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This research article is primarily focused on the investigation of Bayes estimators for the scale parameter of the Weibull distribution. Specifically, we focus on a sampling method known as Ranked Set Sample with Unequal Samples (RSSU), as introduced and studied by Bhoj (2001). Our objective is to derive Bayes estimates for the scale parameter utilizing two distinct loss functions: the squared error loss (SEL) function and the linear exponential (LINEX) loss function. We consider the scenario where the scale parameter follows either a gamma prior distribution or a Jeffreys prior distribution. To evaluate the performance of these estimators, we conduct simulations and analyse their bias and mean squared error (MSE). Our findings show that RSSU-based estimators outperform those based on Simple Random Sampling (SRS) and Ranked Set Sampling (RSS), when either Jeffreys prior or gamma prior distribution is used.

Keywords: Bayes estimation, Bias, Conjugate prior, Jeffreys prior, Mean squared error (MSE), Posterior distribution, Ranked set sampling with unequal samples (RSSU).

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1. Introduction

Ranked set sampling (RSS) is acknowledged as a valuable sampling methodology that enhances the precision and efficiency of statistical procedures, particularly when the variable of interest is costly or challenging to measure, yet can be easily and inexpensively ranked. Ranked set sampling (RSS) concept was introduced by McIntyre (1952) for estimating the mean pasture and forage yields when measurement is costly. The mathematical foundation was provided by Takahasi and Wakimoto (1968). Dell and Clutter (1972) studied theoretical aspects of this technique on the assumption of perfect and imperfect judgement rankings. For an extensive review of RSS methods see Chen, Bai, and Sinha (2004) and Al-Omari and Bouza (2014) and for new developments in this area of research see Bouza and Al-Omari (2018) and references therein. The RSS method, as proposed by McIntyre in (1952), can be succinctly summarized as follows: Initially, a set of n random samples is drawn from the population of interest, with each sample consisting of n units. These units within each sample are then ranked based on a variable of interest, using a cost- effective method, such as visual inspection. Subsequently, the smallest and second smallest units from the first and second samples are chosen for actual measurement. This process is iterated until the largest unit from the nth sample is selected for measurement. Consequently, a total of n measured units, completing one cycle, are obtained through this procedure. This cycle can be replicated k times, resulting in the collection of nk units in total, which collectively constitutes the RSS data set.

The Weibull distribution is widely applicable in various fields, such as reliability analysis for assessing factors like electrical circuit voltage breakdown, physics for the study of crystallization, climatology to investigate tides, and cognitive psychology for analysing task completion times. Rinne (2008) and McCool (2012) provide in-depth accounts of the historical background and development of the Weibull distribution. In this paper, our primary focus is on methods for estimating the scale parameter. Estimating this parameter allows us to compare different data sets or populations and determine whether their failure times or event durations exhibit statistically significant differences. This capability is valuable for making inferences about variations in reliability or survival rates among different groups.

In Bayesian frame work, Al-Saleh and Muttlak (1998) have studied Bayesian estimation for exponential and normal distributions to reduce Bayes risk using RSS. Lavine (1999) examined the RSS procedure in some aspects of Bayesian point of view and explored some optimality questions. Kim and Arnold (1999) considered Bayesian estimation under both balanced and generalized RSS. Al-Saleh et al. (2000) studied Bayesian estimation using RSS and found a Bayes estimator of exponential distribution under conjugate prior and gave an application of real data. Sadek et al. (2009) obtained the Bayes estimator of the scale parameter of exponential distribution based on RSS. Amal Helu et al. (2010) have studied Bayes estimators under squared error loss function using sampling schemes namely, RSS and modified ranked set sampling (MRSS) for shape and scale parameter of Weibull distribution and showed the estimators based on RSS and MRSS are better than SRS. Sadek and Alharbi (2014) have obtained the Bayes estimator of the scale parameter of Weibull distribution under squared error loss (SEL) and LINEX loss functions, respectively. They showed the Bayes estimators based on RSS are less biased and more efficient than the corresponding Bayes estimators based on SRS. Mohie et al. (2015) studied the Bayes estimation and prediction for Pareto distribution based on RSS.

