Journal of Modern Applied Statistical Methods

Volume 22 | Issue 2 Article 6

Length Biased Extended Rayleigh Distribution and its Applications

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Received: 08-10-2023; Revised: 14-11-2023; Accepted: 12-12-2023; Published: 17-12-2023.

Recommended Citation

Abdelfattah Mustafa, M. I. Khan (2023). Length Biased Extended Rayleigh Distribution and its Applications. Journal of Modern Applied Statistical Methods, 22(2), https://doi.org/10.56801/Jmasm.V22.i2.6

Journal of Modern Applied Statistical Methods
2023, Vol. 22, No. 2,
Doi: 10.56801/Jmasm.V22.i2.6

ISSN 1538 – 9472

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The generalization of the Rayleigh distribution has been introduced by several authors in recent years. In this article, generalization of Rayleigh distribution has been derived by length- biased transformer technique named length biased Rayleigh-Rayleigh distribution. (LB-RRD). The statistical properties of LB-RRD have been formulated. Estimation of parameter is carried out by method of maximum likelihood. Three data sets are taken to show the superiority of LB-RRD with other competing distributions.

Keywords: Rayleigh- Rayleigh distribution, Moments, Length- Biased and Estimation.

1. Introduction

Recently improvement of classical distribution has become a common phenomenon. Classical distributions have not fitted with the complex data in many situations, so there is a necessary to introduce a new model for better exploration for the results of the complex data. The new model can be formulated by adding an extra parameter or method of generator, or mixture of base distribution. These techniques came into exitance in the last few decades in statistical theory and continue. The goal of introducing any model is to improve the flexibility and adoptability of complex data or to attain better fits to the compared with related distributions**.**

Several methods of generating new model were given in the literature such as, exponentiated (Gupta et al. [1]), Quadratic rank transmutation (Aryall and Tsokos [2]), alpha power transformation (Mahdavi and Kundu [3]). Another important technique for induction of a new distribution was generated by (Cox, [4]) named Length-Biased (*LB*). This method offers a wide range of practical applications in many industries such as reliability, mathematical financiers, actuarial science, biomedical sciences, and survival analysis.

Rayleigh distribution (RD) named after Lord Rayleigh [5] is a very familiar probability model and has a handful of applications in life testing, applied statistics, reliability analysis, and medicines etc. Since its inception numerous authors have demonstrated the importance of RD in different fields of studies. Some notable contributions on Rayleigh distribution are given by Siddiqui [6] Hoffman and karst [7], Howlader [8], Hirano, K. [9], Lalitha and Mishra [10], Bekker et al. [11], Kundu and Raqab [12], Abd-Elfattah et al. [13], Voda [14], Dey [15], Merovci [16, 17], Merovci and Elbatal [18], Mahmoud and Ghazal [19]), Fundi et al. [20], Ateek et al. [21] and cited references therein.

The random variable (r.v.) X has the extended Rayleigh distribution called Rayleigh-RD if its probability density function (PDF) is given by.

$$g(x) = \frac{x^3}{2\beta^4\sigma^2} \exp\left(-\frac{x^4}{8\beta^4\sigma^2}\right), \quad x > 0, \qquad \beta, \sigma > 0, \tag{1}$$

The cumulative density function (CDF) of (1) is.

$$G(x) = 1 - \exp\left(-\frac{x^4}{8\beta^4\sigma^2}\right), \quad x > 0, \qquad \beta, \sigma > 0. \tag{2}$$

The rth moments is.

$$\mu_r' = 8^{\frac{r}{4}}\beta^r\sigma^{\frac{r}{2}}\Gamma\left(\frac{r}{4} + 1\right). \tag{3}$$

This extension of RD and its novelty over baseline distribution is studied Ateek et al. [21] in detail. More details on R- RD readers are referred to Ateek et al. [21].

The paper is organized as follows: Sect. 1 is based on the introduction. Sect. 2 generates the length biased *R-RD* distribution. Some properties of *LB-RRD* are developed in Sect. 3. Maximum likelihood estimation is taken to estimate the parameters in Sect. 4. The performance of *LB-RRD* with other models is verified by considering three real data sets in Sect. 5. The conclusion is reported in the last section.

2. The LB- RRD Model

Since Cox [4] brought up the idea of LB technique for the first time in literature, there has been a significant growth in the study of LB distribution. The LB models arise when the observations reported from a stochastic process are not assigned an equal chance of being reported. Interested Surveys are given in Box and Cox [22], Gupta and Keating [23], Khatree [24], Gupta and Tripathi [25], Kersey and Oluyede [26], Saghir et al. [27], Mudassir and Ahmad [28] and Ekhosuehi et al. [29] and recently among others.

Several authors utilized the LB method to re-extend Rayleigh distribution; for example, Das and Roy [30] introduced LB Rayleigh distributions. Khadim and Hussein [31] proposed LB exponential and Ryleigh distribution, Ajami and

Jahanshahi [32] presented weighted RD, whereas weighted inverse Rayleigh distribution was introduced by Fatima and Ahmad [33], Shakila and Mujahid [34] investigated a new weighted RD. Moreover, Mustafa and Khan [35] proposed LB powered inverse RD.

