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A new Generalized Nadarajah Haghighi (NGNH) distribution following the Exponential T-X (ETX) family was studied. Structural statistical properties and parameter estimation discussed. Simulation study was executed with impressive results. The applicability and usefulness of NGNH was demonstrated using lifetime data sets. The NGNH distribution was compared with other competing distributions.

Keywords: Exponential T-X., New Generalized Nadarajah Haghighi, Momrents, Probability Weighted Moments, Order Statistics, Entropy, Simulation.

1. Introduction

Probability distributions are extensively used in statistics, probability theory, and various fields of science and engineering to model and analyze random phenomena. They form the foundation for many statistical techniques, such as hypothesis testing, estimation, and simulation. Hybridization of distribution has gained research interest in recent time as a result of complexities in natural phenomena. Nadarajah–Haghighi (NH) distribution was developed by Nadarajah and Haghighi (2011) to enlarge or generalize the exponential distribution in order to increase its flexibility. Moreso, the Nadarajah-Haghighi distribution can comfortably serve as a replacement for some distributions like gamma, exponentiated exponential, Burr III and the popular Weibull distributions. Different authors have studied in details some extensions of the Nadarajah-Haghighi distribution, this includes: Power Inverted Nadarajah-Haghighi distribution (Ahsan-ul-Haq, Ahmend, Albassam and Aslam (2022)) Nadarajah-Haghighi generalized power Weibull (NHGPW) distribution (Abonongo, Luguterah and Nasiru, 2022), Transmuted inverted Nadarajah-Haghighi distribution (Toumaj, Mirmostaface and Hamedani, (2021)), Inverted Nadarajah-Haghighi distribution (Tahir, Corderio, Ali, Dey and Manzoor (2018)), Exponentiated

Generalized Nadarajah-Haghighi Distribution (VedoVatto, Nascimento, et al, 2016), Gamma Nadarajah-Haghighi (GNH) (Ortega et al. (2015)), Exponentiated Nadarajah-Haghighi model (ENH) model (Lemonte, (2013)), etc.

Retrospectively we start by defining the random variable X is said to be a Nadarajah Haghghi random variable if the cumulative distribution function (cdf) and probability density function (pdf) are respectively presented as

$$G(x) = 1 - e^{1-(1+\lambda x)^{\alpha}}; x > 0\ , \tag{1}$$

and $g(x) = \alpha\lambda(1 + \lambda x)^{\alpha-1} e^{1-(1+\lambda x)^{\alpha}}; x > 0,$ (2)

Some standard methods to obtain new families of probability distributions include but not limited to: the method of generator (Eugene, Lee and Famoye, (2002)), method of adding a parameter (Mudholkar and Srivasta (1993)), method of transformation/transmutation (Athayde, Azevedo, Leiva and Antonio (2012)), method of differential equation (Burr,1942), and method of compounding (Adamidi and Loukas (1998)) have been employed which led to some of the following proposed distributions: In this study, we apply the approach proposed by Ahmad, Mahmoudi, Alizadeh, Roozegar, and Afify (2021) which is a method of transformation to obtain a generalized type of the existing distribution thereby improving the characterization and flexibility to the parent distribution. Besides, the kurtosis of the generalized distribution is more flexible and also provides explicit expression for the cdf, survival function and hazard function.

To achieve the purpose of this work which is to proposed a new family of Nadaraja-Haghighi distribution, the cumulative distribution function (cdf) $F(x; B, \eta)$ is presented via the expression

$$F(x; B, \eta) = 1 - \frac{B[1 - G(x; \eta)]}{B - G(x; \eta)}, B > 1, x \epsilon \mathbb{R} \tag{3}$$

and the pdf, $f(x; B, \eta)$ corresponding to (3) is

$$f(x; B, \eta) = \frac{B(B-1)f(x; \eta)}{[B - G(x; \eta)]^2}, x \epsilon \mathbb{R} \tag{4}$$

where, η is a vector of parameters from the baseline distribution

Furthermore, the remaining parts of the paper are scheduled in segments as: Segment 2, development of the new three parameter NGNH distribution and examine some of its mathematical and statistical properties. Segment 3, the maximum likelihood estimation (mle) is explored. Segment 4, the practicability and usefulness of the new model is demonstrated by the using real data sets. Conclusions are presented in segment 5.

