C_{hy}^2 - 2\lambda_h \alpha_h \psi \gamma_h \rho_{hxy} C_{hy} C_{hx} + \alpha_h^2 \psi^2 \gamma_h C_{hx}^2\right\} \tag{22}$$

(vi). Property 6

When $\lambda_h = 1, \alpha_h = \alpha_h$ and $\varphi_h = 0$. The proposed estimator 1 reduces to

$$\tau_{1,6} = \bar{y}_{st} + \alpha_h (\bar{X} - \bar{x}_{st}) \tag{23}$$

with MSE given as

$$MSE(\tau_{1,6}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left\{C_{hy}^2 - 2\alpha_h \psi \rho_{hxy} C_{hy} C_{hx} + \alpha_h^2 \psi^2 C_{hx}^2\right\} \tag{24}$$

(vii) Property 7

When $\lambda_h = \lambda_h, \alpha_h = 1$ and $\varphi_h = 0$. The proposed estimator 1 reduces to

$$\tau_{1,7} = \lambda_h \bar{y}_{st} + (\bar{X} - \bar{x}_{st}) \tag{25}$$

where $\bar{y}_{st} = \sum_{h=1}^{H} w_h \bar{y}_h$; $\bar{x}_{st} = \sum_{h=1}^{H} w_h \bar{x}_h$; $\bar{X} = \sum_{h=1}^{H} w_h \bar{X}_h$

with MSE given as

$$MSE(\tau_{1,7}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \left\{(\lambda_h - 1)^2 + \lambda_h^2 \gamma_h C_{hy}^2 - 2\lambda_h \psi \gamma_h \rho_{hxy} C_{hy} C_{hx} + \psi^2 \gamma_h C_{hx}^2\right\} \tag{26}$$

(viii) Property 8

When $\lambda_h = 1$, $\alpha_h = 1$ and $\varphi_h = 0$. The proposed estimator 1 reduces to

$$\tau_{1,8} = \bar{y}_{st} + (\bar{X} - \bar{x}_{st}) \quad (27)$$

where $\bar{y}_{st} = \sum_{h=1}^{H} w_h \bar{y}_h$; $\bar{x}_{st} = \sum_{h=1}^{H} w_h \bar{x}_h$; $\bar{X} = \sum_{h=1}^{H} w_h \bar{X}_h$

with MSE given as

$$MSE(\tau_{1,8}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \{C_{hy}^2 + \psi^2 C_{hx}^2 - 2\psi \rho_{hxy} C_{hy} C_{hx}\} \quad (28)$$

(ix) Property 9

When $\lambda_h = 1$, $\alpha_h = \alpha_{h,opt}$ and $\varphi_h = 0$. The proposed estimator 1 reduces to

$$\tau_{1,9} = \bar{y}_{st} + \alpha_{h,opt}(\bar{X} - \bar{x}_{st}) \quad (29)$$

with MSE given as

$$MSE(\tau_{1,9}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 (1 - \rho_{hxy}^2) \quad (30)$$

Remark 3: It should be noted from properties [5-9] that irrespective of the values of, λ_h and α_h; whenever $\varphi_h = 0$; the resulting family members of the proposed estimator 1 are always the stratified regression-type estimators of population mean with varying mean square errors (MSEs). Again, it should be noted here that (30) gives the same expression as the minimum variance of the stratified regression estimator of population mean given by:

$$\bar{y}_{st,Reg} = \bar{y}_{st} + b(\bar{X} - \bar{x}_{st}) \quad (31)$$

where b is the sample regression coefficient Cochran [14]. Therefore, when $\varphi_h = 0$, $\lambda_h = 1$; and α_h is optimal, the proposed estimator1 has the same efficiency as the usual stratified regression estimator of mean.

