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Calibration Estimators under Two Auxiliary Variables using Linear and Non-Linear Constraints in Sample Surveys

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The calibration approach based estimators of population parameters improves their efficiency by borrowing strength from available auxiliary variables effectively. In the present article, calibrated estimators of the population mean incorporating two-auxiliary variables under non-linear constraints are proposed and the properties of the estimators are discussed under relevant sampling designs. In addition empirical analysis supported with simulation study is also performed and checked for the superiority of the proposed estimators.

Keywords: Calibration Estimation, Non-linear Constraints, Auxiliary Information, Sampling Designs.

1. Introduction

In sample surveys, the commonly used generalized regression estimator (GREG) incorporates auxiliary information efficiently to enhance the precision of estimates of population parameters e.g., mean, total, ratio, variances, and distribution function. Another frequently used methodology to increase the precision of estimates is "Calibration method of estimation" introduced by Deville and Särndal (1992). Authors have derived modified weights, which are selected in such a way that they minimize the given distance measure between the initial and modified weights satisfying some calibration constraints related to auxiliary variables. It provides a systematic procedure to utilize the auxiliary information efficiently to strengthen the estimates. This has become an important methodological tool now in the statistical literature and accelerated over decades to develop several different statistics to solve real life problems.

Estevao and Särndal (2000) derived calibrated weights by functional form approach. Tracy et al. (2003) suggested the calibration estimators in stratified sampling and double sampling by borrowing strength from available auxiliary information. Subsequently, Singh and Arnab (2006) and Kim et al. (2007) developed the estimators for different population parameters under various sampling designs using different linear constraints. Series of research articles evolved viz. Kim and Park (2010), Singh and Arnab (2011), Singh (2012, 2013), Clement and Enang (2015), Singh and Sedory (2016), Salinas et al. (2018), Enang and Clement (2019), and references there in, among many others are produced in this area. Alam et al. (2020) derived a new calibration estimator for population mean under different sampling designs using a single auxiliary variable and non-linear constraints.

Using single auxiliary variable which is correlated with study variable, many research articles are available to develop calibration estimation technique in the sample survey literature. However, less attention is paid to estimate the character of the study variable incorporating two or multi auxiliary variables and related issues. In this context, previous studies are available for estimation in literature which deals with multi-auxiliary variables. Olkin (1958) has taken multi-auxiliary variables into consideration to introduce the multivariate regression type ratio estimate. Moreover, under single sampling design, Shukla (1965) derived a multivariate regression estimate and also continued it for the double sampling design. Taking advantage of the availability of more than one auxiliary variable, Raj (1965) also proposed a multivariate difference estimator. Recently, Alka et al. (2019) developed a two-step calibration approach for estimating the problem of design weights for two auxiliary variables. The computation problems become very complicated when multivariate cases are considered in real life cause less concern towards this area. Some recent works on calibration method of estimation include Singh et al. (2019), Ozgul et al. (2019), Salinas et al. (2019), Alam and Shabbir (2020), Sisodia and Singh (2020), Wu and Thompson (2020), among many. . Number of studies has been proven the richness of calibration estimation over other available techniques during past couple of decades. In this direction, use of single and multi-auxiliary variables are well taken into the consideration and several dimensions of the estimators have been analyzed under linear constraint defined in literature.

Here, our objective is to develop calibration approach based estimators using two auxiliary variables with linear and non-linear constraints under simple random sampling with replacement (SRSWR), probability proportional to size (PPS) sampling and stratified random sampling designs. These proposed estimators are also checked for their performances using empirical and simulated data.

2. Calibration Estimators: Notations and Terminologies

Suppose that a finite population consists of N units $(\Omega = 1, 2, \dots i, \dots, N)$ from which a random sample s $(s \in \Omega)$ of size n is drawn through a sampling design $p(.)$. It is assumed that the first-order $(\pi_i = P(i \in s)$, and the second-order $(\pi_{ij} = P(i \text{ and} j \in s))$ inclusion probabilities are strictly positive and known. Let us denote the two auxiliary variables as X_1 and X_2 on which the information is assumed to be available and known for every unit associated with the study variable Y. Let $y_i, x_{1i},$ and

x_{2i} represent the values of the i^{th}unit of study and two auxiliary variables respectively and the population total of the auxiliary variables are given as $X_1 = \sum_{i\in\Omega} x_{1i}$and $X_2 = \sum_{i\in\Omega} x_{2i}$ whereas $\bar{y} = \frac{1}{n}\sum_{i\in s} y_i$, $\bar{x}_1 = \frac{1}{n}\sum_{i\in s} x_{1i}$ and $\bar{x}_2 = \frac{1}{n}\sum_{i\in s} x_{2i}$ are the sample mean estimators for study and auxiliary variables considered under study. The aim is to estimate the population mean $\bar{y}$ incorporating information on two auxiliary variables X_1 and X_2 through calibration technique.

An unbiased estimator of population total Y was given by Horvitz and Thompson (1952) as

$$\hat{Y}_{HT} = \sum_{\mathrm{i}\in\mathrm{s}} d_i \mathrm{y_i} \ , \tag{1}$$

where $d_i = \frac{1}{\pi_i}$ are the initial design weights. An improved estimator proposed by Deville and Särndal (1992) is given by

$$\hat{Y}_{ds} = \sum_{\mathrm{i}\in\mathrm{s}} w_i \mathrm{y_i} \tag{2}$$

where w_i are the calibrated weights obtained by minimizing the chi-square distance measure

$$\sum_{\mathrm{i}\in\mathrm{s}} \frac{(w_i-d_i)^2}{q_i d_i} \ , \tag{3}$$

and satisfying the calibration equation

$$\sum_{i\in s} w_i x_i = \sum_{i\in\Omega} X_i. \tag{4}$$

Here q_i used in Eq. (3) are suitable chosen weights which decide the form of estimator. In general, the value of $q_i$is taken as equal i.e. 1, but unequal weights may also be considered [Deville and Särndal (1992)].

The calibrated weights, using the method of Lagrange’s multiplier are obtained a

$$w_i = d_i + \frac{d_i q_i x_i}{\sum_{i\in s} d_i q_i x_i^2}(X - \sum_{i\in s} d_i x_i), \tag{5}$$

and based on these calibrated weights, the improved calibrated estimator is given by Deville and Särndal (1992) as

$$\hat{Y}_{ds} = \sum_{\mathrm{i}\in\mathrm{s}} d_i \mathrm{y_i} + \frac{\sum_{i\in s} d_i q_i x_i \mathrm{y_i}}{\sum_{i\in s} d_i q_i x_i^2}(X - \sum_{i\in s} d_i x_i) = \hat{Y}_{HT} + \hat{\beta}_{ds}(X - \hat{X}_{HT}), \tag{6}$$

where $\hat{\beta}_{ds} = \frac{\sum_{i\in s} d_i q_i x_i \mathrm{y_i}}{\sum_{i\in s} d_i q_i x_i^2}$ is a weighted multiple regression coefficients’ estimator. The form of $\hat{Y}_{ds}$ in Eq. (6) resembles with the GREG estimator of population total using single auxiliary variable.

Next, the expressions for proposed calibration estimators are derived under two auxiliary variables using different sampling designs considered in the study.

2.1 Calibration Estimator under Simple Random Sampling with Replacement (SRSWR) Scheme

The classical unbiased estimator of the population mean under SRS is given by

$$\overline{y}_{srs} = \sum_{i\in s} d_i y_i, \tag{7}$$

where $d_i = \frac{1}{n}$are fixed design weights under SRSWR scheme. The calibration estimator under SRS incorporating information on auxiliary variables $X_1$and$X_2$is considered as

$$\overline{y}_{srs}(c) = \sum_{i\in s} w_i y_i \tag{8}$$

where $w_i$are modified calibrated weights chosen to minimize the chi-square distance function given in Eq.(3), subject to the following linear and non-linear calibration constraints

$$\sum_{i\in s} w_i = 1 , \tag{9}$$

$$\sum_{i\in s} w_i x_{1i} = \bar{X}_1, \tag{10}$$

$$\sum_{i\in s} w_i x_{2i} = \bar{X}_2, \tag{11}$$

$$\sum_{i\in s} w_i^2 (x_{1i} - \bar{x}_1)^2 = S_{x_1}^2, \tag{12}$$

And $\sum_{i\in s} w_i^2 (x_{2i} - \bar{x}_2)^2 = S_{x_2}^2,$ (13)