2. Development of the new three parameter Nadarajah-Haghighi model and its Properties

This segment introduces the new three parameter New Generalized Nadarajah-Haghighi (NGNH) distribution and some of its reliability properties. Assuming that X follows the baseline cumulative distribution in (1), therefore, the cdf of the NGNH distribution can be written as

$$F(x;B,\eta) = 1 - \frac{B\left[1-\left\{1-e^{1-(1+\lambda x)^{\alpha}}\right\}\right]}{B-\left\{1-e^{1-(1+\lambda x)^{\alpha}}\right\}} = 1 - \frac{Be^{1-(1+\lambda x)^{\alpha}}}{B-\left\{1-e^{1-(1+\lambda x)^{\alpha}}\right\}} \tag{5}$$

where $\eta = (\alpha, \lambda)^T$

The probability density, survival function and hazard rate function of the NGNH are respectively given by

$$f(x;B,\eta) = \frac{B(B-1)\alpha\lambda(1+\lambda x)^{\alpha-1}e^{1-(1+\lambda x)^{\alpha}}}{\left[B-\left\{1-e^{1-(1+\lambda x)^{\alpha}}\right\}\right]^2} \tag{6}$$

$$S(x;B,\eta) = \frac{Be^{1-(1+\lambda x)^{\alpha}}}{B-1+e^{1-(1+\lambda x)^{\alpha}}} \tag{7}$$

$$hrf(x;B,\eta) = \frac{\alpha\lambda(B-1)(1+\lambda x)^{\alpha-1}}{B-1+e^{1-(1+\lambda x)^{\alpha}}} \tag{8}$$

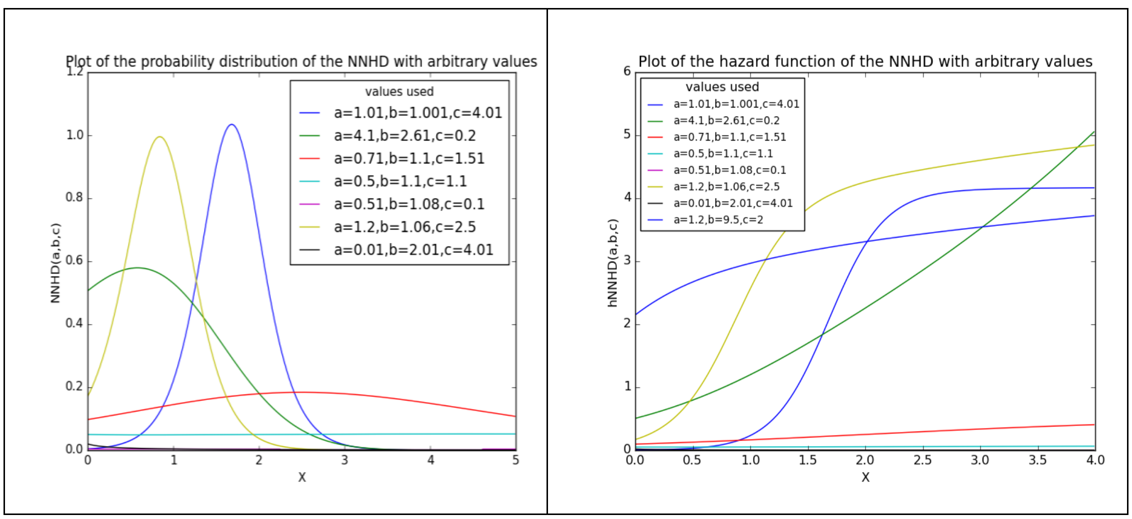

Figure 1: Plots of some possible shapes of the pdf (left) and $hrf(x;B,\eta)$ (right) of the NGNH for some value of parameters

2.1 Linear representation of Density Function

Series expansions can be used to present the probability density function of NGNH in a more elegant manner,

$$f(x;B,\eta)=\frac{B(B-1)\alpha\lambda(1+\lambda x)^{\alpha-1}e^{1-(1+\lambda x)^{\alpha}}}{\left[B-\left\{1-e^{1-(1+\lambda x)^{\alpha}}\right\}\right]^{2}} \tag{9}$$

Using the series

$$\frac{1}{(1-q)^2}=\sum_{n=0}^{\infty}nq^{n-1}$$

$$f(x;B,\eta)=[B^2-B][B-1]^{-2}[\alpha\lambda(1+\lambda x)^{\alpha-1}]\sum_{i=1}^{\infty}n(-1)^{n-1}\left[\frac{1}{B-1}\right]^{n-1}\left[e^{1-(1+\lambda x)^{\alpha}}\right]^{n}$$

2.2 Quantile Function

By inverting equation (5), the quantile function (qf) of the NGNH is computed by inverting $F(x;B,\eta)=p$. $p\sim U(0,1)$ which is a uniform distribution. Consequently, the qf of the NGNH distribution is shown as

$$x_p=\frac{\left\{1-ln\left[\frac{(1-P)(B-1)}{B-1+P}\right]\right\}^{1/\alpha}-1}{\lambda} \tag{10}$$

2.3 Moments

The rth moment of the NGNH distribution is given by,

$$\mu'_r=B[B-1]^{-2}\sum_{m,n,k=0}^{\infty}\frac{(-1)^{r+m+n-1}n^{k-\frac{m}{\alpha}}\Gamma(r+1)\Gamma\left(\frac{m}{\alpha}-k+1\right)}{\lambda^r\Gamma(r+1-m)m!}\left(\frac{1}{B-1}\right)^n\binom{\frac{m}{\alpha}}{k} \tag{11}$$