(x) Property 10

When $\lambda_h = \lambda_{h,opt}$, $\alpha_h = \alpha_{h,opt}$ and $\varphi_h = \varphi_{h,opt}$. The proposed estimator 1 reduces to

$$\tau_{1,10} = \sum_{h=1}^{H} w_h \{\lambda_{h,opt} \bar{y}_h + \alpha_{h,opt} (\bar{X}_h - \bar{x}_h) exp[\varphi_{h,opt} (\bar{X}_h - \bar{x}_h)/(\bar{X}_h + \bar{x}_h)]\} \quad (32)$$

with MSE given as:

$$MSE(\tau_{1,10}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 (1 - \rho_{hxy}^2) \quad (33)$$

4. Proposed Estimator 2

The second suggested estimator is given by:

$$\tau_2 = \sum_{h=1}^{H} w_h \bar{y}_h \left\{\vartheta_h - \xi_h \left(\frac{\bar{x}_h}{\bar{X}_h}\right)^{\nu_h} exp\left[\frac{\delta_h (\bar{x}_h - \bar{X}_h)}{(\bar{x}_h + \bar{X}_h)}\right]\right\} \quad (34)$$

Where ξ_h, ϑ_h and ν_h are suitably chosen scalars such that ξ_h and ϑ_h satisfies the condition $\vartheta_h = 1 + \xi_h$; $-\infty \leq \xi_h \leq \infty$

Expressing (34) in terms of the e's gives

$$\tau_2 = \sum_{h=1}^{H} w_h \bar{Y}_h \left(1 + e_{hy}\right) \left\{ \vartheta_h - \xi_h (1 + e_{hx})^{\nu_h} exp \frac{\delta_h e_{hx}}{2} \left(1 + \frac{1}{2} e_{hx}\right)^{-1} \right\}$$

Now, it is assumed that $|e_{hx}| < 1$, $|e_{hy}| < 1$ so that expanding $(1 + e_{hx})^{\nu_h}$, $\left(1 + \frac{1}{2} e_{hx}\right)^{-1}$ and $exp \frac{\delta_h e_{hx}}{2} \left(1 + \frac{1}{2} e_{hx}\right)^{-1}$ as a series in power of e_{hx}, multiplying out and retaining terms of the e's to the second degree gives

$$\tau_2 = \sum_{h=1}^{H} w_h \bar{Y}_h \left[1 - \frac{\xi_h (2\nu_h + \delta_h) e_{hx}}{2} - \xi_h (2\nu_h + \delta_h) \frac{(2\nu_h + \delta_h - 2) e_{hx}^2}{8} - \xi_h (2\nu_h + \delta_h) e_{hx} e_{hy} + e_{hy} \right]$$

So that

$$\tau_2 - \bar{Y} = \sum_{h=1}^{H} w_h \bar{Y}_h \left[\left\{ -\frac{\xi_h (2\nu_h + \delta_h)}{2} \left(e_{hx} + \frac{(2\nu_h + \delta_h - 2) e_{hx}^2}{4} + e_{hx} e_{hy} \right) \right\} + e_{hy} \right]$$

Therefore, the Bias of estimator 2 is given as:

$$B(\tau_2) = E(\tau_2 - \bar{Y}) \sum_{h=1}^{H} w_h \bar{Y}_h \gamma_h C_{hx}^2 \left[-\frac{\varphi_h (2\alpha_h + \delta_h)}{2} \left(\frac{(2\alpha_h + \delta_h - 2) + 4K_h}{4} \right) \right] \quad (35)$$

Its Mean Square Error is obtained by Taylor's series approximation as:

$$MSE(\tau_2) = E \left[\sum_{h=1}^{H} w_h \bar{Y}_h \left[\left\{ -\frac{\xi_h (2\nu_h + \delta_h)}{2} \left(e_{hx} + \frac{(2\nu_h + \delta_h - 2) e_{hx}^2}{4} + e_{hx} e_{hy} \right) \right\} + e_{hy} \right] \right]^2$$

$$MSE(\tau_2) \simeq \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left\{ C_{hy}^2 + \frac{\xi_h (2\nu_h + \delta_h)}{4} \left[\xi_h (2\nu_h + \delta_h) C_{hx}^2 - 4\rho_{hxy} C_{hx} C_{hy} \right] \right\}$$

$$MSE(\tau_2) \cong \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 \left\{ 1 + \frac{\xi_h (2\nu_h + \delta_h) \rho_{hxy}^2}{4K_h^2} \left[\xi_h (2\nu_h + \delta_h) - 4K_h \right] \right\} \quad (36)$$

4.1 Conditions for optimal efficiency of estimator 2

This section establishes the optimality conditions under which the suggested estimator 2 attains its optimal efficiency.