where $\bar{x}_1$and $\bar{x}_2$ are samples means, while $\bar{X}_1 = \frac{1}{N}\sum_{i\in\Omega} x_{1i}$ and $\bar{X}_2 = \frac{1}{N}\sum_{i\in\Omega} x_{2i}$are the population mean and $S_{x_1}^2 = \frac{1}{N}\sum_{i\in\Omega}(X_{1i} - \bar{X}_1)^2$ and $S_{x_2}^2 = \frac{1}{N}\sum_{i\in\Omega}(X_{2i} - \bar{X}_2)^2$arethe population variance of the auxiliary variables X_1 andX_2respectively. The Lagrange's function is defined as

$$L_1 = \frac{1}{2}\sum_{i\in s}\frac{(w_i - d_i)^2}{q_i d_i} - \lambda_0\left(\sum_{i\in s} w_i - 1\right) - \lambda_1\left(\sum_{i\in s} w_i x_{1i} - \bar{X}_1\right) - \lambda_2\left(\sum_{i\in s} w_i x_{2i} - \bar{X}_2\right) + \frac{\lambda_3}{2}\left(\sum_{i\in s} w_i^2 (x_{1i} - \bar{x}_1)^2 - S_{x_1}^2\right) + \frac{\lambda_4}{2}\left(\sum_{i\in s} w_i^2 (x_{2i} - \bar{x}_2)^2 - S_{x_2}^2\right), \tag{14}$$

where the Lagrange multipliers λ_0, λ_1, λ_2 are linear, as they are attached with the linear constraints [Eqs. (9), (10) and (11)] and λ_3, λ_4 are associated with the non-linear constraints [Eqs. (12) and (13)] in terms ofw_i, so they are non-linear.

For obtaining optimum weights, differentiating Eq. (14) with respect to $w_i$and setting $\frac{\partial L_1}{\partial w_i} = 0$we have,

$$w_i - \frac{d_i}{(1+\lambda_3 q_i d_i (x_{1i} - \bar{x}_1)^2 + \lambda_4 q_i d_i (x_{2i} - \bar{x}_2)^2)} + \lambda_0 \frac{q_i d_i}{(1+\lambda_3 q_i d_i (x_{1i} - \bar{x}_1)^2 + \lambda_4 q_i d_i (x_{2i} - \bar{x}_2)^2)} + \lambda_1 \frac{x_{1i} q_i d_i}{(1+\lambda_3 q_i d_i (x_{1i} - \bar{x}_1)^2 + \lambda_4 q_i d_i (x_{2i} - \bar{x}_2)^2)} + \lambda_2 \frac{x_{2i} q_i d_i}{(1+\lambda_3 q_i d_i (x_{1i} - \bar{x}_1)^2 + \lambda_4 q_i d_i (x_{2i} - \bar{x}_2)^2)} = 0 \tag{15}$$

or, equivalently

$$w_i = A_i + \lambda_0 q_i A_i + \lambda_1 x_{1i} q_i A_i + \lambda_2 x_{2i} q_i A_i \tag{16}$$

where $A_i = \frac{d_i}{(1+\lambda_3 q_i d_i (x_{1i} - \bar{x}_1)^2 + \lambda_4 q_i d_i (x_{2i} - \bar{x}_2)^2)}$.

Here, $\mathrm{w_i}$ is a function of Lagrange multipliers $\lambda_0,\ \lambda_1,\ \lambda_2, \lambda_3$ and λ_4. Since λ_3 and λ_4 are attached with the non-linear constraints, thus the unique solution of λ_3 and λ_4 cannot be obtained by solving the system of equations. So, by solving the system of three linear constraints, the values of $\lambda_0,\ \lambda_1$ and λ_2 can be found after substituting the Eq. (16), i.e.

$$\lambda_0 \sum_{i\epsilon s} \mathrm{q_i} A_i + \lambda_1 \sum_{i\epsilon s} \mathrm{x_{1i}q_i} A_i + \lambda_2 \sum_{i\epsilon s} \mathrm{x_{2i}q_i} A_i = 1 - \sum_{i\epsilon s} A_i \tag{17}$$

$$\lambda_0 \sum_{i\epsilon s} \mathrm{x_{1i}q_i} A_i + \lambda_1 \sum_{i\epsilon s} \mathrm{x}_{1i}^2 \mathrm{q_i} A_i + \lambda_2 \sum_{i\epsilon s} \mathrm{x_{1i}x_{2i}q_i} A_i = \bar{X}_1 - \sum_{i\epsilon s} \mathrm{x_{1i}} A_i \tag{18}$$

$$\lambda_0 \sum_{i\epsilon s} \mathrm{x_{2i}q_i} A_i + \lambda_1 \sum_{i\epsilon s} \mathrm{x_{1i}x_{2i}q_i} A_i + \lambda_2 \sum_{i\epsilon s} \mathrm{x}_{2i}^2 \mathrm{q_i} A_i = \bar{X}_2 - \sum_{i\epsilon s} \mathrm{x_{2i}} A_i \tag{19}$$

or, equivalently

$$\begin{bmatrix} \sum_{i\epsilon s} q_i A_i & \sum_{i\epsilon s} x_{1i} q_i A_i & \sum_{i\epsilon s} x_{2i} q_i A_i \\ \sum_{i\epsilon s} x_{1i} q_i A_i & \sum_{i\epsilon s} x_{1i}^2 q_i A_i & \sum_{i\epsilon s} x_{1i} x_{2i} q_i A_i \\ \sum_{i\epsilon s} x_{2i} q_i A_i & \sum_{i\epsilon s} x_{1i} x_{2i} q_i A_i & \sum_{i\epsilon s} x_{2i}^2 q_i A_i \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 - \sum_{i\epsilon s} A_i \\ \bar{X}_1 - \sum_{i\epsilon s} x_{1i} A_i \\ \bar{X}_{2-} \sum_{i\epsilon s} x_{2i} A_i \end{bmatrix}.$$

On solving the system of equations, we have

$$\lambda_0 = \frac{(d^* f^* - e^{*2})(1 - \sum_{i\epsilon s} A_i) - (b^* f^* - c^* e^*)(\bar{X}_1 - \sum_{i\epsilon s} A_i x_{1i}) + (b^* e^* - c^* d^*)(\bar{X}_{2-} \sum_{i\epsilon s} A_i x_{2i})}{|\mathrm{Z}_1|} = \frac{\Pi_{0s}}{\Pi_s}, \tag{20}$$

$$\lambda_1 = \frac{(c^* e^* - b^* f^*)(1 - \sum_{i\epsilon s} A_i) + (a^* f^* - c^{*2})(\bar{X}_1 - \sum_{i\epsilon s} A_i x_{1i}) - (a^* e^* - b^* c^*)(\bar{X}_{2-} \sum_{i\epsilon s} A_i x_{2i})}{|\mathrm{Z}_1|} = \frac{\Pi_{1s}}{\Pi_s}, \tag{21}$$

$$\lambda_2 = \frac{(b^* e^* - c^* d^*)(1 - \sum_{i\epsilon s} A_i) - (a^* e^* - b^* c^*)(\bar{X}_1 - \sum_{i\epsilon s} A_i x_{1i}) + (a^* d^* - b^{*2})(\bar{X}_{2-} \sum_{i\epsilon s} A_i x_{2i})}{|\mathrm{Z}_1|} = \frac{\Pi_{2s}}{\Pi_s}, \tag{22}$$

where

$$\mathrm{Z}_1 = \begin{bmatrix} a^* & b^* & c^* \\ b^* & d^* & e^* \\ c^* & e^* & f^* \end{bmatrix} = \begin{bmatrix} \sum_{i\epsilon s} q_i A_i & \sum_{i\epsilon s} x_{1i} q_i A_i & \sum_{i\epsilon s} x_{2i} q_i A_i \\ \sum_{i\epsilon s} x_{1i} q_i A_i & \sum_{i\epsilon s} x_{1i}^2 q_i A_i & \sum_{i\epsilon s} x_{1i} x_{2i} q_i A_i \\ \sum_{i\epsilon s} x_{2i} q_i A_i & \sum_{i\epsilon s} x_{1i} x_{2i} q_i A_i & \sum_{i\epsilon s} x_{2i}^2 q_i A_i \end{bmatrix}$$

and $|\mathrm{Z}_1| = a^* d^* f^* - a^* e^{*2} - b^{*2} f^* + 2 b^* c^* e^* - c^{*2} d^*$.