For $r=1$, we get the first moment of the NGNH distribution given by

$$\mu'_1=B[B-1]^{-2}\sum_{m,n,k=0}^{\infty}\frac{(-1)^{m+n}n^{k-\frac{m}{\alpha}}\Gamma\left(\frac{m}{\alpha}-k+1\right)}{\lambda\Gamma(2-m)m!}\left(\frac{1}{B-1}\right)^n\binom{\frac{m}{\alpha}}{k}$$

For $r=2$ we get the 2nd moment of the NGNH distribution given by

$$\mu'_2=B[B-1]^{-2}\sum_{m,n,k=0}^{\infty}\frac{(-1)^{m+n+1}2n^{k-\frac{m}{\alpha}}\Gamma\left(\frac{m}{\alpha}-k+1\right)\lambda^{-2}}{\lambda^2\Gamma(3-m)m!}\left(\frac{1}{B-1}\right)^n\binom{\frac{m}{\alpha}}{k}$$

For $r=3$ we get the 3rd moment of the NGNH distribution given by

$$\mu'_3=B[B-1]^{-2}\sum_{m,n,k=0}^{\infty}\frac{(-1)^{m+n+2}6n^{k-\frac{m}{\alpha}}\Gamma\left(\frac{m}{\alpha}-k+1\right)}{\lambda^3\Gamma(4-m)m!}\left(\frac{1}{B-1}\right)^n\binom{\frac{m}{\alpha}}{k}$$

For $r=4$ we get the 4th moment of the NGNH distribution given by

$$\mu'_4=B[B-1]^{-2}\sum_{m,n,k=0}^{\infty}\frac{(-1)^{4+m}24n^{k-\frac{m}{\alpha}}\Gamma\left(\frac{m}{\alpha}-k+1\right)}{\lambda^4\Gamma(5-m)m!}\left(\frac{1}{B-1}\right)^n\binom{\frac{m}{\alpha}}{k}$$

2.3.1 Moment Based Coefficient of Variation

The relative spread of data points in series of data around the average is measured by moment based coefficient of variation (CV). Moment based coefficient of variation constitutes the ratio of the standard deviation to the mean, and represented mathematically as $CV = \sigma/\mu$

Using equation, we have,

$$CV = \sqrt{\frac{1}{(\mu_1')^2}(\mu_2' - (\mu_1')^2)} \tag{12}$$

2.3.2 Moment Based Coefficient of Skewness

To know whether there are distortions to the left or right in the symmetry of a distribution, moment based skewness is one way of doing so. It checks the asymmetry of a probability model. Moment based coefficient of skewness is defined mathematically as;

$$Coefficient\ of\ Skewness, CS = \frac{\mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3}{\left[\mu_2' - (\mu_1')^2\right]^{3/2}} \tag{13}$$

2.3.3 Moment Based Coefficient of Kurtosis

The peakedness of densities demonstrates how data values are concentrated around the mean of a distribution. Therefore one can use the moment based coefficient of Kurtosis to determine the heaviness of the tails of a distribution. It is set as ratio of the fourth crude moment μ_4' to the square of the variance, $\mu_2' - (\mu_1')^2$, it is expressed mathematically as;

$$Coefficient\ of\ Kurtosis, CK = \frac{\mu_4' - 4\mu_1'\mu_3' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4}{\left[\mu_2' - (\mu_1')^2\right]^2} \tag{14}$$

2.4 Probability Weighted Moments (PWMs) of the NGNH Distribution

When one intends to generalized the normal moments of a probability distribution the probability weighted moments are superior to standard moment estimates. It can be expressed mathematically as:

$$W_{q,s} = E[x^q F(x)^s] = \int_{-\infty}^{\infty} x^q F(x)^s f(x) dx \tag{15}$$

The PWMs for the NGNH distribution as:

$$W_{q,s} = E[x^q F(x)^s]$$
$$= \int_0^\infty x^q \times [[1 - e^{1-(1+\lambda x)^\alpha}] \sum_{n=0}^{\infty} (-1)^n (B-1)^{-(n+2)} [e^{1-(1+\lambda x)^\alpha}]^n]^s [B^2 - B][\alpha\lambda(1+\lambda x)^{\alpha-1}] \sum_{n=0}^{\infty} (-1)^{n-1} n [\frac{1}{B-1}]^{n-1} [e^{1-(1+\lambda x)^\alpha}]^n dx$$

Simplifying further, we have

$$W_{q,s} = \Phi_w \left[\frac{1}{\lambda}\right]^{q+1} \frac{1}{\alpha(m+ns+n)} (-1)^q \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(q+1)}{\Gamma(q+1-j)\, j!} \left[\frac{1}{(m+ns+n)}\right]^{\frac{j}{\alpha}} \int_{m+n+ns}^{\infty} y^{\frac{j}{\alpha}} e^{-y} dy \tag{16}$$

But the upper incomplete gamma $\Gamma(a,c) = \int_c^\infty x^{a-1} e^{-x} dx$

Therefore, $\int_{m+n+ns}^{\infty} y^{\frac{j}{\alpha}} e^{-y} dy = \Gamma(\frac{j}{\alpha}, (m+n+ns))$