Let set $(\nu_h, \delta_h) = (1,1)$ and $\frac{\partial MSE(\tau_2)}{\partial \xi_h}$ so that

$$\xi_h = \left(\frac{2K_h}{3}\right)$$

$$= \xi_{h,opt} \qquad (say) \tag{37}$$

Substituting the value of $\xi_{h,opt}$ in (37) for ξ_h in (34), an asymptotically optimum exponential ratio estimator (AOE) for population mean $(\bar{Y})$ in stratified random sampling is obtained as:

$$\tau_{2,opt} = \sum_{h=1}^{H} w_h \bar{y}_h \left\{1 - \frac{2K_h}{3}\left(\frac{\bar{x}_h}{\bar{X}_h}\right) exp\left[\frac{(\bar{x}_h - \bar{X}_h)}{(\bar{x}_h + \bar{X}_h)}\right]\right\} \tag{38}$$

Similarly, substituting the value of $\xi_{h,opt}$ in (37) for ξ_h in (36), gives the MSE of the asymptotically optimum exponential ratio estimator (AOE) $\tau_{2,opt}$ (or minimum MSE of τ_2) as:

$$MSE(\tau_{2,opt}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 \left(1 - \rho_{hxy}^2\right) \tag{39}$$

Alternatively:

Let set $\frac{\partial MSE(\tau_2)}{\partial \delta_h} = \frac{\partial MSE(\tau_2)}{\partial \nu_h} = \frac{\partial MSE(\tau_2)}{\partial \xi_h} = 0$; so that

$$\xi_h(2\nu_h + \delta_h)\rho_{hxy}^2 = 2(2\nu_h + \delta_h)\rho_{hxy}^2 \tag{40}$$

$$2\xi_h^2 \nu_h \rho_{hxy}^2 = 2K_h \xi_h \rho_{hxy}^2 - \xi_h^2 \delta_h \rho_{hxy}^2 \tag{41}$$

$$\xi_h^2 \delta_h \rho_{hxy}^2 = 2K_h \xi_h \rho_{hxy}^2 - 2\xi_h^2 \nu_h \rho_{hxy}^2 \tag{42}$$

Following from [(40), (41), (42)], the suggested estimator 2 attains its optimal efficiency when the optimal values of ξ_h , ν_h and δ_h are respectively:

$$\xi_h = \left(\frac{2K_h}{2\nu_h + \delta_h}\right)$$

$$= \xi_{h,opt} \qquad (say) \tag{43}$$

$$\nu_h = \left(\frac{2K_h + \delta_h \xi_h}{2\xi_h}\right)$$

$$= \nu_{h,opt} \qquad (say) \tag{44}$$

$$\delta_h = \frac{2[K_h - \nu_h \xi_h]}{\xi_h}$$

$$= \delta_{h,opt} \qquad (say) \tag{45}$$

Remark 4: It should be noted here that the proposed estimator 2 would attain its optimal efficiency with minimum MSE given as:

$$MSE(\tau_{2,opt}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 \left(1 - \rho_{hxy}^2\right)$$

when either of the following optimality conditions is satisfied:

(i) $\nu_{h,opt} = \nu_h, \delta_{h,opt} = \delta_h$ and $\xi_{h,opt} = \left(\frac{2K_h}{2\nu_h + \delta_h}\right)$

(ii) $\nu_{h,opt} = \nu_h$, $\xi_{h,opt} = \xi_h$ and $\delta_{h,opt} = \frac{2[K_h - \nu_h \xi_h]}{\xi_h}$

(iii) $\delta_{h,opt} = \delta_h$, $\xi_{h,opt} = \xi_h$ and $\nu_{h,opt} = \left(\frac{2K_h + \delta_h \xi_h}{2\xi_h}\right)$

Following from the above discussion, theorem 2 is established as follows:

Theorem 2

Given

$$\tau_2 = \sum_{h=1}^{H} w_h \bar{y}_h \left\{ \vartheta_h - \xi_h \left(\frac{\bar{x}_h}{\bar{X}_h}\right)^{\nu_h} exp\left[\frac{\delta_h(\bar{x}_h - \bar{X}_h)}{(\bar{x}_h + \bar{X}_h)}\right] \right\}$$

such that $\vartheta_h = 1 + \xi_h$; $-\infty \leq \xi_h \leq \infty$, $-\infty \leq \delta_h \leq \infty$ and $-\infty \leq \nu_h \leq \infty$