On putting the values of λ_0, λ_1 and λ_2 in Eq.(16), the value of calibrated weights are given by

$$\mathrm{w_i} = A_i + \frac{\Pi_{0s}}{\Pi_s} \mathrm{q_i} A_i + \frac{\Pi_{1s}}{\Pi_s} \mathrm{x_{1i} q_i} A_i + \frac{\Pi_{2s}}{\Pi_s} \mathrm{x_{2i} q_i} A_i \ . \tag{23}$$

These optimum calibrated weights obtained in Eq.(23) rely upon the values of non-linear Lagrange multipliers λ_3 and λ_4 and it can be found by satisfying the following non-linear equations:

$$\sum_{i\epsilon s} \left[A_i + \frac{\Pi_{0s}}{\Pi_s} \mathrm{q_i} A_i + \frac{\Pi_{1s}}{\Pi_s} \mathrm{x_{1i} q_i} A_i + \frac{\Pi_{2s}}{\Pi_s} \mathrm{x_{2i} q_i} A_i \right]^2 (x_{1i} - \bar{x}_1)^2 - S_{x_1}^2 = 0, \tag{24}$$

$$\sum_{i\epsilon s} \left[A_i + \frac{\Pi_{0s}}{\Pi_s} \mathrm{q_i} A_i + \frac{\Pi_{1s}}{\Pi_s} \mathrm{x_{1i} q_i} A_i + \frac{\Pi_{2s}}{\Pi_s} \mathrm{x_{2i} q_i} A_i \right]^2 (x_{2i} - \bar{x}_2)^2 - S_{x_2}^2 = 0, \tag{25}$$

Initially, the wide range values of λ_3 and λ_4 is to be pre-assigned and those values which satisfy Eqs.(24) and (25) will be selected as their optimum values. These optimum values generate optimum calibrated weights in Eq.(23) and based on these weights, the calibrated estimator in Eq.(7) is obtained as,

$$\bar{y}_{srs}(c) = \sum_{i\epsilon s} A_i y_i + \hat{\beta}_{0s}(1 - \sum_{i\epsilon s} A_i) + \hat{\beta}_{1s}(\bar{X}_1 - \sum_{i\epsilon s} A_i x_{1i}) + \hat{\beta}_{2s}(\bar{X}_{2-} \sum_{i\epsilon s} A_i x_{2i}) \quad (26)$$

where $\hat{\beta}_{0s} = \frac{(d^*f^* - e^{*2})\sum_{i\epsilon s} q_i y_i A_i + (c^*e^* - b^*f^*)\sum_{i\epsilon s} q_i x_{1i} y_i A_i + (b^*e^* - c^*d^*)\sum_{i\epsilon s} q_i x_{2i} y_i A_i}{|Z_1|}$,

$\hat{\beta}_{1s} = \frac{(c^*e^* - b^*f^*)\sum_{i\epsilon s} q_i y_i A_i + (a^*f^* - c^{*2})\sum_{i\epsilon s} q_i x_{1i} y_i A_i + (b^*c^* - a^*e^*)\sum_{i\epsilon s} q_i x_{2i} y_i A_i}{|Z_1|}$, and

$\hat{\beta}_{2s} = \frac{(b^*e^* - c^*d^*)\sum_{i\epsilon s} q_i y_i A_i + (b^*c^* - a^*e^*)\sum_{i\epsilon s} q_i x_{1i} y_i A_i + (a^*d^* - b^{*2})\sum_{i\epsilon s} q_i x_{2i} y_i A_i}{|Z_1|}$.

2.2 Calibration Estimator under Probability Proportional to Size Sampling (PPS) Scheme

In PPS sampling design, the probability of selection for units in sample is directly proportional to a given size measure, X which is assumed to be known for all sampling units and highly correlated with the study variable Y. Let, a sample of size $n$is drawn using PPSWR scheme.

An unbiased estimator of the population total $Y = \sum_1^N y_i$is given by Hansen and Hurwitz (1943) as

$$\hat{Y}_{pps} = \sum_{i\in s} d_i y_i \ , \quad (27)$$

where $d_i = \frac{1}{np_i}$ are design weights under PPSWR sampling scheme. The improved calibrated estimator of mean is given as

$$\bar{y}_{pps}(c) = \frac{1}{N}\sum_{i\in s} w_i y_i \ , \quad (28)$$

wherew_i are calibrated weights obtained by minimizing the distance function in Eq.(3) with respect to the following calibration constraints,

$$\sum_{i\in s} w_i = \sum_{i\in s} d_i, \quad (29)$$

$$\sum_{i\in s} w_i x_{1i} = X_1 \ , \quad (30)$$

$$\sum_{i\in s} w_i x_{2i} = X_2, \quad (31)$$

$$N\sum_{i\in s} w_i^2 p_i^\alpha \left(\frac{x_{1i}}{p_i} - \hat{X}_{1,HH}\right)^2 = \frac{1}{n}\sum_i^N P_i \left(\frac{X_{1i}}{P_i} - X_1\right)^2, \quad (32)$$

and $$N\sum_{i\in s} w_i^2 p_i^\alpha \left(\frac{x_{2i}}{p_i} - \hat{X}_{2,HH}\right)^2 = \frac{1}{n}\sum_{i\in U} P_i \left(\frac{X_{2i}}{P_i} - X_2\right)^2, \quad (33)$$

where α is any positive constant, $p_i = \frac{x_i}{X}$, , $i \in s$and $P_i = \frac{X_i}{X}$, $i \in \Omega$ and $\hat{X}_{1,HH}$ and $\hat{X}_{2,HH}$ is the Hansen and Hurwitz (1943) estimator for the two auxiliary totals. The Lagrange's function is defined as

$$L_2 = \frac{1}{2}\sum_{i\in s}\frac{(w_i-d_i)^2}{d_i q_i} - \lambda_5\left(\sum_{i\in s} w_i - \sum_{i\in s} d_i\right) - \lambda_6\left(\sum_{i\in s} w_i\, x_{1i} - X_1\right) - \lambda_7\left(\sum_{i\in s} w_i\, x_{2i} - X_2\right) - \frac{\lambda_8}{2}\left\{N\sum_{i\in s} w_i^2 p_i^{\alpha}\left(\frac{x_{1i}}{p_i} - \hat{X}_{1,HH}\right)^2 - \frac{1}{n}\sum_i^N P_i\left(\frac{X_{1i}}{P_i} - X_1\right)^2\right\} - \frac{\lambda_9}{2}\left\{N\sum_{i\in s} w_i^2 p_i^{\alpha}\left(\frac{x_{2i}}{p_i} - \hat{X}_{2,HH}\right)^2 - \frac{1}{n}\sum_i^N P_i\left(\frac{X_{2i}}{P_i} - X_2\right)^2\right\} \tag{34}$$

On solving $\frac{\partial L_2}{\partial w_i} = 0$ weget,

$$w_i = B_i + \lambda_5 q_i B_i + \lambda_6 x_{1i} q_i B_i + \lambda_7 x_{2i} q_i B_i, \tag{35}$$

where $B_i = \frac{d_i}{1-\lambda_8 N d_i q_i p_i^{\alpha}\left(\frac{x_{1i}}{p_i}-\hat{X}_{1,HH}\right)^2 - \lambda_9 N d_i q_i p_i^{\alpha}\left(\frac{x_{2i}}{p_i}-\hat{X}_{2,HH}\right)^2}.$

Here, the Lagrange multiplier λ_8 and λ_9 are associated with Eqs.(32) and (33) are non-linear in terms ofw_i.Thus,the values of λ_5, λ_6 and λ_7 are to be found by simultaneously solving the system of three linear equations i.e. Eqs.(29), (30) and (31), we have,

$$\lambda_5 = \frac{(j^*l^*-k^{*2})\left(\sum_{i\in s} d_i - \sum_{i\in s} B_i\right) + (i^*k^*-h^*l^*)\left(X_1-\sum_{i\in s} B_i x_{1i}\right) + (h^*k^*-i^*j^*)\left(X_2-\sum_{i\in s} B_i x_{2i}\right)}{|\mathrm{Z}_2|} = \frac{\Pi_{0p}}{\Pi_p}, \tag{36}$$

$$\lambda_6 = \frac{(i^*k^*-h^*l^*)\left(\sum_{i\in s} d_i - \sum_{i\in s} B_i\right) + (g^*l^*-i^{*2})\left(X_1-\sum_{i\in s} B_i x_{1i}\right) + (h^*i^*-g^*k^*)\left(X_2-\sum_{i\in s} B_i x_{2i}\right)}{|\mathrm{Z}_2|} = \frac{\Pi_{1p}}{\Pi_p}, \tag{37}$$

$$\lambda_7 = \frac{(h^*k^*-i^*j^*)\left(\sum_{i\in s} d_i - \sum_{i\in s} B_i\right) + (h^*i^*-g^*k^*)\left(X_1-\sum_{i\in s} B_i x_{1i}\right) + (g^*j^*-h^{*2})\left(X_2-\sum_{i\in s} B_i x_{2i}\right)}{|\mathrm{Z}_2|} = \frac{\Pi_{2p}}{\Pi_p}, \tag{38}$$