Therefore, the probability weighted moment for the NGNH distribution is expressed as

$$W_{q,s} = \Phi_w \left[\frac{1}{\lambda}\right]^{q+1} \frac{1}{\alpha(m+ns+n)} (-1)^q \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(q+1)}{\Gamma(q+1-j)\, j!} \left[\frac{1}{(m+ns+n)}\right]^{\frac{j}{\alpha}} \Gamma(\frac{j}{\alpha}, (m+n+ns)) \tag{17}$$

where

$$\Phi_w = \frac{B^2 - B}{[B-1]^{-2}} \alpha\lambda \sum_{n,m=0}^{\infty} n(-\frac{1}{B-1})^{ns+n-1} \frac{(-1)^m \Gamma(s+1)}{\Gamma(s+1-m)\, m!} e^{m+ns+n}$$

2.5 Entropy

To measure or quantify the level of uncertainty or average information content associated with the outcomes of that random variable, we derived the Renyi Entropy (Rényi, 1961). Named after Considering the events with highest likelihood to happen, Renyi entropy becomes a comprehensive approach to measure the amount of information while maintaining additivity for independent events.

Renyi Entropy is expressed as:

$$y_\sigma(x) = \frac{1}{1-\sigma} log \int_0^q f^\sigma(x)\, dx, \quad \sigma > 0, \quad \sigma \neq 1 \tag{18}$$

For NGNH distribution, the Renyi-entropy can be indicated as:

$$y_\sigma(x) = \frac{1}{1-\sigma} log \left[[B^2 - B]^\sigma \alpha^\sigma \lambda^\sigma \sum_{p,n,j=0}^{\infty} \frac{(B-1)^{-2\sigma-p}[\sigma+p]^n(-1)^{p+j}\Gamma(n+1)}{j!n!\Gamma(n+1-j)} C_p^{2\sigma+p+1} \times \frac{1}{\lambda[\sigma(\alpha-1)+\alpha j+1]} \left\{ (1+qx)^{\sigma(\alpha-1)+\alpha j+1} - 1 \right\} \right]$$

2.6 Order Statistics

Following the work of Alizadeh, Altun, Afify and Ozel (2019), the pdf of the ith order statistics for the NGNH distribution is given as:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \times [B^2 - B][\alpha\lambda(1+\lambda x)^{\alpha-1} e^{1-(1+\lambda x)^\alpha}][B - 1 + e^{1-(1+\lambda x)^\alpha}]^{-2} \times [[B - 1 + e^{1-(1+\lambda x)^\alpha} - Be^{1-(1+\lambda x)^\alpha}][B - 1 + e^{1-(1+\lambda x)^\alpha}]^{-1}]^{i-1} \times [1 - [B - 1 + e^{1-(1+\lambda x)^\alpha} - Be^{1-(1+\lambda x)^\alpha}][B - 1 + e^{1-(1+\lambda x)^\alpha}]^{-1}]^{n-1} \quad (19)$$

Therefore the pdf of the nth order statistics for the NGNH distribution is given as:

$$f_n(x) = [[B - 1 + e^{1-(1+\lambda x)^\alpha} - Be^{1-(1+\lambda x)^\alpha}][B - 1 + e^{1-(1+\lambda x)^\alpha}]^{-1}]^n$$

Therefore, the pdf of the first order statistics for the NGNH distribution is given as:

$$f_1(x) = n[B^2 - B][\alpha\lambda(1+\lambda x)^{\alpha-1} e^{1-(1+\lambda x)^\alpha}][B - 1 + e^{1-(1+\lambda x)^\alpha}]^{-2} \times [1 - [B - 1 + e^{1-(1+\lambda x)^\alpha} - Be^{1-(1+\lambda x)^\alpha}][B - 1 + e^{1-(1+\lambda x)^\alpha}]^{-1}]^{n-1}$$

3. Maximum Likelihood Estimation (MLE)

Here, consideration is given to the maximum likelihood estimation method of parameter estimation over other parameter estimation approaches because of its desirable properties. The very fact that normal approximation for its estimators for situations when samples become increasingly large can be analytically or numerically handled in an easy manner is another advantage. Here, the unknown parameter estimation for the NGNH for complete samples shall be estimated by only maximum likelihood approach only.