The L-B family of r.v. X has the PDF as

$$f_X(x) = \frac{X.\, g_X(x)}{E_g(X)}, \quad E_g(X) \neq 0. \tag{4}$$

From (1) and (3) into (4), then the PDF of LB-RR distribution is.

$$f(x) = \frac{x^4}{2^{-\frac{1}{4}}\beta^5\sigma^{\frac{5}{2}}\Gamma\left(\frac{1}{4}\right)} \exp\left(-\frac{x^4}{8\beta^4\sigma^2}\right), \qquad x > 0\,, \qquad \beta, \sigma > 0. \tag{5}$$

The corresponding CDF of (5) is.

$$F(x) = P[X \leq x] = \int_{-\infty}^{x} f(u)du = \frac{1}{2^{-\frac{1}{4}}\beta^5\sigma^{\frac{5}{2}}\Gamma\left(\frac{1}{4}\right)} \int_0^x u^4 \exp\left(-\frac{u^4}{8\beta^4\sigma^2}\right) du$$

Let $t = \frac{u^4}{8\beta^4\sigma^2} \Longrightarrow \; u = (8\beta^4\sigma^2)^{\frac{1}{4}}\, t^{\frac{1}{4}}$ then $du = \frac{1}{4}(8\beta^4\sigma^2)^{\frac{1}{4}}\, t^{-\frac{3}{4}} dt$ and $t: 0 \to \frac{x^4}{8\beta^4\sigma^2}$

$$F(x) = \frac{1}{2^{-\frac{1}{4}}\beta^5\sigma^{\frac{5}{2}}\Gamma\left(\frac{1}{4}\right)} \int_0^{\frac{x^4}{8\beta^4\sigma^2}} 8\beta^4\sigma^2 t e^{-u} \frac{1}{4}(8\beta^4\sigma^2)^{\frac{1}{4}}\, t^{-\frac{3}{4}} dt$$

$$= \frac{4}{\Gamma\left(\frac{1}{4}\right)} \int_0^{\frac{x^4}{8\beta^4\sigma^2}} t^{\frac{1}{4}} e^{-u}\, dt = \frac{4}{\Gamma\left(\frac{1}{4}\right)}\, \Gamma\left(\frac{5}{4}, \frac{x^4}{8\beta^4\sigma^2}\right). \tag{6}$$

Where $\Gamma(k, t)$ is incomplete gamma function.

The reliability function (RF) of LB-RR distribution is

$$R(t) = P[T > t] = 1 - F(t) = 1 - \frac{4}{\Gamma\left(\frac{1}{4}\right)}\, \Gamma\left(\frac{5}{4}, \frac{t^4}{8\beta^4\sigma^2}\right)$$

The failure rate function (HRF) is given as

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{t^4 \exp\left(-\frac{t^4}{8\beta^4\sigma^2}\right)}{2^{-\frac{1}{4}}\beta^5\sigma^{\frac{5}{2}}\left[\Gamma\left(\frac{1}{4}\right) - 4\,\Gamma\left(\frac{5}{4}, \frac{t^4}{8\beta^4\sigma^2}\right)\right]}$$

Figures 1 and 2 show that the LB-RRD is unimodal and hf is increasing for the chosen parameters.

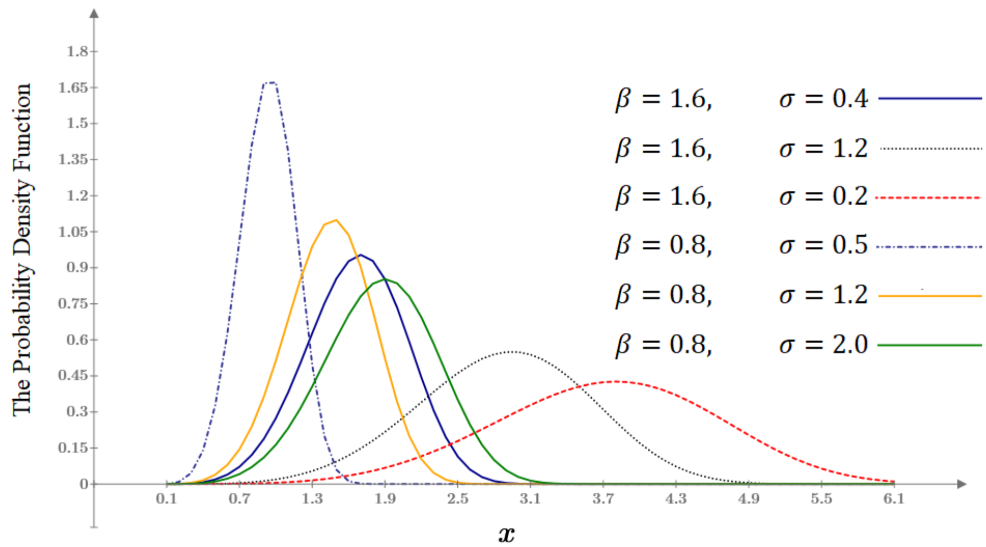

Figure 1. Shapes of density of LB-RRD, for selected values of β and σ.