Then, to the first degree of approximation

$MSE(\tau_{2,opt}) \leq MSE(\tau_2)$

with strict equality holding if any of the following conditions is satisfied:

(i) $\nu_{h,opt} = \nu_h$, $\delta_{h,opt} = \delta_h$ and $\xi_{h,opt} = \left(\frac{2K_h}{2\nu_h + \delta_h}\right)$

(ii) $\nu_{h,opt} = \nu_h$, $\xi_{h,opt} = \xi_h$ and $\delta_{h,opt} = \frac{2[K_h - \nu_h \xi_h]}{\xi_h}$

(iii) $\delta_{h,opt} = \delta_h$, $\xi_{h,opt} = \xi_h$ and $\nu_{h,opt} = \left(\frac{2K_h + \delta_h \xi_h}{2\xi_h}\right)$

where $K_h = \rho_{hxy} C_{hy} / C_{hx}$; $MSE(\tau_{2,opt}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 \left(1 - \rho_{hxy}^2\right)$, and

$$MSE(\tau_2) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 \left\{ 1 + \frac{\xi_h (2\nu_h + \delta_h) \rho_{hxy}^2}{4K_h^2} [\xi_h (2\nu_h + \delta_h) - 4K_h] \right\}$$

4.2 Properties of estimator 2

This section studies the properties of the proposed estimator 2 and identifies some special members of its family and derives their mean square errors (MSEs) under certain prescribed conditions.

(i) Property 1

When $\vartheta_h = 1$, $\xi_h = 0$, $\nu_h = \nu_h$ and $\delta_h = \delta_h$. The proposed estimator 2 reduces to the well-known unbiased stratified random sampling estimator given as:

$$\tau_{2,1} = \sum_{h=1}^{H} w_h \bar{y}_h \tag{46}$$

with MSE given as

$$MSE(\tau_{2,1}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 \tag{47}$$

Remark 5: It should be noted here that irrespective of the values of ν_h and δ_h, whenever $\vartheta_h = 1$ and $\xi_h = 0$; the resulting family member of the proposed estimator 2 is always the unbiased stratified random sampling estimator of population mean given in (46).

(ii) Property 2

When $\vartheta_h = 0$, $\xi_h = -1$, $\nu_h = 0$ and $\delta_h = -1$. The proposed estimator 2 reduces to the Bahl and Tuteja [1] exponential ratio-type estimator in stratified random sampling given as:

$$\tau_{2,2} = \sum_{h=1}^{H} w_h \bar{y}_h \ exp\left[\frac{(\bar{X}_h - \bar{x}_h)}{(\bar{X}_h + \bar{x}_h)}\right] \tag{48}$$

with MSE given as:

$$MSE\left(\tau_{2,2}\right) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left[C_{hy}^2 + \frac{C_{hx}^2}{4}(1 - 4K_h)\right] \tag{49}$$

(iii) Property 3

When $\vartheta_h = 0$, $\xi_h = -1$, $\nu_h = -\nu_h$ and $\delta_h = 0$. The proposed estimator 2 reduces to

$$\tau_{2,3} = \sum_{h=1}^{H} w_h \bar{y}_h \left(\frac{\bar{X}_h}{\bar{x}_h}\right)^{\nu_h} \tag{50}$$

with MSE given as

$$MSE\left(\tau_{2,3}\right) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left[C_{hy}^2 + \nu_h C_{hx}^2(\nu_h - 2K_h)\right] \tag{51}$$

(iv) Property 4

When $\vartheta_h = 0$, $\xi_h = -1$, $\nu_h = -2$ and $\delta_h = 0$. The proposed estimator 2 reduces to the Kadilar and Cingi [18] chain ratio-type estimator in stratified random sampling given as:

$$\tau_{2,4} = \sum_{h=1}^{H} w_h \bar{y}_h \left(\frac{\bar{X}_h}{\bar{x}_h}\right)^2 \tag{52}$$

with MSE given as

$$MSE\left(\tau_{2,4}\right) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left[C_{hy}^2 + 4C_{hx}^2(1 + K_h)\right] \tag{53}$$