In recent years, there have been several modifications, proposed to the traditional ranked set sampling method, specifically addressing situations with varying sample sizes. Notable contributions include the works of Bhoj (2001), Al-Odat and Al-Saleh (2001), and Biradar and Santosha (2014). They have introduced important adaptations like Ranked Set Sampling with Unequal Samples (RSSU), Moving Ranked Set Samples (MERSS), and Maximum Ranked Set Sampling with Unequal Samples (MaxRSSU). Al-Hadhrami and Al-Omari (2012) addressed Bayesian estimation of the mean of the normal distribution using MERSS, while Al-Hadhrami and Al-Omari (2012) studied Bayesian estimation of the mean of the normal distribution using MERSS. Furthermore, Al-Hadhrami and Al-Omari (2014) extended Bayesian estimation to the mean of the exponential distribution by employing MERSS, demonstrating that MERSS estimators outperform SRS estimators. Biradar and Shivanna (2016) also contributed by deriving Bayesian estimators for the scale parameter of the Weibull distribution based on the MaxRSSU. These adaptations find natural applications in various scenarios, such as assessing commuters on different public buses or tracking patients waiting in doctors' waiting rooms with varying group sizes.

Bhoj (2001) proposed a ranked set sampling procedure with unequal set sizes (RSSU) to estimate the population mean, and showed that the estimators based on RSSU are more efficient than the estimators based on SRS, RSS and median ranked set sampling (MRSS, see Muttlak 1997) when the distributions under considerations are symmetrical or moderately skewed. In RSSU, we draw n samples, where the size of the i-th sample is $2i-1$, for $i = 1, 2, \ldots, n$. The steps in RSSU are the same as in RSS. This process is repeated k times in order to get a RSSU of size nk. For one cycle RSSU can be obtained as follows: Let $\{X_{j1}, X_{j2}, \ldots, X_{j(2j-1)}\}$, $j = 1, 2, \ldots, n$, be n sets of random samples from a distribution with pdf $f(x, \theta)$ and cdf $F(x, \theta)$, where θ is an unknown parameter and F is known. Let Y_j denote j-th ordered observation from j-th sample of size $2j-1$, for $j = 1, 2, \ldots, n$. Then $Y_1, Y_2, \ldots, Y_n$ constitute a RSSU of size n, where these n observations are independently distributed. Then Y_j has the same distribution as the j-th order statistic of a SRS of size $(2j-1)$ from $f(x, \theta)$ i.e. the pdf of Y_j is

$$f_j(y, \theta) = [B(j, j)]^{-1} [F(y, \theta)]^{j-1} [1 - F(y, \theta)]^{j-1} f(y, \theta), \quad (1)$$

and cdf

$$F_{j:2j-1}(y, \theta) = \sum_{i=j}^{2j-1} \binom{2j-1}{i} F^i(y, \theta)[1 - F(y, \theta)]^{2j-1-i}.$$

Here $B(j, j) = \frac{(j-1)!(j-1)!}{(2j-1)!}$.

Zhang et al. (2014) proposed sign test based on RSSU. They have provided weighted sign test and it is shown that optimal weighted sign test under RSSU is more efficient than optimal sign test under RSS and MRSS. Biradar and Shivanna (2023) have studied Bayesian estimation of the mean of the exponential distribution by employing RSSU, demonstrating that RSSU estimators outperform SRS estimators. Recently, Biradar (2022) developed maximum likelihood estimators for the location-scale family of distributions based on RSSU. As far as we know, no Bayes estimators based on RSSU for Weibull distribution has been studied, therefore, our main objective of this study is to develop Bayes estimators based on RSSU and explore various properties.

In this paper, Bayes estimator based on SRS is given in Section 2. Bayes estimator based on RSSU is discussed in Section 3. In Section 4, we conduct a simulation study and perform numerical comparisons to evaluate the effectiveness of these proposed estimators. Finally, the paper is concluded in Section 5.

2. Bayes estimates of scale parameter α based on SRS

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (iid)random variables following a Weibull distribution with the probability density function (pdf) given by

$$f(x,\alpha,\beta) = \begin{cases} \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, \; x \geq 0, \; \alpha > 0, \; \beta > 0 \\ 0, \; \text{otherwise} \end{cases} \tag{2}$$

and the cumulative distribution function (cdf) is given by

$$F(x,\alpha,\beta) = 1 - e^{-\alpha x^{\beta}}, \; x \geq 0, \; \alpha > 0, \; \beta > 0.$$

We assume that shape parameter β is known and our objective is to obtain the Bayes estimates of the scale parameter α. When the shape parameter is known, the scale parameter has a conjugate gamma prior distribution with pdf given by

$$\pi(\alpha \mid a,b) = \begin{cases} \frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha}, & \alpha > 0 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

where $a > 0$, $b > 0$, are hyperparameters and $T(a) = \int_0^{\infty} x^{a-1}e^{-x}dx$.