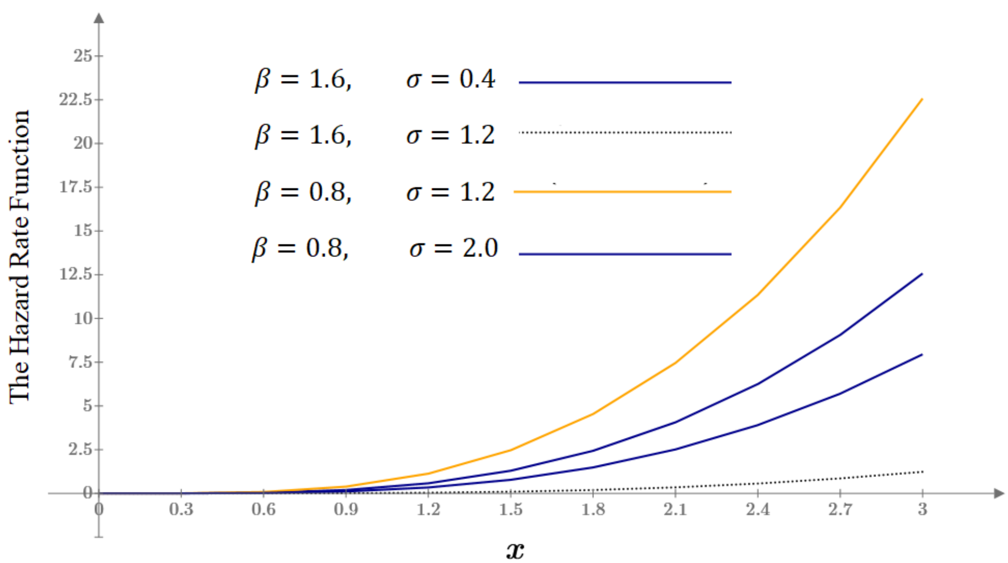

Figure 2. Shapes of h(t) for different values of β and σ.

This study aims to introduce *LBRR* distribution using the length biased method. This model will be more flexible in examining the lifetime data. We are kin to introduce the *(LBRR)* distribution because of:

(1) It unifies well-known lifetime distributions.

(2) It is a unimodal and increasing HRF.

(3) It can be viewed as a good model for fixed skewed data that may not be suitably fitted by other distributions and conclusions extracted from them are shown to be quite comprehensive. It has several uses in industrial reliability, survival analysis, sports and many more.

3. Basic Properties

Some basic properties of the LB-RRD are investigated in Section 3.

Theorem 1. Let $X \sim LBRRD(\beta, \sigma)$, then r^{th} moments of (5) is.

$$\mu_r' = \frac{4(8\beta^4\sigma^2)^{\frac{r}{4}}}{\Gamma\left(\frac{1}{4}\right)}\Gamma\left(\frac{r+1}{4}+1\right). \tag{7}$$

Proof. Since

$$\mu_r' = E[X^r] = \int_0^\infty x^r f(x)dx$$

$$= \frac{1}{2^{-\frac{1}{4}}\beta^5\sigma^{\frac{5}{2}}\Gamma\left(\frac{1}{4}\right)}\int_0^\infty x^{r+4}\exp\left(-\frac{x^4}{8\beta^4\sigma^2}\right)dx$$

Let $u = \frac{x^4}{8\beta^4\sigma^2} \Longrightarrow \ x = (8\beta^4\sigma^2)^{\frac{1}{4}}\, u^{\frac{1}{4}}$.

Then,

$dx = \frac{1}{4}(8\beta^4\sigma^2)^{\frac{1}{4}}\, u^{-\frac{3}{4}}du$ and u: $0 \to \infty$.

Therefore,

$$\mu_r' = E[X^r] = \frac{1}{2^{-\frac{1}{4}}\beta^5\sigma^{\frac{5}{2}}\Gamma\left(\frac{1}{4}\right)}\int_0^\infty \frac{1}{4}(8\beta^4\sigma^2)^{\frac{r+5}{4}}\, u^{\frac{r+1}{4}}e^{-u}\ du$$

$$= \frac{4\ (8\beta^4\sigma^2)^{\frac{r}{4}}}{\Gamma\left(\frac{1}{4}\right)}\int_0^\infty u^{\frac{r+1}{4}}e^{-u}\ du = \frac{4(8\beta^4\sigma^2)^{\frac{r}{4}}}{\Gamma\left(\frac{1}{4}\right)}\Gamma\left(\frac{r+1}{4}+1\right).$$

From (7), the mean of the LB-RR is.

$$\mu = E[X] = \frac{4(8\beta^4\sigma^2)^{\frac{1}{4}}}{\Gamma\left(\frac{1}{4}\right)}\Gamma\left(\frac{1}{2}+1\right) = \frac{2^{\frac{7}{4}}\beta\sqrt{\pi\sigma}}{\Gamma\left(\frac{1}{4}\right)}.$$