(v) Property 5

When $\vartheta_h = 0$, $\xi_h = -1$, $\nu_h = -\nu_h$ and $\delta_h = -\delta_h$. The proposed estimator 2 reduces to

$$\tau_{2,5} = \sum_{h=1}^{H} w_h \bar{y}_h \left\{ \left(\frac{\bar{X}_h}{\bar{x}_h} \right)^{\nu_h} exp \left[\frac{\delta_h (\bar{X}_h - \bar{x}_h)}{(\bar{X}_h + \bar{x}_h)} \right] \right\} \tag{54}$$

with MSE given as:

$$MSE(\tau_{2,5}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left[C_{hy}^2 + \frac{(2\nu_h + \delta_h) C_{hx}^2}{4} [(2\nu_h + \delta_h) - 4K_h] \right] \tag{55}$$

(vi) Property 6

When $\vartheta_h = 0$, $\xi_h = -1$, $\nu_h = -2$ and $\delta_h = -\delta_h$. The proposed estimator 2 reduces to

$$\tau_{2,6} = \sum_{h=1}^{H} w_h \bar{y}_h \left\{ \left(\frac{\bar{X}_h}{\bar{x}_h} \right)^{2} exp \left[\frac{\delta_h (\bar{X}_h - \bar{x}_h)}{(\bar{X}_h + \bar{x}_h)} \right] \right\} \tag{56}$$

with MSE given as:

$$MSE(\tau_{2,6}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left[C_{hy}^2 + (4 + \delta_h) C_{hx}^2 \left[\frac{\delta_h}{4} + (1 - K_h) \right] \right] \tag{57}$$

(vii) Property 7

When $\vartheta_h = 0$, $\xi_h = -1$, $\nu_h = -2$ and $\delta_h = -2$. The proposed estimator 2 reduces to

$$\tau_{2,7} = \sum_{h=1}^{H} w_h \bar{y}_h \left\{ \left(\frac{\bar{X}_h}{\bar{x}_h} \right)^{2} exp \left[\frac{2(\bar{X}_h - \bar{x}_h)}{(\bar{X}_h + \bar{x}_h)} \right] \right\} \tag{58}$$

with MSE given as:

$$MSE(\tau_{2,7}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left[C_{hy}^2 + 3C_{hx}^2 (3 - 2K_h) \right] \tag{59}$$

(viii) Property 8

When $\vartheta_h = 0$, $\xi_h = -1$, $\nu_h = -\nu_h$ and $\delta_h = -2$. The proposed estimator 2 reduces to

$$\tau_{2,8} = \sum_{h=1}^{H} w_h \bar{y}_h \left\{ \left(\frac{\bar{X}_h}{\bar{x}_h} \right)^{\nu_h} exp \left[\frac{2(\bar{X}_h - \bar{x}_h)}{(\bar{X}_h + \bar{x}_h)} \right] \right\} \tag{60}$$

with MSE given as:

$$MSE(\tau_{2,8}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left[C_{hy}^2 + (\nu_h + 1) C_{hx}^2 [(\nu_h + 1) - 2K_h] \right] \tag{61}$$

(ix) Property 9

When $\vartheta_h = 2$, $\xi_h = 1$, $\nu_h = \nu_h$ and $\delta_h = \delta_h$. The proposed estimator 2 reduces to the Solanki et *al*. [28] class of ratio-type estimators in stratified random sampling given as:

$$\tau_{2,9} = \sum_{h=1}^{H} w_h \bar{y}_h \left\{ 2 - \left(\frac{\bar{x}_h}{\bar{X}_h} \right)^{\nu_h} exp \left[\frac{\delta_h (\bar{x}_h - \bar{X}_h)}{(\bar{X}_h + \bar{x}_h)} \right] \right\} \tag{62}$$

with MSE given as

$$MSE(\tau_{2,9}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left[C_{hy}^2 + \frac{(2\nu_h+\delta_h)C_{hx}^2}{4} [(2\nu_h + \delta_h) - 4K_h] \right] \tag{63}$$

5. Analytical Study

5.1 Efficiency comparisons

This section compares the optimal MSEs of the proposed classes of estimators with some existing estimators who are special members of these classes of estimators.