where $\mathrm{Z}_2 = \begin{bmatrix} g^* & h^* & i^* \\ h^* & j^* & k^* \\ i^* & k^* & l^* \end{bmatrix} = \begin{bmatrix} \sum_{i\epsilon s} q_i B_i & \sum_{i\epsilon s} q_i x_{1i} B_i & \sum_{i\epsilon s} q_i x_{2i} B_i \\ \sum_{i\epsilon s} q_i x_{1i} B_i & \sum_{i\epsilon s} q_i x_{1i}^2 B_i & \sum_{i\epsilon s} q_i x_{1i} x_{2i} B_i \\ \sum_{i\epsilon s} q_i x_{2i} B_i & \sum_{i\epsilon s} q_i x_{1i} x_{2i} B_i & \sum_{i\epsilon s} q_i x_{2i}^2 B_i \end{bmatrix}$

and $|\mathrm{Z}_2| = g^*j^*l^* - k^{*2}g^* - h^{*2}l^* + 2h^*i^*k^* - i^{*2}j^*$. Now, after solving the system of linear equation and on putting the values of lambdas in Eq. (35), the calibrated weights are obtained as

$$\mathrm{w_i} = B_i + \frac{\Pi_{0p}}{\Pi_p}\mathrm{q_i} B_i + \frac{\Pi_{1p}}{\Pi_p}\mathrm{x_{1i} q_i} B_i + \frac{\Pi_{2p}}{\Pi_p}\mathrm{x_{2i} q_i} B_i. \tag{39}$$

On substituting calibrated weights w_i from Eq.(39) in Eq.(32) and Eq.(33), we get

$$N\sum_{i\epsilon s}\left[B_i + \frac{\Pi_{0p}}{\Pi_p}\mathrm{q_i} B_i + \frac{\Pi_{1p}}{\Pi_p}\mathrm{x_{1i} q_i} B_i + \frac{\Pi_{2p}}{\Pi_p}\mathrm{x_{2i} q_i} B_i\right]^2 p_i^{\alpha}\left(\frac{x_{1i}}{p_i} - \hat{X}_{1,HH}\right)^2 = \frac{1}{n}\sum_i^N P_i\left(\frac{X_{1i}}{P_i} - X_1\right)^2, \tag{40}$$

$N\sum_{i\in s}\left[B_i+\frac{\Pi_{0p}}{\Pi_p}\mathrm{q_i}B_i+\frac{\Pi_{1p}}{\Pi_p}\mathrm{x_{1i}q_i}B_i+\frac{\Pi_{2p}}{\Pi_p}\mathrm{x_{2i}q_i}B_i\right]^2 p_i^{\alpha}\left(\frac{x_{2i}}{p_i}-\hat{X}_{2,HH}\right)^2=$
$\frac{1}{n}\sum_{i\in U}P_i\left(\frac{X_{2i}}{P_i}-X_2\right)^2,$ (41)

Here, we can obtain the optimum range for non-linear Lagrange multiplier i.e. the values of λ_8 and λ_9 by selecting those values which satisfy the Eqs. (40) and (41).The form of calibrated estimator Eq.(28), based on the improved calibrated weights as in Eq. (39) is given by

$\bar{y}_{pps}(c)=\frac{1}{N}\left[\sum_{i\in s}y_iB_i+\hat{\beta}_{0p}(\sum_{i\in s}d_i-\sum_{i\in s}B_i)+\hat{\beta}_{1p}(X_1-\sum_{i\in s}x_{1i}B_i)+\hat{\beta}_{2p}(X_2-\right.$
$\left.\sum_{i\in s}x_{2i}B_i)\right],$ (42)

where,

$\hat{\beta}_{0p}=\frac{(j^*l^*-k^{*2})\sum_{i\epsilon s}q_iy_iB_i+(i^*k^*-h^*l^*)\sum_{i\epsilon s}q_ix_{1i}y_iB_i+(h^*k^*-i^*j^*)\sum_{i\epsilon s}q_ix_{2i}y_iB_i}{|\mathrm{Z}_2|},$

$\hat{\beta}_{1p}=\frac{(i^*k^*-h^*l^*)\sum_{i\epsilon s}q_iy_iB_i+(g^*l^*-i^{*2})\sum_{i\epsilon s}q_ix_{1i}y_iB_i+(h^*i^*-g^*k^*)\sum_{i\epsilon s}q_ix_{2i}y_iB_i}{|\mathrm{Z}_2|},$

$\hat{\beta}_{2p}=\frac{(h^*k^*-i^*j^*)\sum_{i\epsilon s}q_iy_iB_i+(h^*i^*-g^*k^*)\sum_{i\epsilon s}q_ix_{1i}y_iB_i+(g^*j^*-h^{*2})\sum_{i\epsilon s}q_ix_{2i}y_iB_i}{|\mathrm{Z}_2|}.$

2.3 Calibration Estimator Under Stratified Random Sampling Scheme

Now, let us divide Ωof N individual units into L homogeneous subgroups referred as strata such that the h^{th} stratum consist of N_h units, where *h = (1, 2,...,L)* and $\sum_{h=1}^{L}N_h=N$. Further using SRSWR a sample of size n_h is draw from each stratum. The new calibrated estimator for population mean under stratified random sampling is considered as,

$\bar{y}_{st}(c)=\sum_{h=1}^{L}w_h\bar{y}_h,$ (43)

where $w_h$are the improved weights obtained by minimizing Eq.(3) satisfying the following calibration constraints,

$\sum_{h=1}^{L}w_h=\sum_{h=1}^{L}d_h,$ (44)

$\sum_{h=1}^{L}w_h\bar{x}_{1h}=\bar{X}_1,$ (45)

$\sum_{h=1}^{L}w_h\bar{x}_{2h}=\bar{X}_2,$ (46)

$\sum_{h=1}^{L}w_h^2\frac{s_{hx_1}^2}{n_h}=\sum_{h=1}^{L}d_h^2\frac{S_{hx_1}^2}{n_h},$ (47)

$\sum_{h=1}^{L}w_h^2\frac{s_{hx_2}^2}{n_h}=\sum_{h=1}^{L}d_h^2\frac{S_{hx_2}^2}{n_h},$ (48)

where$d_h=\frac{N_h}{N}$are the stratified design weights,$\bar{x}_h=\sum_{h=1}^{L}\frac{x_{hi}}{n_h}$ and $\bar{X}=\sum_{h=1}^{L}d_h\bar{X}_h$are the sample mean and population mean, while $s_{hx}^2=\sum_{i=1}^{n_h}\frac{(x_{hi}-\bar{x}_h)^2}{n_h-1}$ and $S_{hx}^2=\sum_{i=1}^{N_h}\frac{(X_{hi}-\bar{X}_h)^2}{N_h-1}$are the sample variance and population variance of the auxiliary variable in the h^{th} stratum, respectively.

The Lagrange function for this problem of minimization is defined as,

$$L_3 = \frac{1}{2}\sum_{h=1}^{L}\frac{(w_h-d_h)^2}{d_h q_h} - \lambda_{10}\left(\sum_{h=1}^{L} w_h - \sum_{h=1}^{L} d_h\right) - \lambda_{11}\left(\sum_{h=1}^{L} w_h\bar{x}_{1h} - \bar{X}_1\right) - \lambda_{12}\left(\sum_{h=1}^{L} w_h\bar{x}_{2h} - \bar{X}_2\right) + \frac{\lambda_{13}}{2}\left(\sum_{h=1}^{L} w_h^2\frac{s_{hx_1}^2}{n_h} - \sum_{h=1}^{L} d_h^2\frac{S_{hx_1}^2}{n_h}\right) + \frac{\lambda_{14}}{2}\left(\sum_{h=1}^{L} w_h^2\frac{s_{hx_2}^2}{n_h} - \sum_{h=1}^{L} d_h^2\frac{S_{hx_2}^2}{n_h}\right), \tag{49}$$

where λ_i, $(i = 10, 11, 12, 13, 14)$are Lagrange multipliers. On solving$\frac{\partial L_3}{\partial w_i} = 0$, we have

$$w_h = C_h(1 + \lambda_{10}q_h + \lambda_{11}\bar{x}_{1h}q_h + \lambda_{12}\bar{x}_{2h}q_h), \tag{50}$$

where $C_h = \frac{d_h}{1+\lambda_{13}d_h q_h\frac{s_{x_{1h}}^2}{n_h}+\lambda_{14}d_h q_h\frac{s_{x_{2h}}^2}{n_h}}$.