Also, from (7), the 2^{nd} moment is.

$$E(X^2) = \frac{4(8\beta^4\sigma^2)^{\frac{2}{4}}}{\Gamma\left(\frac{1}{4}\right)}\Gamma\left(\frac{3}{4}+1\right) = \frac{6\sqrt{2}\beta^2\sigma}{\Gamma\left(\frac{1}{4}\right)}\Gamma\left(\frac{3}{4}\right)$$

Therefore, the variance is.

$$\sigma^2 = E[X^2] - (E[X])^2 = \frac{2\beta^2\sigma\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)}\left[3\Gamma\left(\frac{3}{4}\right) - \frac{4\pi}{\Gamma\left(\frac{1}{4}\right)}\right].$$

The mode of the LB-RR distribution is derived from.

$$\frac{d}{dx}f(x) = 0$$

Then

$$\frac{d}{dx}f(x) = \frac{1}{2^{-\frac{1}{4}}\beta^5\sigma^{\frac{5}{2}}\Gamma\left(\frac{1}{4}\right)}\left[4x^3 e^{-\frac{x^4}{8\beta^4\sigma^2}} - \frac{4x^7}{8\beta^4\sigma^2}e^{-\frac{x^4}{8\beta^4\sigma^2}}\right]$$

$$= \frac{4x^3}{2^{-\frac{1}{4}}\beta^5\sigma^{\frac{5}{2}}\Gamma\left(\frac{1}{4}\right)}\left(1 - \frac{x^4}{8\beta^4\sigma^2}\right)e^{-\frac{x^4}{8\beta^4\sigma^2}} = 0.$$

Then $x = 0$, or $1 - \frac{x^4}{8\beta^4\sigma^2} = 0$, thus the mode is

$$Mode = (8\beta^4\sigma^2)^{\frac{1}{4}} = 2^{\frac{3}{4}}\beta\sqrt{\sigma}.$$

The coefficient of skewness of the LB-RR can be obtained as follows.

$$sk = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3}{\sigma^3} = \frac{\mu_3' - 3\mu_1'\mu_2' + 2{\mu_1'}^3}{\left(\mu_2' - {\mu_1'}^2\right)^{\frac{3}{2}}}.$$

Where μ_r' is the rth moments about the origin and μ_r is the rth moments about the mean.

From (7) we can calculate the coefficient of skewness as follows.

$$sk = \frac{2\left[2\left[\Gamma\left(\frac{1}{4}\right)\right]^2 - 9\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) + 8\pi\sqrt{\pi}\right]}{\left[3\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) - 4\pi\right]^{\frac{3}{2}}} = -0.11$$

Therefore, the LB-RRD is negative skewed and nearly symmetric, $-0.5 < SK < 0.5$. Also, the Sk for LB-RR distribution is independent of the parameters' values.

And the coefficient of kurtosis of the LB-RR can be obtained as follows.

$$ku = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{\mu_4}{\sigma^4} = \frac{\mu_4' - 4\mu_1'\mu_3' + 6{\mu_1'}^2\mu_2' - 3{\mu_1'}^4}{\left(\mu_2' - {\mu_1'}^2\right)^2}$$

From (7) we can calculate the coefficient of kurtosis as follows.

$$ku = \frac{\frac{5}{4}\left(\Gamma\left(\frac{1}{4}\right)\right)^4 - 32\sqrt{\pi}\,\left(\Gamma\left(\frac{1}{2}\right)\right)^2 + 72\pi\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) - 48\pi^2}{\left[3\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) - 4\pi\right]^2} = 2.819$$

The LB-RRD is Platykurtic or short-tailed distribution, $ku < 3.0$. Also, the ku for LB-RR distribution is independent of the parameters' values.

Theorem 2. The moment generating function (mgf) from LB-RR distribution is.

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{4(8\beta^4\sigma^2)^{\frac{r}{4}}}{\Gamma\left(\frac{1}{4}\right)} \Gamma\left(\frac{r+1}{4} + 1\right). \tag{8}$$

Proof. The mgf is given by.

$$M_X(t) = E(e^{tX})$$

$$= \int_0^{\infty} e^{tx} f(x,\beta,\sigma)dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x,\beta,\sigma)dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'. \tag{9}$$

Put the expression given in (7) into (9). We obtained the (8).