The likelihood function of the observed SRS $\underline{x} = (x_1, \ldots, x_n)$, is

$$L(\alpha|\underline{x}) = \alpha^n\beta^n\left(\prod_{i=1}^{n} x_i\right)^{\beta-1} exp\left[-\alpha\sum_{i=1}^{n} x_i^{\beta}\right], \tag{4}$$

where β is known and α has a prior pdf given in (3).

By combining the prior distribution given in equation (3) and the likelihood function of α given in (4), the posterior distribution of α given the observed SRS sample $\underline{x} = (x_1, \ldots, x_n)$ is denoted as $\pi(\alpha|\underline{x})$, which can be expressed as

$$\Pi(\alpha|\underline{x}) = \frac{\alpha^{(n+\alpha-1)}\, e^{-\alpha\left(\sum_{i=1}^{n} x_i^{\beta}+b\right)}\left(\sum_{i=1}^{n} x_i^{\beta}+b\right)^{(n+a)}}{T(n+a)} \tag{5}$$

It's worth noting that the posterior distribution of α follows a gamma distribution with parameters$(n + a_1 \sum_{i=1}^{n} x_i^{\beta} + b)$. Therefore, the Bayes estimate of α under the squared error loss (SEL) function is

$$\hat{\alpha}_{Sel}(\underline{x}) = \frac{n+a}{\sum_{i=1}^{n} x_i^{\beta}+b} \tag{6}$$

It should be noted that the use of the SEL function is appropriate only when the losses are symmetric. For instance, in the case of estimating the survival function, the symmetric loss function may not be suitable. Therefore, asymmetric loss functions have been explored in the literature. One of the commonly used asymmetric loss functions is the linear exponential (LINEX) loss function, which is a natural extension of SEL function. This loss function was introduced by Varian (1975) and was popularized by Zellner (1986). The LINEX loss function for the parameter α can be expressed as

$$L(\Delta, c) \quad = \quad d(e^{c\Delta} - c\Delta - 1),$$

where $\Delta = (\hat{\alpha} - \alpha)$; $\hat{\alpha}$ is an estimate of α and c ≠ 0, c and d are shape and scale parameters. The sign and magnitude of the shape parameter c indicate that the direction and degree of symmetry. The Bayes estimator of α under the LINEX loss function, denoted by $\hat{\alpha}_{Lnx}$, is the value which minimizes $E[(L(\hat{\alpha} - \alpha)]$ and is given by

$$\hat{\alpha}_{Lnx} = -\frac{1}{c} lnE(e^{-c\alpha}) \tag{7}$$

Therefore, the Bayes estimator of α under LINEX loss function is

$$\hat{\alpha}_{Lnx}(\underline{x}) = -\frac{1}{c}\ln\left[\frac{\int_0^\infty \alpha^{(n+a-1)} e^{-\alpha\left(\sum_{i=1}^n x_i^\beta + b + c\right)} \left(\sum_{i=1}^n x_i^\beta + b\right)^{(n+a)} d\alpha}{\Gamma(n+a)}\right]$$

$$= \left(\frac{n+a}{c}\right) ln\left[\frac{c}{\sum_{i=1}^n x_i^\beta + b}\right] \tag{8}$$

Now consider the Jeffreys prior distribution of α and its pdf is given by

$$\pi(\alpha) \propto \frac{1}{\alpha},\ \alpha > 0 \tag{9}$$

Then we can obtain posterior density of α given the SRS sample as

$$\pi(\alpha \mid \underline{x}) = \frac{\alpha^{n-1} e^{-\alpha \sum_{i=1}^n x_i^\beta} \left(\sum_{i=1}^n x_i^\beta\right)^n}{\Gamma(n)}$$

We can observe that the posterior distribution of α is a gamma distribution with parameters $(n, \Sigma_i^n = 1\, x_i^\beta)$. The Bayes estimator of α with Jeffreys prior distribution under SEL function is

$$\hat{\alpha}_{Sel}^{j}(\underline{x}) = \frac{n}{\sum_{i=1}^n x_i^\beta} \tag{10}$$

The Bayes estimator of α under LINEX loss function is given by

$$\hat{\alpha}_{Lnx}^{J}(\underline{x}) = \left(\frac{n}{c}\right) \ln\left[1 + \frac{c}{\sum_{i=1}^n x_i^\beta}\right] \tag{11}$$

3. Bayes estimates of scale parameter α based on RSSU

Suppose that the random variable X has a Weibull distribution with pdf given by (2) with $\theta = (\alpha, \beta)$, then from (1) the pdf of Y_j is

$$f_j(y, \alpha, \beta) = \alpha\beta y^{\beta-1} \sum_{k=0}^{j-1} a_k(j) e^{-\alpha(k+j)y^{\beta}}, \quad y > 0, \alpha > 0, \tag{12}$$

Where $a_k(j) = \binom{j-1}{k}(-1)^k[B(j,j)]^{-1}$, and the shape parameter β is known and the scale parameter α is unknown.