Let $MSE(\tau_{1,Opt}) = MSE(\tau_{2,opt}) = MSE(\tau_{*,Opt}) = \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 (1 - \rho_{hxy}^2)$

Since the two classes of exponential ratio estimators have equal optimal efficiency.

(i) Stratified random sampling estimator

The proposed estimators would be more efficient than the stratified random sampling estimator if:

$$MSE(\tau_{*,Opt}) < MSE(\bar{y}_{st})$$

$$\sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 (1 - \rho_{hxy}^2) < \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2$$

So that

$$\rho_{hxy} > 0$$

(ii) Bahl and Tuteja [1] exponential ratio estimator

The proposed estimators would be more efficient than the Bahl and Tuteja [1] exponential ratio estimator if:

$$MSE(\tau_{*,Opt}) < MSE(\bar{y}_{st,BT})$$

$$\sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 (1 - \rho_{hxy}^2) < \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left(C_{hy}^2 + \frac{C_{hx}^2}{4} [1 - 4K] \right)$$

This is equivalent to the inequalities

$$(2C_{hy}\rho_{hxy} - C_{hx})^2 > 0$$

So that

$$\rho_{hxy} > \frac{C_{hx}}{2C_{hy}}$$

(iii) Kadillar and Cingi [18] chain ratio estimator

The proposed estimators would be more efficient than the Kadillar and Cingi [18] chain ratio estimator if:

$$MSE(\tau_{*,Opt}) < MSE(\bar{y}_{st,KC})$$

$$\sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2 (1 - \rho_{hxy}^2) < \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h (C_{hy}^2 + 4C_{hx}^2 [1 + 4K])$$

This is equivalent to the inequalities
$\left(C_{hy}\rho_{hxy} + 2C_{hx}\right)^2 > 0$
So that
$\rho_{hxy} > \frac{-2C_{hx}}{C_{hy}}$

(iv) Solanki et. al [28] exponential ratio estimator
The proposed estimators would be more efficient than the Solanki *et. al* [29] exponential ratio estimator if:
$MSE\left(\tau_{*,Opt}\right) < MSE\left(\bar{y}_{st,SO}\right)$
$$\sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2\left(1-\rho_{hxy}^2\right) < \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left(C_{hy}^2 + \frac{3C_{hx}^2}{4}[3-4K]\right)$$
This is equivalent to the inequalities
$\left(2C_{hy}\rho_{hxy} - 3C_{hx}\right)^2 > 0$
So that
$\rho_{hxy} > \frac{3C_{hx}}{2C_{hy}}$

(v) Stratified combined regression estimator
The proposed estimators would be more efficient than the stratified combined regression estimator if:
$MSE\left(\tau_{*,Opt}\right) < MSE\left(\bar{y}_{st,REG}\right)$
$$\sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2\left(1-\rho_{hxy}^2\right)$$
$$< \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left(C_{hy}^2 + b^2\theta_h^2 C_{hx}^2 - b\theta_h \rho_{hxy} C_{hy} C_{hx}\right)$$
This is equivalent to the inequalities
$\left(C_{hy}\rho_{hxy} + b\theta_h C_{hx}\right)^2 > 0$
So that
$\rho_{hxy} > \frac{b\theta_h C_{hx}}{C_{hy}}$
where $\theta_h = \frac{\bar{X}_h}{\bar{Y}_h}$ and $b = \frac{S_{xy}}{S_x^2}$

(vi) Stratified combined ratio estimator
The proposed estimators would be more efficient than the stratified combined ratio estimator if:
$MSE\left(\tau_{*,Opt}\right) < MSE\left(\bar{y}_{st,RC}\right)$
$$\sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h C_{hy}^2\left(1-\rho_{hxy}^2\right)$$
$$< \sum_{h=1}^{H} w_h^2 \bar{Y}_h^2 \gamma_h \left(C_{hy}^2 + R^2\theta_h^2 C_{hx}^2 - R\theta_h \rho_{hxy} C_{hy} C_{hx}\right)$$