Similarly, characteristic function (cf) of LB- RRD is given as

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{4(8\beta^4\sigma^2)^{\frac{r}{4}}}{\Gamma\left(\frac{1}{4}\right)} \Gamma\left(\frac{r+1}{4} + 1\right). \tag{10}$$

and cumulant generating function (cgf) is given as

$$K(t) = \log \phi_X(t)$$

$$= \log\left[\sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{4(8\beta^4\sigma^2)^{\frac{r}{4}}}{\Gamma\left(\frac{1}{4}\right)} \Gamma\left(\frac{r+1}{4} + 1\right)\right]. \tag{11}$$

4. Estimation

Let $x_1, x_2, \cdots, x_n$ denote a random sample from the LB-RR distribution, which has the parameters (β, σ). The likelihood function (LF) is.

$$L(\beta,\sigma)=\prod_{i=1}^{n} f(x_i,\beta,\sigma)=\prod_{i=1}^{n}\left[\frac{x^4}{2^{-\frac{1}{4}}\beta^5\sigma^{\frac{5}{2}}\Gamma\left(\frac{1}{4}\right)}\exp\left(-\frac{x^4}{8\beta^4\sigma^2}\right)\right]. \tag{12}$$

The log- LF is.

$$\mathcal{L}=-n\ln\left(2^{-\frac{1}{4}}\Gamma\left(\frac{1}{4}\right)\right)-n\ln\left(\beta^5\sigma^{\frac{5}{2}}\right)+4\sum_{i=1}^{n}\ln(x_i)-\frac{1}{8\beta^4\sigma^2}\sum_{i=1}^{n}x_i^4. \tag{13}$$

The first partial derivatives (FPD) of (13) are as follows.

$$\frac{\partial\mathcal{L}}{\partial\beta}=-\frac{5n}{\beta}+\frac{1}{2\beta^5\sigma^2}\sum_{i=1}^{n}x_i^4,$$

$$\frac{\partial\mathcal{L}}{\partial\sigma}=-\frac{5n}{2\sigma}+\frac{1}{4\beta^4\sigma^3}\sum_{i=1}^{n}x_i^4,$$

The MLEs of β and σ can be obtained by equating the FPD with zero and solve the obtained equations with respect to β and σ.

$$-\frac{5n}{\beta}+\frac{1}{2\beta^5\sigma^2}\sum_{i=1}^{n}x_i^4=0, \tag{14}$$

$$-\frac{5n}{2\sigma}+\frac{1}{4\beta^4\sigma^3}\sum_{i=1}^{n}x_i^4=0. \tag{15}$$

From (14) and (15) we have

$$10\beta^4\sigma^2=\mu_4'. \tag{16}$$

This equation has no closed form solution in β and σ. To use a numerical program system to achieve its solution with respect to parameters.

The MLE estimators are asymptotically normally distributed with multivariate normal distribution. We derive asymptotic confidence intervals (C.I.) of unknown parameters using variance covariance(V-C) matrix V**,** see, Lawless [36].

$$\left(\hat{\beta},\hat{\sigma}\right)\sim N_2(\Theta,V),$$

where $\Theta=(\beta,\sigma)$ and V is

$$\begin{pmatrix}-\frac{\partial^2\mathcal{L}}{\partial\beta^2} & -\frac{\partial^2\mathcal{L}}{\partial\beta\partial\sigma}\\ -\frac{\partial^2\mathcal{L}}{\partial\sigma\partial\beta} & -\frac{\partial^2\mathcal{L}}{\partial\sigma^2}\end{pmatrix}^{-1}_{\Theta\to\hat{\Theta}}.$$

where,

$$\frac{\partial^2 \mathcal{L}}{\partial \beta^2} = \frac{5n}{\beta^2} - \frac{5}{2\beta^6\sigma^2}\sum_{i=1}^{n} x_i^4,$$

$$\frac{\partial^2 \mathcal{L}}{\partial \sigma \partial \beta} = -\frac{1}{\beta^5\sigma^3}\sum_{i=1}^{n} x_i^4,$$

$$\frac{\partial^2 \mathcal{L}}{\partial \sigma^2} = \frac{5n}{2\sigma^2} - \frac{3}{4\beta^4\sigma^4}\sum_{i=1}^{n} x_i^4,$$

Large sample $(1-\delta)100\%$ C. I. for $\Theta = (\beta, \sigma)$ has the following explanation.

$\hat{\beta} \pm z_{\frac{\delta}{2}}\sqrt{var(\hat{\beta})}$, and $\hat{\sigma} \pm z_{\frac{\delta}{2}}\sqrt{var(\hat{\sigma})}$

where $z_{\delta/2}$ is the upper $100\frac{\delta}{2}$-th percentile of $N(0,1)$, and $var(\widehat{\Theta}_i)$ is the i^{th} element in V diagonally.

5. Real Data Applications

The purpose of a new model is to surge its adaptability and suitability, which confirms its usefulness in various domain of studies, particularly in lifetime analysis. In this section, we have explored the comparative performance of LB-RRD with two existing distributions: Rayleigh distribution (RD) by Lord Rayleigh [5], An extension of Rayleigh distribution (RRD) by Ateek et al. [21].

Goodness of fit measures as follows:

- Akaike information criterion ($\Delta_1 = AIC$) by Akaike [37]

  $\Delta_1 = 2m - 2\ell,$

- Akaike information criterion with correction ($\Delta_2 = AICC$) by Hurvich and Tsai [38].

  $$\Delta_2 = \Delta_1 + \frac{2m(m+1)}{k-m+1},$$

- Bayesian information criterion ($\Delta_3 = BIC$) by Schwarz [39].