Given $X_1, X_2, \cdots, X_n$ random samples from the NGNH distribution with parameter vector $\vartheta = (\alpha, \lambda, B)^T$. Let the estimator of ϑ be $\hat{\vartheta}$, the log likelihood function for the vector of parameters ϑ can be written as

$$logL(\vartheta|x_i) = mln(\alpha) + mln(\lambda) + mln(B^2 - B) + (\alpha - 1)\sum_{i=1}^{m} ln(1+\lambda x) + \sum_{i=1}^{m}[1 - (1+\lambda x)^\alpha] - 2\sum_{i=1}^{m} ln\left[B - 1 + e^{1-(1+\lambda x)^\alpha}\right] \quad (20)$$

Taking the partial derivate and differentiate w.r.t α, λ, and B

$$\frac{\partial L(\vartheta|x_i)}{\partial\alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m} ln(1+\lambda x) - \sum_{i=1}^{m}(1+\lambda x)^\alpha ln(1+\lambda x) - 2\sum_{i=1}^{m} \frac{e^{1-(1+\lambda x)^\alpha} ln(1+\lambda x)}{B - 1 + e^{1-(1+\lambda x)^\alpha}}$$

$$\frac{\partial L(\vartheta|x_i)}{\partial \lambda} = \frac{m}{\lambda} + (\alpha - 1)\sum_{i=1}^{m}\frac{x}{1+\lambda x} - \alpha\sum_{i=1}^{m} x(1+\lambda x)^{\alpha-1} + 2\sum_{i=1}^{m}\frac{\alpha x e^{1-(1+\lambda x)^{\alpha}}(1+\lambda x)^{\alpha-1}}{B-1+e^{1-(1+\lambda x)^{\alpha}}}$$

$$\frac{\partial L(\vartheta|x_i)}{\partial B} = m\left(\frac{2B-1}{B^2-B}\right) - 2\sum_{i=1}^{m}\frac{1}{B-1+e^{1-(1+\lambda x)^{\alpha}}}$$

To access the maximum likelihood estimates of α, λ, and B.

Table 1. Numerical result for the simulation

n	MSE	parameters	Average	Bias	Variance
100	16.966	α	2.9701	1.6701	14.1771
	0.0254	B	1.1572	0.0572	0.0221
	27.032	λ	1.9267	1.2267	25.5266
200	2.9818	α	1.8331	0.5331	2.6976
	0.0116	B	1.1318	0.0318	0.0106
	2.5435	λ	1.0272	0.3272	2.4364
350	0.9186	α	1.5369	0.2369	0.8625
	0.0052	B	1.1163	0.0163	0.0050
	0.5008	λ	0.8702	0.1702	0.4718
450	0.6279	α	1.5313	0.2313	0.5744
	0.0043	B	1.1176	0.0176	0.0040
	0.3461	λ	0.7976	0.0976	0.3366
600	0.3244	α	1.4367	0.1367	0.3057
	0.0028	B	1.1112	0.0112	0.0027
	0.2272	λ	0.7787	0.0787	0.2210
800	0.1273	α	1.3851	0.0851	0.1200
	0.0019	B	1.1087	0.0087	0.0018

	0.1406	λ	0.7537	0.0537	0.1378
1000	0.0740	α	1.3438	0.0438	0.0721
	0.0013	B	1.1038	0.0038	0.0012
	0.0933	λ	0.7525	0.0525	0.0905

From the above table, we could see from the Average column that for the parameters α, λ, while n was set as 100, the value deviated hugely from the original value used for the simulation compared to when n was set as 200. It is also evident that as n gets larger, the value gets closer to the values used for the simulation.

For the column of bias, variance, and MSE, it is evident that the values when n was set as 100 deviate largely from zero, but as n gets larger, the values tend to zero