This is equivalent to the inequalities

$$\left(C_{hy}\rho_{hxy} - R\theta_h C_{hx}\right)^2 > 0$$

So that

$$\rho_{hxy} > \frac{R\theta_h C_{hx}}{C_{hy}}$$

Where $R = \frac{\sum_{h=1}^{H} W_h \bar{Y}_h}{\sum_{h=1}^{H} W_h \bar{X}_h}$

5.2 The percent relative efficiency $(\boldsymbol{PRE})$

The percent relative efficiency (PRE) of an estimator Ω with respect to the conventional stratified combined ratio estimator $(\bar{y}_{st,RC})$ is defined by

$$PRE(\Omega, \bar{y}_{st,RC}) = \frac{Var(\bar{y}_{st,RC})}{Var(\Omega)} \times 100$$

6. Empirical Study

In this section, the performance of the proposed classes of estimators is assessed with that of the existing estimators and members of their classes. The merits of the suggested classes of estimators 1 and 2 over the existing members of their classes were judged using natural populations adapted from Singh and Chaudhary [25] and Gamze and Ozel [16]. The description of the populations and the required values of the parameters are respectively shown in the Tables 1 and 2.

Table 1: Population I adapted from Singh and Chaudhary [25]

Stratum	1	2	3	Total
N_h	6	8	11	$N = 25$
n_h	3	3	4	$n = 10$
$\bar{Y}_h$	417.330	503.375	340.000	$\bar{X} = 8.3792$
$\bar{X}_h$	6.813	10.12	7.967	$\bar{Y} = 410.84$
C_{hx}	0.586584	1.138124	0.778190	$S_x^2 = 59.73676$
C_{hy}	0.655239	1.011238	0.754947	$S_y^2 = 1237702$
S_{hx}^2	15.9712	132.66	38.438	$\rho = 0.9285341$
S_{hy}^2	74775.467	259443.70	65885.60	$S_{xy} = 2524.79$
ρ_h	0.9215191	0.9737715	0.8826909	
γ_h	0.16666667	0.2083333	0.1590909	

w_h^2	0.0576	0.1024	0.1936
θ_h	0.0163	0.0201	0.0234

Table 2: Population II adapted from Gamze and Ozel (2015)

Stratum	1	2	3	4	5	6
N_h	127	117	103	170	205	201
n_h	31	21	29	38	22	39
$\bar{Y}_h$	703.74	413.00	573.17	424.66	267.03	393.84
$\bar{X}_h$	20804.59	9211.79	14309.30	9478.85	5569.95	12997.59
C_{hx}	1.465	1.648	1.925	1.922	1.526	1.777
C_{hy}	1.256	1.562	1.803	1.909	1.512	1.807
S_{hx}	30486.751	15180.769	2754.9697	18218.931	8497.776	23094.141
S_{hy}	883.835	644.922	1033.467	810.585	403.654	711.723
ρ_{hxy}	0.936	0.996	0.994	0.983	0.989	0.965
γ_h	0.024	0.039	0.025	0.020	0.041	0.021
w_h^2	0.019	0.016	0.013	0.034	0.049	0.048
θ_h	29.5629	22.3046	24.9652	22.3210	20.8589	33.0022
	$S_x^2 = 19{,}740.0345$		$S_{xy} = 829.0814$			

Table 3: MSE and PRE values for the estimators

S/No. Estimator	*Population I*		*Population II*	
	MSE	*PRE*	*MSE*	*PRE*
1. $\bar{y}_{st,RC}$	1137.1827	100	218.7898	100
2. $\tau_{*,opt}$ [Proposed]	842.6005	134.9610	107.7099	203.1288
3. $\bar{y}_{st,REG}$	948.9747	119.8328	201.8586	108.3877
4 $\bar{y}_{st,BT}$	2211.3893	51.4239	576.8330	37.9295
5. $\bar{y}_{st,SO}$	4675.4565	24.3223	901.0174	24.2825

6. $\bar{y}_{st,KC}$	81,041.9121	1.4032	21,171.0350	1.0334

7. Results and Discussion

Analytical results for the efficiency comparisons showed that all the optimality conditions for the efficiency of the proposed classes of estimators are satisfied as evidenced in the MSE values of the respective estimators in Table 3 for the two populations considered.