  $\Delta_3 = n\ln(k) - 2\ell,$

- Hannan-Quinn information criterion ($\Delta_4 = HQIC$) by Hannan and Quinn [40]

  $\Delta_4 = 2n\ln[\ln(k)] - 2\ell,$

- Kolmogorov Smirnov ($\Delta_5 = K - S$) statistic,

  $\Delta_5 = sup_x\left|F_k(x) - \hat{F}(x)\right|,$

- The determination coefficient ($\Delta_6 = R^2$):

$$\Delta_6 = \frac{\sum_{j=1}^{k}\left(\hat{F}(x_j) - \bar{F}(x_j)\right)^2}{\sum_{j=1}^{k}\left(\hat{F}(x_j) - \bar{F}(x_j)\right)^2 + \sum_{j=1}^{k}\left(F_k(x_j) - \hat{F}(x_j)\right)^2}$$

- The root mean square error ($\Delta_7 = RMSE$):

$$\Delta_7 = \left[\frac{1}{k}\sum_{j=1}^{k}\left(F_k(x_j) - \hat{F}(x_j)\right)^2\right]^{\frac{1}{2}}$$

have been used, where m represents the number of parameters and k represents the sample size, $\hat{F}(x)$ is estimated CDF and $F_k(x)$ is the empirical distribution function,

$$\bar{F}(x) = \frac{1}{k}\sum_{j=1}^{k}\hat{F}(x_j), \qquad F_k(x) = \frac{1}{k}\sum_{j=1}^{k} I\left(x_{(j)} \leq x\right)$$

and

$$I\left(x_{(j)} \leq x\right) = \begin{cases} 1, & \text{if } x_{(j)} \leq x \\ 0, & \text{otherwise} \end{cases}$$

If the proposed model has the lowest value of goodness of fit measures among all fitted model, then this new model can be opted as the best fit.

Dataset 1: The following data is extracted from Nicholas and Padgett [41]. These are breaking stress of carbon fibers of 50 mm length (GPa).

0.39	0.85	1.08	1.25	1.47	1.57	1.61	1.61	1.69	1.80	1.84
1.87	1.89	2.03	2.03	2.05	2.12	2.35	2.41	2.43	2.48	2.50
2.53	2.55	2.55	2.56	2.59	2.67	2.73	2.74	2.79	2.81	2.82
2.85	2.87	2.88	2.93	2.95	2.96	2.97	3.09	3.11	3.11	3.15
3.15	3.19	3.22	3.22	3.27	3.28	3.31	3.31	3.33	3.39	3.39
3.56	3.60	3.65	3.68	3.70	3.75	4.20	4.38	4.42	4.70	4.90

Table 1 describes MLEs and goodness of fit measures for dataset 1.

Table 1. MLEs and goodness of fit measures

Models	MLEs		Goodness of fit measures								
	$\hat{\beta}$	$\hat{\sigma}$	$\mathcal{L}$	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	p -value
RD	2.049	-	-98.208	198.417	198.479	200.606	199.282	0.227	0.760	0.119	0.002
RRD	0.966	2.897	-96.498	196.995	197.186	201.375	198.726	0.628	0.347	0.378	1.01E-23
LB-RR	1.322	1.761	-91.602	187.204	187.394	191.583	188.934	0.117	0.966	0.057	0.314

It concluded that we get a more significant result as compared to others. Hence, LB-RR distribution is best fit for the considered data.

The V-C matrix is given as

$$V = \begin{pmatrix} 1.216 \times 10^4 & -3.241 \times 10^4 \\ -3.241 \times 10^4 & 8.636 \times 10^4 \end{pmatrix}$$

Then the 95% C. I. for β and σ for LB-RR distribution are (0, 217.475) and (0, 577.756), respectively.

It is shown that the LF has a unique solution by Figures 3-5.