As Eqs.(44), (45) and (46) are linear constraints with respect to w_h, so we get the values of λ_{10}, λ_{11} and λ_{12} by substituting $w_h$from Eq.(50) in Eqs.(44), (45) and (46), respectively. Now, solving the system of equations, we obtain

$$\lambda_{10} = \frac{(p^*r^*-q^{*2})\left(\sum_{h=1}^{L} d_h-\sum_{h=1}^{L} C_h\right)+(o^*q^*-n^*r^*)\left(\bar{X}_1-\sum_{h=1}^{L}\bar{x}_{1h}C_h\right)+(n^*q^*-o^*p^*)\left(\bar{X}_2-\sum_{h=1}^{L}\bar{x}_{2h}C_h\right)}{|\mathrm{Z}_3|} = \frac{\Pi_{0st}}{\Pi_{st}}, \tag{51}$$

$$\lambda_{11} = \frac{(o^*q^*-n^*r^*)\left(\sum_{h=1}^{L} d_h-\sum_{h=1}^{L} C_h\right)+(m^*r^*-o^{*2})\left(\bar{X}_1-\sum_{h=1}^{L}\bar{x}_{1h}C_h\right)+(n^*o^*-m^*q^*)\left(\bar{X}_2-\sum_{h=1}^{L}\bar{x}_{2h}C_h\right)}{|\mathrm{Z}_3|} = \frac{\Pi_{1st}}{\Pi_{st}}, \tag{52}$$

$$\lambda_{12} = \frac{(n^*q^*-o^*p^*)\left(\sum_{h=1}^{L} d_h-\sum_{h=1}^{L} C_h\right)+(n^*o^*-m^*q^*)\left(\bar{X}_1-\sum_{h=1}^{L}\bar{x}_{1h}C_h\right)+(m^*p^*-n^{*2})\left(\bar{X}_2-\sum_{h=1}^{L}\bar{x}_{2h}C_h\right)}{|\mathrm{Z}_3|} = \frac{\Pi_{2st}}{\Pi_{st}}, \tag{53}$$

where$\mathrm{Z}_3 = \begin{bmatrix} m^* & n^* & o^* \\ n^* & p^* & q^* \\ o^* & q^* & r^* \end{bmatrix} = \begin{bmatrix} \sum_{h=1}^{L} q_h C_h & \sum_{h=1}^{L} q_h\bar{x}_{1h}C_h & \sum_{h=1}^{L} q_h\bar{x}_{2h}C_h \\ \sum_{h=1}^{L} q_h\bar{x}_{1h}C_h & \sum_{h=1}^{L} q_h\bar{x}_{1h}^2 C_h & \sum_{h=1}^{L} q_h\bar{x}_{1h}\bar{x}_{2h}C_h \\ \sum_{h=1}^{L} q_h\bar{x}_{2h}C_h & \sum_{h=1}^{L} q_h\bar{x}_{1h}\bar{x}_{2h}C_h & \sum_{h=1}^{L} q_h\bar{x}_{2h}^2 C_h \end{bmatrix}$

and $|\mathrm{Z}_3| = m^*p^*r^* - m^*q^{*2} - n^{*2}r^* + 2n^*o^*q^* - p^*o^{*2}$.

On substituting the values of λ_{10}, λ_{11} and λ_{12} in Eq.(50), the optimum calibrated weights are obtained as

$$w_h = C_h + \frac{\Pi_{0st}}{\Pi_{st}} q_h C_h + \frac{\Pi_{1st}}{\Pi_{st}} \bar{x}_{1h} q_h C_h + \frac{\Pi_{2st}}{\Pi_{st}} q_h \bar{x}_{2h} C_h . \tag{54}$$

Now, on putting the value of w_h from Eq.(54) in Eq.(47) and Eq.(48), we get

$$\sum_{h=1}^{L} \left[C_h + \frac{\Pi_{0st}}{\Pi_{st}} q_h C_h + \frac{\Pi_{1st}}{\Pi_{st}} \bar{x}_{1h} q_h C_h + \frac{\Pi_{2st}}{\Pi_{st}} q_h \bar{x}_{2h} C_h \right]^2 \frac{s_{hx_1}^2}{n_h} = \sum_{h=1}^{L} d_h^2 \frac{S_{hx_1}^2}{n_h}, \tag{55}$$

$$\sum_{h=1}^{L} \left[C_h + \frac{\Pi_{0st}}{\Pi_{st}} q_h C_h + \frac{\Pi_{1st}}{\Pi_{st}} \bar{x}_{1h} q_h C_h + \frac{\Pi_{2st}}{\Pi_{st}} q_h \bar{x}_{2h} C_h \right]^2 \frac{s_{hx_2}^2}{n_h} = \sum_{h=1}^{L} d_h^2 \frac{S_{hx_2}^2}{n_h} , \tag{56}$$

The calibrated weights so derived in Eq.(54) are the function of non-linear Lagrange multipliers λ_{13} and λ_{14}, which can be obtained by pre-assigning a wide range of values of λ which satisfy Eqs.(55) and (56). Thus, the expression for improved calibrated estimator can be obtained by putting the optimum weights from Eq.(54) in Eq.(43), we have

$$\bar{y}_{st}(c) = \sum_{h=1}^{L} C_h \bar{y}_h + \hat{\beta}_{0st}\left(\sum_{h=1}^{L} d_h - \sum_{h=1}^{L} C_h\right) + \hat{\beta}_{1st}\left(\bar{X}_1 - \sum_{h=1}^{L} \bar{x}_{1h} C_h\right) + \hat{\beta}_{2st}\left(\bar{X}_2 - \sum_{h=1}^{L} \bar{x}_{2h} C_h\right) \tag{57}$$

where

$$\hat{\beta}_{0st} = \frac{(p^* r^* - q^{*2}) \sum_{h=1}^{L} q_h \bar{y}_h C_h + (o^* q^* - n^* r^*) \sum_{h=1}^{L} q_h \bar{x}_{1h} \bar{y}_h C_h + (n^* q^* - o^* p^*) \sum_{h=1}^{L} q_h \bar{x}_{2h} \bar{y}_h C_h}{|Z_3|},$$

$$\hat{\beta}_{1st} = \frac{(o^* q^* - n^* r^*) \sum_{h=1}^{L} q_h \bar{y}_h C_h + (m^* r^* - o^{*2}) \sum_{h=1}^{L} q_h \bar{x}_{1h} \bar{y}_h C_h + (n^* o^* - m^* q^*) \sum_{h=1}^{L} q_h \bar{x}_{2h} \bar{y}_h C_h}{|Z_3|},$$

$$\hat{\beta}_{2st} = \frac{(n^* q^* - o^* p^*) \sum_{h=1}^{L} q_h \bar{y}_h C_h + (n^* o^* - m^* q^*) \sum_{h=1}^{L} q_h \bar{x}_{1h} \bar{y}_h C_h + (m^* p^* - n^{*2}) \sum_{h=1}^{L} q_h \bar{x}_{2h} \bar{y}_h C_h}{|Z_3|} .$$

We have obtained the expression of the calibrated estimator under SRSWR, PPS, and stratified random sampling designs incorporating two auxiliary variables. Now, to analyze the efficiency of these proposed calibrated estimators, an empirical study supported by a simulation study is also provided in further sections.

3. Empirical Study

In this Section, an empirical study is carried out for comparing the efficacy of proposed calibrated estimators with existing estimators based on single auxiliary variable under three different sampling techniques (i.e. SRSWR, PPS, and Stratified Random Sampling). The two auxiliary variables based proposed calibrated estimators $\bar{y}_{srs}(c)$, $\bar{y}_{pps}(c)$ and $\bar{y}_{st}(c)$ defined in Eqs. (26), (42) and (57) are compared with the existing single auxiliary variable based calibrated estimators $\bar{y}_{srs}^*(c)$, $\bar{y}_{pps}^*(c)$ and $\bar{y}_{st}^*(c)$[Alam et. al. (2020)],respectively, as defined below:

(1) $$\bar{y}_{srs}^*(c) = \frac{1}{n} \sum_{i\epsilon s} \left(\frac{y_i}{1 + \lambda_2^* q_i \frac{(x_i - \bar{x})^2}{n}} \right) + \hat{\beta}_{1s}^* \left(\bar{X} - \frac{1}{n} \sum_{i\epsilon s} \frac{x_i}{1 + \lambda_2^* q_i \frac{(x_i - \bar{x})^2}{n}} \right) + \hat{\beta}_{2s}^* \left(1 - \frac{1}{n} \sum_{i\epsilon s} \frac{1}{1 + \lambda_2^* q_i \frac{(x_i - \bar{x})^2}{n}} \right), \tag{58}$$

where

$$\hat{\beta}_{1s}^{*}=\frac{1}{\Delta_s}\left\{\left(\frac{1}{n}\sum_{i\epsilon s}\frac{q_i}{1+\lambda_2^{*}q_i\frac{(x_i-\bar{x})^2}{n}}\right)\left(\frac{1}{n}\sum_{i\epsilon s}\frac{q_ix_iy_i}{1+\lambda_2^{*}q_i\frac{(x_i-\bar{x})^2}{n}}\right)-\left(\frac{1}{n}\sum_{i\epsilon s}\frac{q_ix_i}{1+\lambda_2^{*}q_i\frac{(x_i-\bar{x})^2}{n}}\right)\left(\frac{1}{n}\sum_{i\epsilon s}\frac{q_iy_i}{1+\lambda_2^{*}q_i\frac{(x_i-\bar{x})^2}{n}}\right)\right\},$$

$$\hat{\beta}_{2s}^{*}=\frac{1}{\Delta_s}\left\{\left(\frac{1}{n}\sum_{i\epsilon s}\frac{q_ix_i^2}{1+\lambda_2^{*}q_i\frac{(x_i-\bar{x})^2}{n}}\right)\left(\frac{1}{n}\sum_{i\epsilon s}\frac{q_iy_i}{1+\lambda_2^{*}q_i\frac{(x_i-\bar{x})^2}{n}}\right)-\left(\frac{1}{n}\sum_{i\epsilon s}\frac{q_ix_i}{1+\lambda_2^{*}q_i\frac{(x_i-\bar{x})^2}{n}}\right)\left(\frac{1}{n}\sum_{i\epsilon s}\frac{q_ix_iy_i}{1+\lambda_2^{*}q_i\frac{(x_i-\bar{x})^2}{n}}\right)\right\},$$

$$\Delta_s=\left(\frac{1}{n}\sum_{i\epsilon s}\frac{q_ix_i^2}{1+\lambda_2^{*}q_i\frac{(x_i-\bar{x})^2}{n}}\right)\left(\frac{1}{n}\sum_{i\epsilon s}\frac{q_i}{1+\lambda_2^{*}q_i\frac{(x_i-\bar{x})^2}{n}}\right)-\left(\frac{1}{n}\sum_{i\epsilon s}\frac{q_ix_i}{1+\lambda_2^{*}q_i\frac{(x_i-\bar{x})^2}{n}}\right)^2.$$

(2)

$$\bar{y}_{pps}^{*}(c)=\frac{1}{N}\left\{\sum_{i\epsilon s}\left(\frac{y_i}{np_i-N\lambda_2^{**}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)+\hat{\beta}_{1p}^{*}\left(\frac{1}{n}\sum_{i\epsilon s}\frac{1}{p_i}-\frac{q_i}{np_i-N\lambda_2^{**}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)+\hat{\beta}_{2p}^{*}\left(X-\frac{x_i}{np_i-N\lambda_2^{**}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)\right\}, \tag{59}$$

where

$$\hat{\beta}_{1p}^{*}=\frac{1}{\Delta_p}\left\{\left(\sum_{i\epsilon s}\frac{y_i}{np_i-N\lambda_5^{*}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)\left(\sum_{i\epsilon s}\frac{q_ix_i^2}{np_i-N\lambda_5^{*}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)-\left(\sum_{i\epsilon s}\frac{q_ix_i}{np_i-N\lambda_5^{*}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)\left(\sum_{i\epsilon s}\frac{q_ix_iy_i}{np_i-N\lambda_5^{*}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)\right\},$$

$$\hat{\beta}_{2p}^{*}=\frac{1}{\Delta_p}\left\{\left(\sum_{i\epsilon s}\frac{q_ix_iy_i}{np_i-N\lambda_2^{**}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)\left(\sum_{i\epsilon s}\frac{q_i}{np_i-N\lambda_2^{**}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)-\left(\sum_{i\epsilon s}\frac{q_iy_i}{np_i-N\lambda_2^{**}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)\left(\sum_{i\epsilon s}\frac{q_ix_i}{np_i-N\lambda_2^{**}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)\right\},$$

$$\Delta_p = \left(\sum_{i\epsilon s}\frac{q_i x_i^2}{np_i - N\lambda_2^{**}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)\left(\sum_{i\epsilon s}\frac{q_i}{np_i - N\lambda_2^{**}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right) - \left(\sum_{i\epsilon s}\frac{q_i x_i}{np_i - N\lambda_2^{**}p_i^{\alpha}q_i\left(\frac{x_i}{p_i}-\hat{X}_{HH}\right)^2}\right)^2.$$

(3) $$\bar{y}_{st}^{*}(c) = \sum_{h=1}^{L}\left(\frac{d_h\bar{y}_h}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right) + \hat{\beta}_{1st}\left(1-\sum_{h=1}^{L}\frac{d_h}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right) + \hat{\beta}_{2st}\left(\bar{X} - \sum_{h=1}^{L}\frac{d_h\bar{x}_h}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right), \tag{60}$$

where

$$\hat{\beta}_{1st}^{*} = \frac{1}{\Delta_{st}}\left\{\left(\frac{d_h q_h \bar{x}_h^2}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right)\left(\frac{d_h q_h \bar{y}_h}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right) - \left(\frac{d_h q_h \bar{x}_h}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right)\left(\frac{d_h q_h \bar{x}_h\bar{y}_h}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right)\right\},$$

$$\hat{\beta}_{2st}^{*} = \frac{1}{\Delta_{st}}\left\{\left(\frac{d_h q_h \bar{x}_h\bar{y}_h}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right)\left(\frac{d_h q_h}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right) - \left(\frac{d_h q_h \bar{x}_h}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right)\left(\frac{d_h q_h \bar{y}_h}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right)\right\},$$

$$\Delta_{st} = \left(\frac{d_h q_h}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right)\left(\frac{d_h q_h \bar{x}_h^2}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right) - \left(\frac{d_h q_h \bar{x}_h}{1+\lambda_2^{***}d_h q_h\frac{s_{hx}^2}{n_h}}\right)^2,$$

where λ_2^{*}, $\lambda_2^{**}$and λ_2^{***} are the non-linear lagrange's multiplier under SRS, PPS and Stratified sampling, respectively.

The evaluation of performances of these estimators is done by using the average of absolute relative bias (ARB), simulated relative squared error (SRSE), and percent relative efficiency (PRE) (expressed as the ratio of the mean square error of the two estimators). These three criterion are defined below:

$$ARB(T) = \frac{1}{M}\left(\sum_{r=1}^{M}\left|\frac{T_r-\theta}{\theta}\right|\right)\times 100\% \tag{61}$$

$$SRSE = \frac{1}{\theta}\sqrt{\frac{1}{M}\sum_{r=1}^{M}(T_r-\theta)^2} \tag{62}$$

$$PRE = \frac{MSE(\theta)}{MSE(T_r)}\times 100\% \tag{63}$$

where $r = 1, 2, \dots, M$ denotes the number of replications, T_r is the proposed calibration estimator under two auxiliary variables andθ is the existing calibration estimator under single auxiliary variable.

In order to check the efficacy of the proposed calibration estimator under the SRS and PPS sampling designs, we considered the Forced Expiratory Volume (FEV) dataset used by Alka et.al. (2019). The dataset contains the information of 654 children aged 3 to 19 years' old who had childhood respiratory diseases. In our study, the FEV is considered as a study variable whereas, the two auxiliary variables are the child's age (3-19 years old) and the child's height (inches) considered as X_1 and X_2, respectively. For this data set, $\bar{Y} = 2.636, \bar{X}_1 = 9.931$, $\bar{X}_2 = 61.143$,the correlation coefficient values,$\rho(X_1, Y) = 0.756$, $\rho(X_2, Y) = 0.868$, and $\rho(X_1, X_2) = 0.791$. The aim is to estimate the population mean, Y (assumed unknown) for the known values of X_1, X_2.

Random samples of proportion ranging from 5% to 30% are drawn with an increment of 5% using SRSWR and PPSWR. Also, the values of non-linear Lagrange multipliers $\lambda_3$and $\lambda_4$taken from -1 to 1 and -1.5 to 1 with an increment of 0.05under SRS. Whereas, under PPS sampling design, the values of non-linear Lagrange multipliers $\lambda_8$and λ_9 are taken with an increment of 0.5 ranging from – 2 to 3 and -1.5 to 2, respectively. The whole procedure is replicated 2000 times using R- software. The values of the ARB, SRSE, and PRE of the calibration estimators incorporating single and two-auxiliary variables under SRS and PPS are shown in Table 1, Table 2 and Table 3, respectively.