Let $Y_1, \ldots, Y_n$ be RSSU sample of size n from the Weibull distribution with known shape parameter β and unknown scale parameter α, then the likelihood function of the RSSU sample $\underline{y} = (y_1, \ldots, y_n)$ is

$$L(\alpha \mid \underline{y}) = \sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \ldots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] \alpha^n \beta^n e^{-\alpha \sum_{j=1}^{n} (i_j+j) y_j^{\beta}} \tag{13}$$

3.1 Conjugate prior for *α*

We assume that the scale parameter α follows a gamma prior distribution with pdf given by (3). Then the posterior density of α given the RSSU sample becomes

$$\pi(\alpha \mid \underline{y}) = \frac{\alpha^{n+a-1} \sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] e^{-\alpha\left(\sum_{j=1}^{n} (i_j+j) y_j^{\beta} + b\right)}}{\sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] \Gamma(n+a) \left[\sum_{j=1}^{n} (i_j+j) y_j^{\beta} + b\right]^{-(n+a)}}. \tag{14}$$

Thus, the Bayes estimator of the parameter α based on RSSU under the SEL function can be simplified to

$$\hat{\alpha}_{Sel}(\underline{y}) = \frac{(n+a) \sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] \left[\sum_{j=1}^{n} (i_j+j) y_j^{\beta} + b\right]^{-(n+a+1)}}{\sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] \left[\sum_{j=1}^{n} (i_j+j) y_j^{\beta} + b\right]^{-(n+a)}}. \tag{15}$$

In order to obtain the Bayes estimator of α, assuming a gamma prior distribution based on RSSU under the LINEX loss function, we find that

$$E(e^{-c\alpha}) = \frac{\sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] \left[\sum_{j=1}^{n} y_j^{\beta}(i_j+j) + b + c\right]^{-(n+a)}}{\sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] \left[\sum_{j=1}^{n} y_j^{\beta}(i_j+j) + b\right]^{-(n+a)}} \tag{16}$$

Using expression for $E(e^{-c\alpha})$ in the following equation we obtain Bayes estimator for α under the LINEX function

$$\hat{\alpha}_{Lnx}\left(\underline{y}\right) = -\frac{1}{c} \ln[E(e^{-c\alpha})]. \tag{17}$$

3.2 Jeffreys prior for *α*

Now assume that the parameter α follows a Jeffreys prior distribution with its pdf given by (9). Subsequently, the posterior density of α given a RSSU sample $\underline{y}$ can be simplified to

$$\pi(\alpha \mid \underline{y}) = \frac{\sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] \alpha^{n-1} e^{-\alpha\left(\sum_{j=1}^{n} (i_j+j) y_j^{\beta}\right)}}{\sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] \Gamma(n) \left(\sum_{j=1}^{n} (i_j+j) y_j^{\beta}\right)^{-n}}. \tag{18}$$

Considering the Jeffreys prior the Bayes estimator of the parameter α under the SEL function simplifies to

$$\hat{\alpha}_{Sel}^{J}(\underline{y}) = \frac{\sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] n \left(\sum_{j=1}^{n} (i_j+j) y_j^{\beta}\right)^{-(n+1)}}{\sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] \left(\sum_{j=1}^{n} (i_j+j) y_j^{\beta}\right)^{-n}}. \tag{19}$$

Finally, the Bayes estimator of α under the LINEX loss function with Jeffreys prior distribution can be obtained as

$$\hat{\alpha}_{Lnx}^{J}(\underline{y}) = -\frac{1}{c} ln \left[\frac{\sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] \left(\sum_{j=1}^{n} (i_j+j) y_j^{\beta}+c\right)^{-n}}{\sum_{i_1=0}^{0} \sum_{i_2=0}^{1} \cdots \sum_{i_n=0}^{n-1} \left[\prod_{j=1}^{n} a_{i_j}(j) y_j^{\beta-1}\right] \left[\sum_{j=1}^{n} (i_j+j) y_j^{\beta}\right]^{-n}}\right]. \tag{20}$$