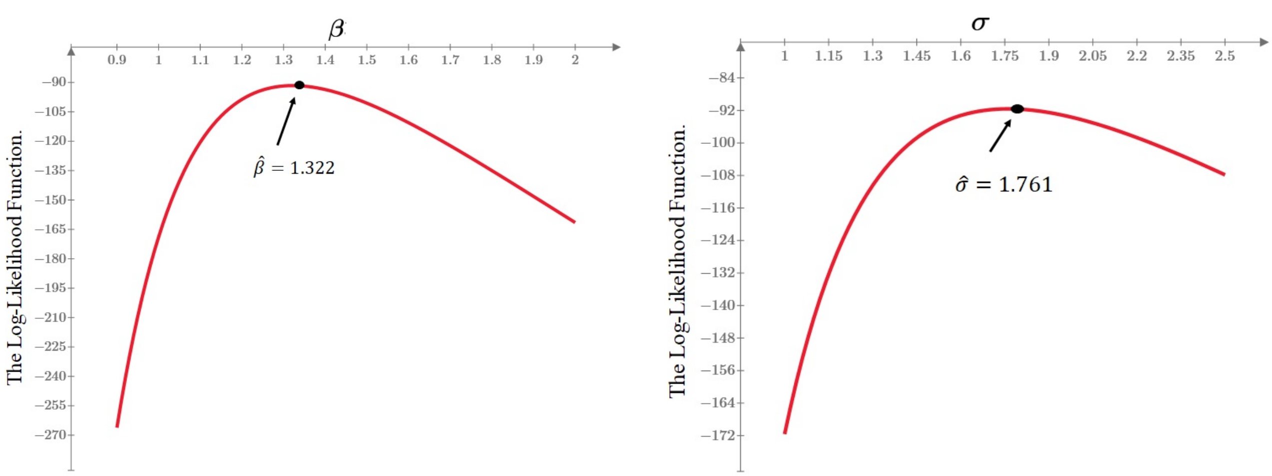

Figure 3: The depiction of log-LF.

Dataset 2: The second data set is taken by workers at the UK National Physical Laboratory on the strengths of 1.5 cm glass fibers. These data are also analyzed by Smith and Naylor [42] and Bourguignon et al. [43]. The values are as follows:

0.55	0.74	0.77	0.81	0.84	0.93	1.04	1.11	1.13	1.24	1.25	1.27
1.28	1.29	1.3	1.36	1.39	1.42	1.48	1.48	1.49	1.49	1.5	1.5
1.51	1.52	1.53	1.54	1.55	1.55	1.58	1.59	1.6	1.61	1.61	1.61
1.61	1.62	1.62	1.63	1.64	1.66	1.66	1.66	1.67	1.68	1.68	1.69
1.7	1.7	1.73	1.76	1.76	1.77	1.78	1.81	1.82	1.84	1.84	1.89
2	2.01	2.24									

Table 2 describes MLEs and goodness of fit measures for dataset 2.

Table 2. MLEs and goodness of fit measures

Models	**MLEs**		**Goodness of fit measures**								
	$\hat{\beta}$	$\hat{\sigma}$	$\mathcal{L}$	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	p -value
RD	1.089	--	-49.791	101.582	101.647	103.725	102.425	0.334	0.415	0.188	9.851E-07
RRD	1.545	0.375	-20.839	45.678	45.878	49.965	47.364	0.664	0.243	0.416	2.434E-25
LB-RR	1.432	0.391	-18.118	40.235	40.435	44.522	41.921	0.213	0.890	0.088	0.006

It has been indicated from Table 2 that LB-RR distribution is a better fit than other distributions for dataset 2.

The V-C matrix is given as

$$V = \begin{pmatrix} 3.516 \times 10^3 & -1.919 \times 10^3 \\ -1.919 \times 10^3 & 1.048 \times 10^3 \end{pmatrix}$$

Then the 95% C. I. for β and σ for LB-RR distribution are (0, 117.645) and (0, 63.841), respectively.

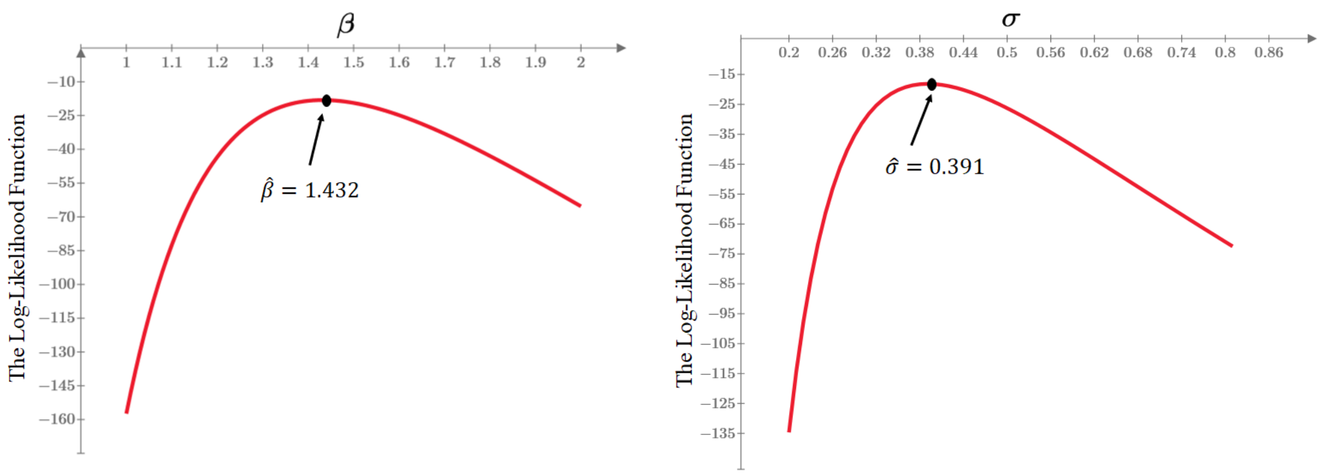

Figure 4: The depiction of log-LF.

Dataset 3: The real data is recorded for 100 Australian female athletes (height in cm). See (Raid et al. [44]).