Table 1. ARB, SRSE and PRE values for the calibrated estimator under SRS

Sample Proportion (%)	ARB		SRSE		PRE
	$\bar{y}^*_{srs}(c)$	$\bar{y}_{srs}(c)$	$\bar{y}^*_{srs}(c)$	$\bar{y}_{srs}(c)$	
5	0.937	0.932	3.957	2.844	193.641
10	0.758	0.548	2.654	1.887	197.816
15	0.586	0.388	2.130	1.506	200.055
20	0.479	0.318	1.774	1.264	196.877
25	0.380	0.240	1.492	1.068	195.279
30	0.351	0.209	1.384	0.974	202.087

For checking the efficiency of the proposed estimator under stratified random sampling design, we consider the data set on Abalone which was used by Alamet al. (2020). The complete dataset of 4177 values is divided into three strata with respect to the sex of abalone, with stratum sizes 1528, 1307 and 1342, for male, female, and infant, respectively. In our study, the study variable Y is considered to be the number of rings (for calculating the age of abalone) whereas the diameter (mm) and whole weight are taken as an auxiliary variables (X_1, X_2). The correlation coefficient 0.574 is observedbetween Y and X_1, 0.540 between Y and X_2 whereas 0.925 between X_1 and $X_2$for this dataset. Under stratified random sampling, thevalues of non-linear Lagrange multipliers $\lambda_{13}$and λ_{14} are taken with an increment of 0.5 ranging from – 1.8to 1.8 and -1.5 to 1.5, respectively.Simple random samples of sample proportions ranging from

5% to 30% with an increment of 5% are selected with replacement from each stratum and analysed using R software for 2000 iterations.

Table 2. ARB, SRSE and PRE values for the calibrated estimator under PPS

Sample Proportion (%)	ARB		SRSE		PRE
	$\bar{y}^*_{pps}(c)$	$\bar{y}_{pps}(c)$	$\bar{y}^*_{pps}(c)$	$\bar{y}_{pps}(c)$	
5	2.262	2.164	4.539	5.577	65.893
10	0.766	0.406	2.816	3.576	87.877
15	0.288	0.256	2.925	3.476	93.476
20	0.729	0.483	3.585	3.084	135.082
25	0.671	0.313	4.176	3.007	192.801
30	0.138	0.305	3.845	2.448	246.651

Table 3. ARB, SRSE and PRE values for the calibrated estimator under Stratified Random Sampling

Sample Proportion (%)	ARB		SRSE		PRE
	$\bar{y}^*_{st}(c)$	$\bar{y}_{st}(c)$	$\bar{y}^*_{st}(c)$	$\bar{y}_{st}(c)$	
5	0.036	0.033	1.923	1.789	107.452
10	0.017	0.015	1.331	1.274	104.495
15	0.010	0.010	1.073	1.020	105.204
20	0.025	0.013	0.874	0.805	108.574
25	0.021	0.011	0.808	0.739	109.236
30	0.026	0.016	0.702	0.641	109.516

4. Simulation Study

A simulation study is also carried out to assess the performance of the proposed calibration estimator incorporating two auxiliary variables with the existing calibration estimator under single auxiliary variable. This study is divided into three different parts based on different sampling designs as discussed below:

4.1 Under Simple Random Sampling

Under this sampling design, a finite population of size $N = 1500$is generated using normal distribution with mean 5 and variance 1 i.e. $y_i \sim N(5,1)$. The two auxiliary variables are also generated for various values of degree of correlation ranging from 0.5 to 0.8 between Y and $X_1$and 0.6 to 0.85 between Y and X_2. The two auxiliary variables are chosen such that a positive correlation exists between them. Based on different correlation coefficient, four sets of populations are generated and random samples of proportions 5%, 10%, 15%, 20%, 25%, and 30% of N, are drawn using

SRSWR. The program is simulated for 500 times using R software and different estimates of population mean, ARB, SRSE and PRE are calculated. The value of λ_2^* is defined in the range of – 0.5 to 0.85. The maximum and minimum value of $\lambda_3$is taken as -1 and 1 whereas for λ_4, -1.5 and 1 is considered for all generated population sets. The results are produced in Table 4.

Table 4. Simulated ARB, SRSE and PRE values for the calibrated estimator under SRSWR

Sample Proportion (%)	**Correlation coefficient**	**ARB**		**SRSE**		**PRE**
		$\bar{y}^*_{srs}(c)$	$\bar{y}_{srs}(c)$	$\bar{y}^*_{srs}(c)$	$\bar{y}_{srs}(c)$	
5	$r_1 = 0.50$ $r_2 = 0.60$	0.347	0.007	2.789	0.559	2468.998
10		0.074	0.006	1.921	0.151	16111.007
15		0.221	0.002	1.391	0.080	30140.674
20		0.420	0.003	1.246	0.064	37207.010
25		0.279	0.002	1.045	0.039	71044.305
30		0.211	0.001	0.934	0.032	82466.129
5	$r_1 = 0.60$ $r_2 = 0.70$	0.580	0.008	2.558	0.089	2874.157
10		0.277	0.009	1.645	0.210	6109.493
15		0.085	0.005	1.378	0.232	3526.469
20		0.085	0.001	1.209	0.085	19766.452
25		0.265	0.001	1.007	0.106	8883.001
30		0.210	0.001	0.846	0.021	155887.344
5	$r_1 = 0.70$ $r_2 = 0.80$	0.021	0.107	2.346	0.307	5837.191
10		0.076	0.028	1.502	0.466	1038.776
15		0.174	0.001	1.121	0.030	134183.693
20		0.042	0.002	0.946	0.143	4359.812
25		0.260	0.003	0.908	0.055	27125.616
30		0.169	0.001	0.740	0.024	101537.756
5	$r_1 = 0.80$ $r_2 = 0.85$	0.429	0.122	1.973	0.213	8564.719
10		0.144	0.008	1.170	0.214	2979.811
15		0.008	0.007	0.915	0.112	6653.248
20		0.057	0.005	0.808	0.123	4280.487
25		0.046	0.008	0.710	0.170	1733.063
30		0.023	0.003	0.669	0.051	16693.155

4.2 Under Probability Proportional to Size (PPS) Sampling

Under PPS sampling scheme, a finite population of size $N = 1500$ is generated using normal distribution with mean 4 and variance 1 i.e. $y_i \sim N(4,1)$. The two auxiliary variables are also generated for various values of degree of correlation ranging from 0.5 to 0.8 (r_1) between Y and $X_1$and 0.6 to 0.85 (r_2) between Y and X_2 as in case for

SRS. A total of 500 random pprswr samples of proportions 5%, 10%, 15%, 20%, 25%, and 30% of population size is drawn using cumulative method and estimates are calculated for the two cases of auxiliary variable (see Table 5). Also, the values of $\lambda_2^{**}, \lambda_8$ and λ_9 are defined in the range (-1, 1), (-2, 2) and (-1.5, 1) with an increment of 0.05, 0.5 and 0.5, respectively. The optimum value of α is taken in between -2 to 2 with an increment of 0.2.

Table 5. Simulated ARB, SRSE and PRE values for the calibrated estimator under PPS

Sample Proportion (%)	Correlation coefficient	ARB		SRSE		PRE
		$\bar{y}_{pps}^{*}(c)$	$\bar{y}_{pps}(c)$	$\bar{y}_{pps}^{*}(c)$	$\bar{y}_{pps}(c)$	
5	$r_1 = 0.50$ $r_2 = 0.60$	0.145	0.010	2.738	0.200	18574.633
10		0.135	0.011	2.094	0.285	5385.252
15		0.090	0.002	2.039	0.219	8659.648
20		0.016	0.005	2.662	0.369	5201.659
25		0.063	0.006	5.599	0.142	153738.469
30		0.010	0.005	2.123	0.276	5909.991
5	$r_1 = 0.60$ $r_2 = 0.70$	0.033	0.009	2.445	0.203	14426.570
10		0.070	0.001	2.247	0.061	135053.616
15		0.036	0.003	3.087	0.075	165171.170
20		0.381	0.016	4.907	0.244	40283.255
25		0.526	0.037	6.602	0.735	8059.432
30		0.031	0.011	2.969	0.194	23342.565
5	$r_1 = 0.70$ $r_2 = 0.80$	0.093	0.001	2.148	0.042	254825.244
10		0.124	0.004	1.719	0.142	14562.679
15		0.105	0.005	1.411	0.120	13718.158
20		0.007	0.006	5.078	0.164	95788.045
25		0.054	0.010	2.472	0.172	20505.312
30		0.012	0.001	2.026	0.073	77068.083
5	$r_1 = 0.80$ $r_2 = 0.85$	0.165	0.002	1.752	0.083	44552.757
10		0.126	0.011	3.505	0.204	29409.167
15		0.095	0.001	2.218	0.042	411218.092
20		0.037	0.001	1.879	0.008	4706442.798
25		0.301	0.049	2.557	0.133	7748.485
30		0.066	0.004	2.137	0.221	9347.985

4.3 Under Stratified Random Sampling

Here, we generated a finite population of size $N = 1500$, constituting three different strata with stratum sizes $N_h = 500$ for $h = 1, 2, 3$. The study and auxiliary variables are generated using gamma distribution such that Y and X_1 have the correlation coefficient in the range of 0.5-0.55, 0.6-0.68, 0.7-0.78 and 0.8-0.85 whereas Y and X_2

have 0.6-0.65, 0.7-0.79, 0.8-0.88, and 0.87-0.92 for the population sets 1, 2, 3 and 4, respectively. Samples of proportions 5%, 10%, 15%, 20%, 25%, and 30% of N_h, are drawn from each stratum using SRSWR. The program is simulated for 500 times for the different values ofλ_2^{***} lying in the range (-1, 1) andλ_{13}, λ_{14} in between -0.5 to 0.5 with an increment of0.05 and 0.1, respectively for all sets. The simulated ARB, SRSE and PRE values are noted down in Table 6 for the calibrated estimator under single and two- auxiliary variables.