4. Simulation study

In this section, we carry out a Monte Carlo simulation to assess and compare the effectiveness of Bayes estimators based on RSSU with their counterparts based on RSS and SRS. For simulation studies we assume that prior distribution of α follows Gamma (1, 0.5). Sadek and Alharbi (2014) have obtained Bayes estimators based RSS sample for the scale parameter α of Weibull distribution with gamma prior and Jeffreys prior when the shape parameter β is known. We generate SRS, RSS and RSSU samples of sizes n = 2(1)5 for 1000 simulation runs from a Weibull distribution with gamma prior and Jeffreys prior distribution for the scale parameter α when the shape parameter β is known. We compute bias and mean squared error (MSE) of theses Bayes estimators for $\alpha = 0.5$ and 1 when, $\beta = .5$ and $c = 1, -1$. We compute bias and MSE of these estimators as under:

1. Bias is computed as bias $(\hat{\alpha}, \alpha) = \bar{\hat{\alpha}} - \alpha$, where $\bar{\hat{\alpha}}$ is the average of the 1000 estimates of α is the value that is used in the simulation.

2. MSE of an estimator is computed from the simulated sample as $\text{MSE}(\hat{\alpha}_j) = \frac{1}{100} \sum_{i=1}^{1000} (\hat{\alpha}_{ij} - \alpha)^2$, where $\hat{\alpha}_{ij}$ is the Bayes estimate of α for the ith simulated data for the jth estimate, j = 1, 2, ..., 12.

3. Let e_1 to e_4 denote relative efficiencies of Bayes estimators of α based on RSS w.r.t. SRS and RSSU w.r.t. SRS respectively, under SEL loss function, can be expressed as

$$e_1 = \frac{MSE_{SRS}(\hat{\alpha}^{J}{}_{Sel})}{MSE_{RSS}(\hat{\alpha}^{J}{}_{Sel})}, e_2 = \frac{MSE_{SRS}(\hat{\alpha}_{Sel})}{MSE_{RSS}(\hat{\alpha}_{Sel})},$$

$$e_3 = \frac{MSE_{SRS}(\hat{\alpha}^J{}_{Sel})}{MSE_{RSSU}(\hat{\alpha}^J{}_{Sel})}, e_4 = \frac{MSE_{SRS}(\hat{\alpha}_{Sel})}{MSE_{RSSU}(\hat{\alpha}_{Sel})}.$$

Similarly, e_5 to e_8 represent relative efficiencies of Bayes estimator of α based on RSS w.r.t. SRS and RSSU w.r.t. SRS respectively, under LINEX loss function, can be expressed as

$$e_5 = \frac{MSE_{SRS}(\hat{\alpha}^J_{Lnx})}{MSE_{RSS}(\hat{\alpha}^J_{Lnx})}, e_6 = \frac{MSE_{SRS}(\hat{\alpha}_{Lnx})}{MSE_{RSS}(\hat{\alpha}_{Lnx})},$$

$$e_7 = \frac{MSE_{SRS}(\hat{\alpha}^J_{Lnx})}{MSE_{RSSU}(\hat{\alpha}^J_{Lnx})}, e_8 = \frac{MSE_{SRS}(\hat{\alpha}_{Lnx})}{MSE_{RSSU}(\hat{\alpha}_{Lnx})}.$$

Note that $e_1 > 1$ implies that Bayes estimator of α based on RSS performs better than the corresponding estimator based on SRS. A similar interpretation can be applied to e_i for $i = 2(1)8$. The numerical results on bias, MSE and relative efficiencies of the Bayes estimators based on SRS, RSS and RSSU are presented in Tables 1 - 6. All the computations has been performed using R software.

Upon examining Tables 1 and 2, it becomes evident that the bias and MSE of the Bayes estimators for the scale parameter α, using RSSU sampling scheme under SEL and LINEX loss functions and adopting either the gamma or the Jeffrey prior distributions, are significantly smaller compared to the corresponding estimators based on SRS. Additionally, the Bayes estimator demonstrates a marginal reduction in bias and MSE compared to the estimators based on RSS, except for the case $n = 2$, where the MSE differs.