148.9	149.0	156.0	156.9	157.9	158.9	162.0	162.0	162.5	163.0
163.9	165.0	166.1	166.7	167.3	167.9	168.0	168.6	169.1	169.8
169.9	170.0	170.0	170.3	170.8	171.1	171.4	171.4	171.6	171.7
172.0	172.2	172.3	172.5	172.6	172.7	173.0	173.3	173.3	173.5
173.6	173.7	173.8	174.0	174.0	174.0	174.1	174.1	174.4	175.0
180.2	180.5	180.5	180.9	181.0	181.3	182.1	182.7	183.0	183.3
177.3	177.5	177.5	177.8	177.9	178.0	178.2	178.7	178.9	179.3
175.0	175.0	175.3	175.6	176.0	176.0	176.0	176.0	176.8	177.0
183.3	184.6	184.7	185.0	185.2	186.2	186.3	188.7	189.7	193.4
179.5	179.6	179.6	179.7	179.7	179.8	179.9	180.2	177.3	195.9

Table 3 describes MLEs and goodness of fit measures for dataset 3.

Table 3. MLEs and goodness of fit measures

Models	MLEs		Goodness of fit measures								
	$\hat{\beta}$	$\hat{\sigma}$	$\mathcal{L}$	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	p -value
RD	123.593	--	-547.265	1097	1097	1099	1098	0.529	0.015	0.285	2.552E-25
RRD	15.747	43.746	-479.249	962.499	962.623	967.709	964.608	0.899	0.001	0.544	2.498E-72
LB-RR	29.256	11.336	-466.966	937.931	938.055	943.142	940.04	0.430	0.092	0.242	2.238E-11

Table 3 reveals that LB-RR distribution can be taken as a best fit.

The V-C. Matrix is given as

$$V = \begin{pmatrix} 1.538 \times 10^5 & -1.192 \times 10^5 \\ -1.192 \times 10^5 & 9.236 \times 10^4 \end{pmatrix}$$

Then the 95% C. I. for β and σ for LB-RR distribution are (0, 797.908) and (0, 606.994), respectively.

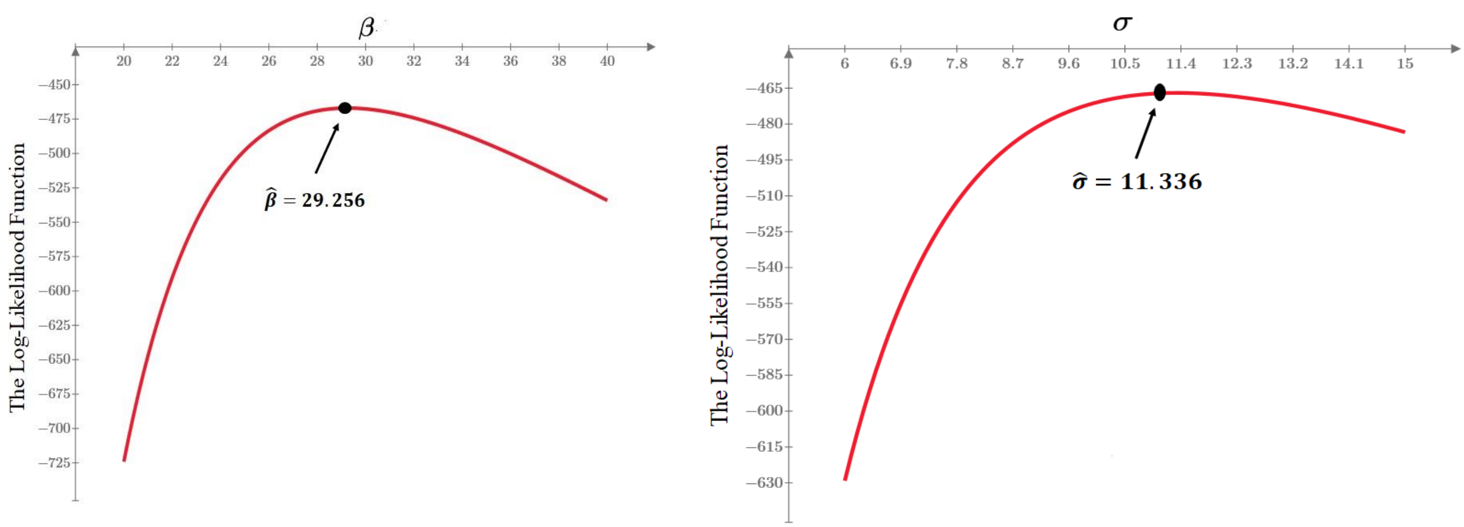

Figure 5: The depiction of log-LF.

6. Conclusion

The LB-RRD has been introduced. The basic statistical characteristics are successfully attained. The model has increasing failure rates depending on numeric value of the parameters. The MLE for complete sample data has been applied to estimate the unknown parameters. Numerical studies are applied for LB-RRD. We notice that the performance of LB-RRD is better than the competitive distributions**.**

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