(i) Stratified random sampling estimator
The proposed classes of estimators are more efficient than the stratified random sampling estimator under the two populations respectively as:
 (a) $MSE(\tau_{*,Opt}) = 842.6005 < MSE(\bar{y}_{st}) = 8474.2512$
 (b) $MSE(\tau_{*,Opt}) = 107.7099 < MSE(\bar{y}_{st}) = 2247.9474$

(ii) Bahl and Tuteja [1]) exponential ratio estimator
The proposed classes of estimators are more efficient than the Bahl and Tuteja [1] exponential ratio estimator under the two populations respectively as:
 (a) $MSE(\tau_{*,Opt}) = 842.6005 < MSE(\bar{y}_{st,BT}) = 2211.3893$
 (b) $MSE(\tau_{*,Opt}) = 107.7099 < MSE(\bar{y}_{st,BT}) = 576.8330$

(iii) Kadillar and Cingi [18] chain ratio estimator
The proposed classes of estimators are more efficient than the Kadillar and Cingi [18] chain ratio estimator under the two populations respectively as:
 (a) $MSE(\tau_{*,Opt}) = 842.6005 < MSE(\bar{y}_{st,KC}) = 81{,}041.9121$
 (b) $MSE(\tau_{*,Opt}) = 107.7099 < MSE(\bar{y}_{st,KC}) = 21{,}171.0350$

(iv) Solanki et. al [28] exponential ratio estimator
The proposed classes of estimators are more efficient than the Solanki *et. al* [28] exponential ratio estimator under the two populations respectively as:
 (a) $MSE(\tau_{*,Opt}) = 842.6005 < MSE(\bar{y}_{st,SO}) = 4675.4565$
 (b) $MSE(\tau_{*,Opt}) = 107.7099 < MSE(\bar{y}_{st,SO}) = 901.0174$

(v) Stratified combined regression estimator
The proposed classes of estimators are more efficient than the stratified combined regression estimator under the two populations respectively as:
 (a) $MSE(\tau_{*,Opt}) = 842.6005 < MSE(\bar{y}_{st,REG}) = 948.9747$
 (b) $MSE(\tau_{*,Opt}) = 107.7099 < MSE(\bar{y}_{st,REG}) = 201.8586$

(vi) Stratified combined ratio estimator
The proposed classes of estimators are more efficient than the stratified combined ratio estimator under the two populations respectively as:

(a) $MSE(\tau_{*,Opt}) = 842.6005 < MSE(\bar{y}_{st,RC}) = 1137.1827$
(b) $MSE(\tau_{*,Opt}) = 107.7099 < MSE(\bar{y}_{st,RC}) = 218.7898$

Numerical results for the percent relative efficiency (*PREs*) in Table 3 reveals that the proposed estimators $(\tau_{*,opt})$ have 35 percent gains and 103 percent gains in efficiency respectively for population I and population II while the regression estimator $(\bar{y}_{st,REG})$ has 20 percent gains and 8 percent gains in efficiency respectively for population I and population II. This shows that the proposed estimators $(\tau_{*,opt})$ are 15 percent more efficient than the regression estimator $(\bar{y}_{st,REG})$ under population I and 8 percent more efficient than it under population II.

Similarly, the proposed estimators $(\tau_{*,opt})$ are 84 percent more efficient than the Bahl and Tuteja [1] exponential ratio estimator $(\bar{y}_{st,BT})$ under population I and 165 percent more efficient than it under population II.

In using the proposed classes of estimators $(\tau_{*,opt})$ one will have 111 percent efficiency gain over the Solanki et al [28] exponential ratio estimator $(\bar{y}_{st,SO})$ under population I and will have 179 percent efficiency gain over it under population II.

8. Conclusion

This paper introduces new classes of exponential ratio estimators for estimating population mean $\bar{Y}$ in stratified random sampling and derives expressions for their biases and mean square errors (MSEs) under the large sample approximation. Asymptotic optimum estimator (AOE) and its approximate MSE are derived for each class of estimators. The properties of the proposed classes of estimators are studied and some special members of these families of estimators are identified. Analytically, the proposed classes of estimators have been shown to have equal optimal efficiency under certain prescribed conditions.

It is well known that the ratio and product estimators have the limitation of having efficiency not exceeding that of the linear regression estimator. Analytical and numerical evidences in this study showed that at optimal conditions, the proposed classes of estimators are more efficient than the stratified regression estimator.

Generally, the proposed classes of estimators fare better than all the modified existing estimators under review for both populations used in the study with appreciable efficiency gains. Consequently, they should be preferred in practice as they provide consistent and more precise parameter estimates.

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