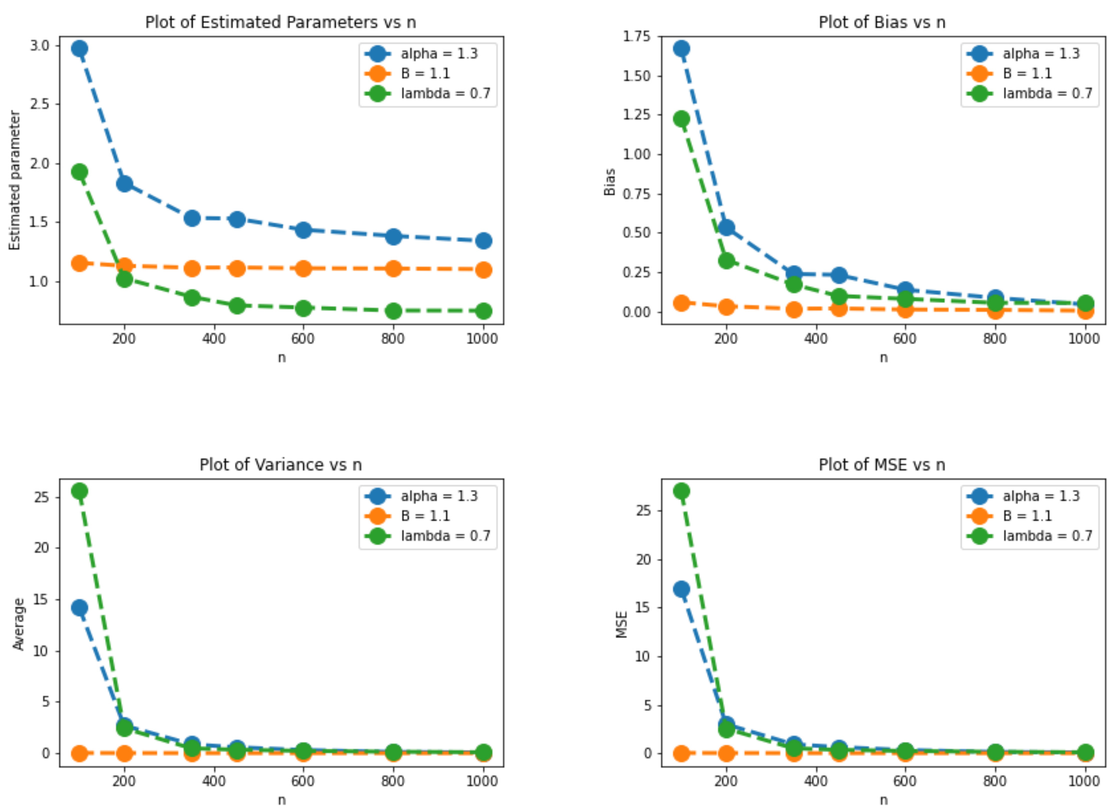

Figure 2. Visual exhibition of the simulation results of average, bias, variance and mean square error (MSE).

4. Applications

Applicability and flexibility of the proposed NGNH model can be demonstrated numerically and graphically by using two real data sets. The fit of NGNH model is compared with other model fits such as (i) Nadarajah Haghighi distribution (NH) (Nadarajah and Haghigh (2011)), (ii) inverted Nadarajah Haghighi (INH) (Tahir, Cordeiro, Ali, Dey and Manzoor (2018)), (iii) Power inverted Nadarajah Haghighi (PINH), (iv) weighted Nadarajah Haghighi (WNH), (v) transmuted Inverted Nadarajah Haghighi (TINH) (Toumaj, Mirmostatae and Hamedani (2021)), (vi) Exponential (Exp) and (vii) two parameter Burr Hatke Exponential (2BHE) (Gao and Gui (2023)) distributions represented as

$$(i) f(x) = \alpha\lambda(1+\lambda x)^{\alpha-1} exp(1-(1+\lambda x)^{\alpha})$$

$$(ii) f(x) = \frac{\alpha\gamma}{x^2}\left(1+\frac{\gamma}{x}\right)^{\alpha-1} exp\left(1-\left(1+\frac{\gamma}{x}\right)^{\alpha}\right)$$

$$(iii)\ f(x) = \alpha\delta\beta x^{-\alpha-1}\left(1+\frac{\beta}{x^{\delta}}\right)^{\alpha-1} exp\left(1-\left(1+\frac{\beta}{x^{\delta}}\right)^{\alpha}\right)$$

$$(iv) f(x) = \frac{2\alpha\pi(1+\pi x)^{\alpha-1} exp(1-(1+\pi x)^{\alpha})}{[1+exp(1-(1+\pi x)^{\alpha})]^2}$$

$$(v)\ f(x) = \frac{\delta\beta}{x^2}\left(1+\frac{\beta}{x}\right)^{\delta-1} exp\left(1-\left(1+\frac{\beta}{x}\right)^{\delta}\right)\left\{1+\lambda-2\lambda exp\left(1-\left(1+\frac{\beta}{x}\right)^{\delta}\right)\right\}$$

$$(vi)\ f(x) = \lambda exp(\lambda x)$$

$$(vii) f(x) = \alpha\tau exp(-\alpha\tau x)\frac{2+\tau x}{(1+\tau x)^{\alpha+1}}$$

Data set 1

Complete data set presented by Nichols and Padgett on the breaking stress of carbon fibers (in Gba) (see Ahmed, Nofal, Osman, 2021)

Table 2. Complete data set presented by Nichols and Padgett on the breaking stress of carbon fibers (in Gba)

3.60, 3.11 , 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 3.70, 2.74, 2.73, 2.50, 1.89, 2.88, 2.82, 2.05, 4.38, 2.93, 3.22, 3.39, 2.81, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03

Table 3. Complete data set presented by Nichols and Padgett on the breaking stress of carbon fibers.

Model	MLE(Standard Error)	L	C-von-M	AD	K-S	P-value	AIC	CAIC	BIC	HQIC
NGNH	$\alpha = 1.87(1.02)$ $B = 1.01(0.01)$ $\lambda = 0.48(0.47)$	**84.81**	**0.03**	**0.25**	**0.06**	**0.96**	**175.63**	**176.02**	**182.20**	**178.23**
NH	$\alpha = 39.75(30.24)$ $\lambda = 0.006(0.00)$	114.74	0.13	0.71	0.33	0.00	233.49	233.68	237.87	235.22
INH	$\alpha = 0.90(0.35)$ $\gamma = 2.00(0.58)$	132.25	0.96	5.38	0.34	0.00	268.62	268.68	272.87	270.22
PINH	$\alpha = 71.60(31.25)$ $\theta = 0.00(0.00)$	134.33	0.24	1.30	0.37	0.00	272.65	272.84	277.03	274.38
WNH	$\alpha = 0.01(0.00)$ $\pi = 33.06(18.53)$	102.28	0.08	0.49	0.27	0.00	208.57	208.76	212.95	210.30
TINH	$\lambda = 14.15(8.55)$ $\delta = 2.01(0.74)$ $\beta = 1.36(0.39)$	108.24	0.66	3.73	0.19	0.01	222.48	222.87	229.05	225.07
Exp	$\lambda = 0.36(0.04)$	132.99	0.24	1.33	0.35	0.00	267.98	268.05	270.17	268.85
2BHE	$\alpha = 71.60(31.25)$ $\tau = 0,004(0.002)$	134.32	0.24	1.30	0.36	0.00	272.65	272.84	277.03	274.38