Table 1. Bias of the Bayes estimators based on SRS, RSS and RSSU for α = 0.5, (when β = 0.5, a = 1, b = 0.5)

	Bias($\hat{\alpha}_{Sel}$)			Bias($\hat{\alpha}_{Sel}$)				Bias($\hat{\alpha}_{Lnx}$)			Bias($\hat{\alpha}_{Lnx}$)		
	JP			GP				JP			GP		
n	SRS	RSS	RSSU	SRS	RSS	RSSU	c	SRS	RSS	RSSU	SRS	RSS	RSSU
2	0.4591	0.2613	0.2334	0.4909	0.3530	0.3243	1 -1	0.1850 0.5859	-0.5541 1.1260	0.1219 0.3985	0.3139 0.8153	-0.8515 1.6489	0.2271 0.4996
3	0.2487	0.1065	0.1035	0.3420	0.1705	0.1645	1 -1	0.1379 0.4071	0.0675 0.1588	0.0679 0.1485	0.2433 0.5318	0.1308 0.2211	0.1283 0.2089
4	0.1585	0.0734	0.0580	0.2355	0.1143	0.0963	1 -1	0.0914 0.2454	0.0443 0.0858	0.0405 0.0773	0.1733 0.3207	0.0852 0.1282	0.0783 0.1162

5	0.1255	0.0432	0.0376	0.1936	0.0718	0.0635	1 -1	0.0794 0.1831	0.0313 0.0560	0.0270 0.0488	0.1481 0.2536	0.0594 0.0850	0.0526 0.0750

Table 2. MSE of the Bayes estimators based on SRS, RSS and RSSU for α = 0.5, (when β = 0.5, a = 1, b = 0.5)

	MSE($\hat{\alpha}_{Sel}$)			MSE($\hat{\alpha}_{Sel}$)				MSE($\hat{\alpha}_{Lnx}$)			MSE($\hat{\alpha}_{Lnx}$)		
	Jeffreys prior			Gamma prior				Jeffreys prior			Gamma prior		
n	SRS	RSS	RSSU	SRS	RSS	RSSU	c	SRS	RSS	RSSU	SRS	RSS	RSSU
2	1.8610	0.2613	0.4746	0.7696	0.4165	0.3791	1 -1	0.3395 2.4829	0.4758 2.1225	0.1840 1.0394	0.3348 2.7866	0.8793 3.6327	0.2083 1.1047
3	0.4193	0.1179	0.1047	0.3899	0.1368	0.1233	1 -1	0.1849 1.0296	0.0828 0.1931	0.0756 0.1574	0.2179 1.2535	0.0993 0.2026	0.0914 0.1749
4	0.2850	0.1046	0.0446	0.2470	0.1247	0.0530	1 -1	0.1389 0.4709	0.0397 0.0602	0.0376 0.0539	0.1524 0.4577	0.0474 0.0716	0.0445 0.0639
5	0.1584	0.0282	0.0253	0.1696	0.0329	0.0291	1 -1	0.1004 0.2593	0.0249 0.0322	0.0227 0.0285	0.1170 0.2823	0.0288 0.0378	0.0259 0.0330

Table 3. Bias of the Bayes estimators based on SRS, RSS and RSSU for α = 1, (when β = 0.5, a = 1, b = 0.5)

	Bias($\hat{\alpha}_{Sel}$)			Bias($\hat{\alpha}_{Sel}$)				Bias($\hat{\alpha}_{Lnx}$)			Bias($\hat{\alpha}_{Lnx}$)		
	Jeffreys prior			Gamma prior				Jeffreys prior			Gamma prior		
n	SRS	RSS	RSSU	SRS	RSS	RSSU	c	SRS	RSS	RSSU	SRS	RSS	RSSU
2	0.9182	-0.2387	-0.2666	0.6050	-0.1470	-0.1757	1 -1	0.1453 0.5509	-1.0541 0.6260	-0.3781 -0.1015	0.2305 1.3181	-1.3515 1.1489	-0.2729 -0.0004
3	0.4975	-0.3935	-0.3965	0.4778	0.4778	-0.3355	1 -1	0.1369 0.8876	-0.4325 -0.3412	-0.4321 -0.3515	0.2209 1.0272	-0.3692 -0.2789	-0.3717 -0.2911
4	0.3171	-0.4291	-0.4420	0.3422	0.3422	-0.4037	1 -1	0.0883 0.7232	-0.4557 -0.4142	-0.4595 -0.4227	0.1626 0.6499	-0.4148 -0.3718	-0.4217 -0.3838
5	0.2510	-0.4568	-0.4624	0.2939	0.2939	-0.4365	1 -1	0.0864 0.5139	-0.4687 -0.4440	-0.4730 -0.4512	0.1523 0.5115	-0.4406 -0.4150	-0.4474 -0.4250

Table 4. MSE of the Bayes estimators based on SRS, RSS and RSSU for α = 1, (when β = 0.5, a = 1, b = 0.5)