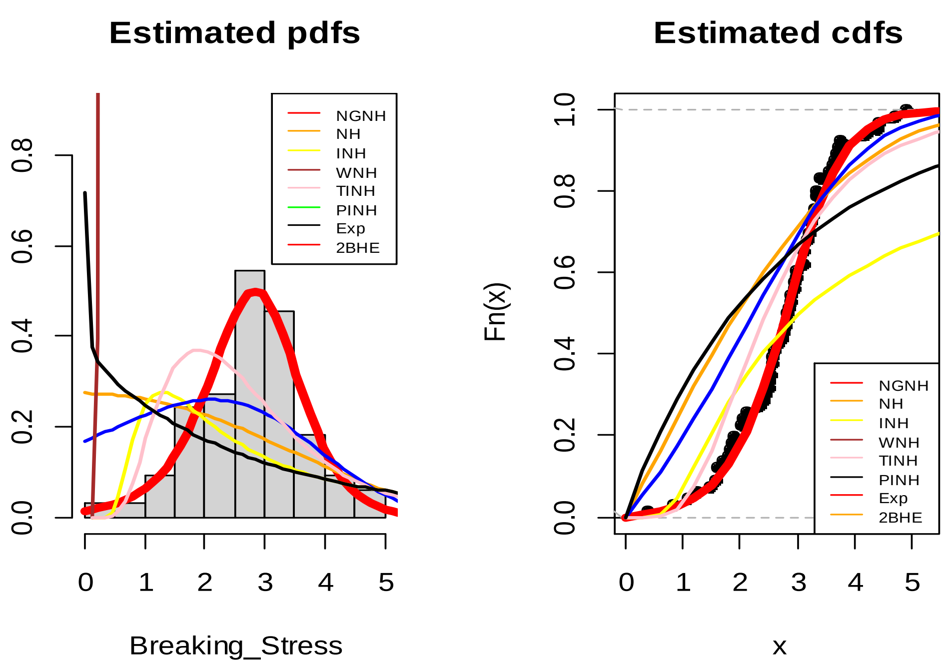

Figure 3. Graphical representation of estimated pdfs and estimated cdfs for breaking stress of Carbon fibers and model fits.

Data Set 2

Table 4. Random sample of 128 Bladder Cancer Patients showing the remission times of Cancer patients as reported in Lee and Wang (2003).

4.980, 6.97, 9.020, 13.290, 0.40, 2.260, 3.57,5.06, 7.090, 9.22,13.80, 25.740, 0.50, 2.46, 3.640, 5.09, 7.260, 9.47, 14.240, 25.820, 0.510, 2.54, 3.70, 5.170, 5.41, 7.62,10.750, 16.62, 43.010, 1.19, 2.750, 4.26, 5.410, 7.63, 17.120, 7.87, 11.640, 17.36, 1.40, 3.020, 4.34, 5.710, 7.28, 9.74, 14.760, 26.31, 0.810, 2.62, 3.82, 5.320, 7.32,10.060, 14.77, 32.150, 2.64, 3.88, 5.320, 7.39,10.34, 14.830, 34.26, 0.90 , 2.690, 4.180, 5.340, 7.590, 10.66, 15.960, 36.660, 11.980, 19.13, 1.760, 3.25, 4.50, 6.250, 8.37, 12.02, 2.020, 0.080, 2.09, 3.480, 4.870, 6.940, 8.660, 13.110, 23.63, 0.20, 2.230, 3.52, 46.12, 1.260, 2.83, 4.330, 5.49, 7.660, 11.25, 17.140, 79.05, 1.350, 2.87, 5.620, 1.05, 2.690, 4.23, 7.930, 11.79, 18.10, 1.460, 4.40, 5.850, 8.26, 3.310, 4.51, 6.54, 8.53, 12.030, 20.28, 2.02, 3.360, 6.76, 12.070, 21.730, 2.07, 3.360, 6.930, 8.650, 12.630, 22.690

Table 5. Likelihood estimates, goodness of fit and information criteria for Cancer patient data