	MSE($\hat{\alpha}_{Sel}$)			MSE($\hat{\alpha}_{Sel}$)				MSE($\hat{\alpha}_{Lnx}$)			MSE($\hat{\alpha}_{Lnx}$)		
	Jeffreys prior			Gamma prior				Jeffreys prior			Gamma prior		
n	SRS	RSS	RSSU	SRS	RSS	RSSU	c	SRS	RSS	RSSU	SRS	RSS	RSSU
2	7.4439	0.4972	0.4911	1.2351	0.3134	0.3048	1 -1	0.6239 2.7370	1.2899 1.2464	0.3121 0.8909	0.3681 3.1726	1.9908 2.2338	0.2312 0.8551
3	1.6772	0.2614	0.2512	0.8381	0.2163	0.2088	1 -1	0.4288 2.6104	0.2653 0.2843	0.2577 0.2588	0.3299 2.8436	0.2185 0.2315	0.2131 0.2160
4	1.1400	0.2490	0.2366	0.5929	0.2023	0.2066	1 -1	0.3550 2.0419	0.2454 0.2244	0.2471 0.2266	0.2771 1.7406	0.2122 0.1934	0.2122 0.1977
5	0.6337	0.2350	0.2357	0.4605	0.2011	0.2057	1 -1	0.2852 1.6544	0.2437 0.2162	0.2457 0.2257	0.2464 1.0991	0.2014 0.1828	0.2033 0.1680

Table 5. Relative Efficiencies of Bayes estimators under SEL function when α = 0.5 and α = 1.

	eff-Jeffreys(Sel)				eff-Gamma(Sel)			
	$\alpha = 0.5$		$\alpha = 1$		$\alpha = 0.5$		$\alpha = 1$	
n	e_1	e_3	e_1	e_3	e_2	e_4	e_2	e_4
2	7.1220	3.9212	14.9716	15.1576	1.8478	2.0300	3.9409	4.0522
3	3.5564	4.0048	6.4162	6.6768	2.8501	3.1623	3.8747	4.0139
4	2.7247	6.3901	4.5783	4.8183	1.9807	4.6604	2.9308	2.8698
5	5.6170	6.26087	2.6966	2.6886	5.1550	5.8282	2.2899	2.2387

From Table 3 we can see that, for the case of scale parameter $\alpha = 1$, the bias values of the Bayes estimator based on RSS under SEL function are smaller than the bias values of the corresponding estimators based on SRS and RSSU when $n = 2, 3$. However, for sample sizes of $n = 4, 5$, the bias values of the estimator using SRS are smaller than the corresponding estimators based on both the ranked set sampling schemes (RSS and RSSU). When $c = 1$ for sample sizes between $2 \leq n \leq 5$ the Bayes estimators based on SRS under LINEX loss function show smaller bias compared to the estimators based on RSS and RSSU schemes. In case of $c = -1$ bias values of estimators based on RSS are smaller than the estimators based on RSSU and SRS schemes except for $n = 2$.

In Table 4, we can observe that for the sample sizes $n = 2, 3$, MSE of the Bayes estimators based on RSSU scheme are smaller than the MSE's of their competing estimators based on RSS and SRS schemes. However, for sample sizes $n = 4, 5$, the MSE's of the Bayes estimators based on both ranked set sampling schemes (RSS and RSSU) are close to each other, and they are significantly smaller than the MSE's of the corresponding estimators based on the SRS scheme.

From Table 5, we can observe that e_1 to e_4 values are larger than 1, which indi- cates that Bayes estimators based on RSS and RSSU outperforms the corresponding estimators based on SRS scheme. Moreover, e_1 and e_3 values are significantly larger than the corresponding values of e_2 and e_4 for $\alpha = .5$, 1 for all values of $2 \le n \le 5$. This indicates that Bayes estimators of α with Jeffreys prior distribution are more efficient than the corresponding estimators with gamma prior distribution under SEL function. Further, we can note that for all values of $2 \le n \le 5$ and $\alpha = 0.5$, 1, except when $n = 5$ and $\alpha = 1$, it holds true that e_3 is greater than e1 and e4 is greater than e_2. This suggests that when utilizing RSSU, the Bayes estimator of α under SEL loss function tends to outperform its counterparts derived from RSS and SRS, except in the specific case where $n = 5$ and $\alpha = 1$.