Model	**MLE(Standard)**	**LL**	**W***	**A***	**K-S**	**P-value**	**AIC**	**CAIC**	**BIC**	**HQIC**
NGNH	$\alpha = 0.36(0.053)$ $B = 1.03(0.026)$ $\lambda = 9.37(9.973)$	**410.76**	**0.05**	**0.30**	**0.04**	**0.96**	**826.93**	**827.12**	**835.49**	**830.41**
NH	$\alpha = 0.92(0.151)$ $\lambda = 0.12(0.034)$	414.22	0.10	0.61	0.09	0.23	832.45	832.54	830.15	834.76
INH	$\lambda = 10.59(2.32)$ $\theta = 0.50(0.04)$	431.05	0.35	2.28	0/12	0.03	866.11	866.21	871.82	868.43
PINH	$\alpha = 164.57(85.57)$ $\theta = 1,84(0.167)$ $\gamma = 0.249(0.027)$	417.36	0.132	0.921	0.09	0.17	840.73	840.92	849.28	844.20
WNH	$\alpha = 0.29(0.076)$ $\pi = 0.70(0.086)$	413.05	0.09	0.57	0.08	0.37	830.10	830.20	835.80	832.42
TINH	$\lambda = 5.37(1.244)$ $\delta = 0.50(0.068)$ $\beta = -0.86(0.109)$	426.53	0.28	1.78	0.12	0.05	859.08	859.27	867.63	82.55
Exp	$\lambda = 0.10(0.004)$	414.34	0.11	0.71	0.08	0.31	830.68	830.67	833.53	831.84
2BHE	$\alpha = 13.57(5.913)$ $\tau = 0.01(0.0031)$	415.73	0.74	4.71	0.09	0.15	835.46	835.56	841.16	837.78

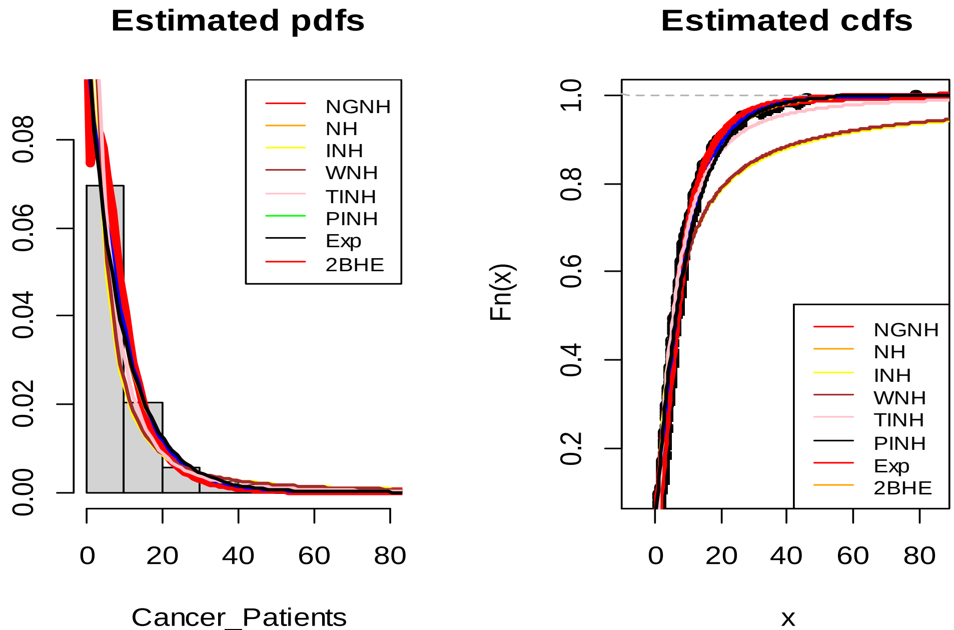

Figure 4: Graphical representation of estimated pdfs fits and estimated cdfs fits for Cancer Patients**.**

The comparative evaluation for the proposed NGNH model with its competing models (NH, INH, PINH, WNHD, TINH, Exp and 2BHE) has been carried out. The MLEs, goodness of fit test and information criteria to buttress the competitiveness of the models are presented in Table 4, and Table 5. The model with the least information criteria values; AIC, CAIC, BIC and HQIC together with the least goodness of fit values; log-likelihood (LL), Cramer Von Mises, Anderson Darling K-S statistics and high P-value of K-S statistics is adjudged the best model for that data set. Graphical illustrations to support our assertion are shown in Figure 3, and Figure 4. It is worthy of note that the proposed model provided a better fit in terms of the information criteria and goodness of fit test and can be seen as a novel distribution in today's class of distributions.

5. Concluding Remarks

A modified Nadarajah Haghighi distribution called the New Generalised Nadarajah Haghighi (NGNH) distribution has been proposed. Some important statistical features of the NGNH has been reported and discussed. We obtained the quantile function which assisted us to carry out the simulation study. The simulation results demonstrated that the models behaviour was impressive. Comparison with other well-known models also proved the usefulness and attractiveness of the NGNH model. It is our hope that since the model is well behaved and had a better fit than the competing models, it will be a veritable tool in the kit of researchers.

Data Availability

Data used in this article can be found within the articles and properly referenced.

Conflict of Interest

We, the authors declare that there is no known conflict of interest.

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