Table 6. Relative efficiencies of Bayes estimators under LINEX loss function when α = 0.5 and α = 1.

		eff-Jeffreys(Lnx)				eff-Gamma(Lnx)			
		$\alpha = 0.5$		$\alpha = 1$		$\alpha = 0.5$		$\alpha = 1$	
n	c	e_5	e_7	e_5	e_7	e_6	e_8	e_6	e_8
2	1 -1	0.7135 1.1698	1.8451 2.3887	0.4837 2.1959	1.9990 3.0723	0.3808 0.7671	1.6073 2.5224	0.1849 1.4203	1.5923 3.7102
3	1 -1	2.2331 5.3320	2.4452 6.5418	1.6163 9.1834	1.6638 10.0848	2.1943 6.1870	2.3843 7.1673	1.5100 12.2833	1.5482 13.1669
4	1 -1	3.4987 7.8222	3.6932 8.7319	1.4468 9.0994	1.4367 9.0101	3.2151 6.3925	3.4272 7.1601	1.3059 8.9984	1.3058 8.8033
5	1 -1	4.0321 8.0528	4.4229 9.0982	1.1702 7.6522	1.1608 7.3301	4.0625 7.4683	4.5174 8.5545	1.2234 6.0126	1.2120 6.5422

From Table 6 we can examine that $e_7 > e_5$ and $e_8 > e_6$ for all values of $2 \leq n \leq 5$ and α = .5 and $c = -1, 1$ indicates that Bayes estimators based on RSSU scheme under LINEX loss function when the scale parameter follows either gamma or Jef- freys prior distribution outperforms its competing estimators based on RSS and SRS schemes. However, when $\alpha = 1$ and $c = -1, 1$ the same inequality holds true initially but the inequality reverses for $4 \leq n \leq 5$. More specifically when $c = -1$ denoting an exponential increase in the LINEX loss function, the Bayes estimator based on the RSSU and RSS schemes significantly outperforms the estimators based on the SRS scheme.

Table 7. Relative efficiencies of Bayes Estimators based on RSSU w.r.t. RSS under SEL function when α = 0.5 and α = 1.

	eff-Jeffrey$_{(Sel)}$		eff-Gamma$_{(Sel)}$	
	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$
n	e_3/e_1	e_3/e_1	e_4/e_2	e_4/e_2
2	0.5506	1.0124	1.0986	1.0282
3	1.1261	1.0406	1.1095	1.0359
4	2.3453	1.0524	2.3528	0.9792
5	1.1146	0.9970	1.1306	0.97764

Table 8. Relative efficiencies of Bayes estimators baed on RSSU w.r.t.RSS under LINEX loss function when α = 0.5 and α = 1.

		eff-Jeffreys$_{(Lnx)}$		eff-Gamma$_{(Lnx)}$	
		$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$
n	c	e_7/e_5	e_7/e_5	e_8/e_6	e_8/e_6
2	1	2.5859	4.1330	4.2213	8.611
	-1	2.0420	1.3990	3.2884	2.6123
3	1	1.0952	1.0295	1.0864	1.0253
	-1	1.2268	1.0985	1.1584	1.0718
4	1	1.0559	0.9931	1.0652	1.0000
	-1	1.1169	0.9903	1.1205	0.9782
5	1	1.0969	0.9919	1.1120	0.9907
	-1	1.1298	0.9579	1.1455	1.0881

Tables 7 and 8 display the efficiencies of Bayes estimators employing RSSU as opposed to RSS, under the SEL and LINEX loss functions. Notably, the results indicate that, for the majority of cases, the estimators based on RSSU outperform those based on RSS. However, its worth noting an exception to this trend, which occurs when the sample size $n \geq 4$ is and the value of $\alpha = 1$.

We can conclude that Bayes estimators using ranked set sampling (RSS, RSSU) schemes outperform the estimators based on simple random sampling (SRS) scheme. Specifically, the Bayes estimator employing the RSSU scheme demonstrates superior performance compared to estimators based on the RSS scheme in the majority of cases examined in this study.

5. Summary and conclusions

In this paper, Bayesian estimation of the scale parameter of Weibull distribution based on SRS, RSS and RSSU sampling schemes are considered. Bayes estimators are derived under SEL and LINEX loss functions, assuming the scale parameter follows either gamma prior or Jeffreys prior distribution. A simulation study is conducted to compare the different estimators. Bayes estimators of α with Jeffreys prior distribution under SEL loss function based on RSS and RSSU are found to perform better than corresponding Bayes estimators of α with gamma prior distribution under SEL loss function. The Bayes estimator based on RSS and RSSU methods is observed to perform more effectively when using a LINEX loss function with a shape parameter of −1, in comparison to the case when the shape parameter is 1. This paper considered only the case of Bayesian estimation of scale parameter. Therefore, in our upcoming research, we will study Bayesian estimation for Weibull distribution parameters, including cases where the scale and shape parameters are both unknown, using ranked set samples with unequal sample sizes.

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