Table 6. Simulated ARB, SRSE and PRE values for the calibrated estimator under Stratified Random Sampling

Sample Proportion (%)	Correlation coefficient	ARB		SRSE		PRE
		$\bar{y}_{st}^{*}(c)$	$\bar{y}_{st}(c)$	$\bar{y}_{st}^{*}(c)$	$\bar{y}_{st}(c)$	
5	$r_1 = 0.5$-0.6, $r_2 = 0.6$-0.65	0.110	0.014	4.985	4.880	104.351
10		0.372	0.218	5.184	4.906	111.647
15		0.106	0.006	4.900	4.809	103.812
20		0.403	0.314	5.744	5.450	111.063
25		0.202	0.053	4.930	4.726	108.844
30		0.255	0.065	5.555	5.230	112.820
5	$r_1 = 0.6$-0.68, $r_2 = 0.7$-0.79	0.285	0.118	5.060	4.861	108.350
10		0.303	0.114	4.782	4.653	105.595
15		0.301	0.128	5.110	4.975	105.494
20		0.235	0.059	5.214	5.089	104.981
25		0.499	0.277	5.552	5.136	116.879
30		0.303	0.126	5.143	5.049	103.772
5	$r_1 = 0.7$-0.78, $r_2 = 0.8$-0.88	0.294	0.218	5.262	5.146	104.576
10		0.661	0.393	5.822	5.337	118.974
15		0.152	0.101	5.845	5.352	119.269
20		0.138	0.298	5.502	5.155	113.929
25		0.528	0.297	5.668	5.298	114.467
30		0.362	0.143	5.436	4.964	119.925
5	$r_1 = 0.8$-0.85, $r_2 = 0.87$-0.92	0.296	0.064	5.098	4.792	113.149
10		0.185	0.064	5.306	5.030	111.254
15		0.412	0.118	5.049	4.821	109.687
20		0.182	0.079	4.929	4.816	104.781
25		0.210	0.125	5.490	5.171	112.690
30		0.167	0.066	5.004	4.786	109.307

5. Results and Discussion

Tables 1, 2 and 3, represent the values of ARB, SRSE, and PRE using empirical dataset under SRS, PPS with replacement scheme, and stratified random sampling, respectively. It can be seen from Table 1 that, the PRE values vary closely from 193.641% to 202.087%. We can also observe that the ARB and SRSE values decrease with an increase in the sample proportions and maximum gain efficiency is202.087% at 30% sample proportion under two auxiliary variables compared to existing Alam et al.(2020) estimator under single auxiliary variable. From Table 2, we can observe that the PRE values are increasing as sample proportion increases and maximum gained efficiency is 246.651% at 30% sample proportion over existing estimator under PPS using single auxiliary variable a similar trend as SRSWR design. However, the PRE values are not found favorable for the proposed estimator with some variations in case of small sample proportions i.e. 5%, 10%, and 15%. Table 3 highlights very less variation in PRE values with respect to change of sample size. The maximum and minimum gained efficiency are 109.516% and 104.495% at 30% and 10% sample proportions, respectively.

Table 4; conclude about the results obtained through simulated data. The values of ARB, SRSE, and PRE under SRSWR design for estimators under single and two auxiliary variables, it shows that all average relative biases are very small and nearly unbiased for some sample proportions. The maximum percentage relative efficiency gained is 101537.756% for upper sample proportion when the correlation coefficient is 0.7 and 0.8 between(X_1, Y) and (X_2, Y), respectively. It shows a similar tendency of the estimator as in case of empirical analysis.

Table5; shows simulated ARB, SRSE, and PRE values of both types of estimators under PPSWR sampling scheme, we can observe that the percentage relative efficiency varies from 5201.696% (20% sample proportions, $r_1 = 0.5$, $r_2 = 0.6$) to 4706443.798% (20% sample proportions, $r_1 = 0.8$, $r_2 = 0.85$)within the complete population sets. Average relative biases are found to be very small in both the cases and may be ignored.

Table6; shows the values of three performance criteria in case of stratified random sampling, here also the average relative biases are very small and negligible. There is less fluctuations of PRE value with respect to sample proportions are obtained. The maximum PRE observed is 119.925% for 30% sample proportions ($r_1 = 0.7 - 0.78$, $r_2 = 0.8 - 0.88$) and the minimum is 103.772% for 30% sample proportions with varying correlation coefficient as $r_1 = 0.6$ to 0.68 and$r_2 = 0.7$ to 0.79.

6. Conclusions

In this study, different new calibrated weights are derived for estimating population mean under calibration approach using two auxiliary variables with non-linear constraints and three different sampling designs, SRSWR, Stratified and PPS. On the basis of real and simulated data analysis, the proposed calibrated estimators for mean under PPSWR performed better for large sample proportions with substantial increase in PRE values than that for the other two sampling designs. Also, all the proposed mean calibration estimators under three different sampling designs incorporating two-auxiliary variables using non-linear constraints are found to be more efficient than that for single auxiliary variable based calibrated estimators [i.e. Alam et. al. (2020) estimators] in terms of ARB (%), SRSE (%) and PRE (%) and recommended to prefer more effectively in sample surveys.

References

Alam, S., Singh, S., and J. Shabbir. (2020). Calibrated estimators using non-linear calibration constraints. Journal of Statistical Computation and Simulation, 90 (3), 489–514.

Alam, S., and Shabbir, J. (2020). Calibration estimation of mean by using double use of auxiliary information. Communications in Statistics-Simulation and Computation, 1-19.

Alka, Rai, P. K., and Qasim, M. (2019). Two-step calibration of design weights under two auxiliary variables in sample survey. Journal of Statistical Computation and Simulation, 89, 2316–2327.

Clement, E. P., and Enang, E. I. (2015). Calibration approach alternative ratio estimator for population mean in stratified sampling. International Journal of Statistics and Economics, 16(1), 83-93.

Deville, J. C, and Särndal, C. E. (1992). Calibration estimators in survey sampling. Journal of the American Statistical Association, 87(418), 376–382.

Enang, E. I., and Clement, E. P. (2019). An efficient class of calibration ratio estimators of domain mean in survey sampling. Communications in Mathematics and Statistics, 1-15.

Estevao, V. M.,and Särndal, C. E. (2000). A functional form approach to calibration. Journal of Official Statistics,16(4), 379–399.

Hansen, M. H.,and Hurwitz, W. N. (1943). On the theory of sampling from finite populations. Annals of Mathematical Statistics, 14(4), 333–362.

Horvitz, D. G.,and Thompson, D. J. (1952). A generalization of sampling without replacement from a finite universe. Journal of the American Statistical Association, 47, 663–685.

Kim, J. M., Sungur., E. A.,and Heo, T. Y. (2007). Calibration approach estimators in stratified sampling. Statistics probability letters, 77, 99–103.

Kim, J. K., and Park, M. (2010). Calibration estimation in survey sampling. International Statistical Review, 78, 21-39

Olkin, I. (1958). Multivariate ratio estimation for finite populations. Biometrika, 45, 154–165.

Ozgul, N. (2019). New calibration estimator based on two auxiliary variables in stratified sampling. Communications in Statistics-Theory and Methods, 48(6), 1481-1492.

Raj, D. (1965). On a method of using multi-auxiliary information in sample surveys. Journal ofthe American Statistical Association, 60, 270–277.

Salinas, V. I., Sedory, S. A., and Singh, S. (2018). Calibration using power transformation. Communications in Statistics-Simulation and Computation, 0, 1-31.

Salinas, V. I., Sedory, S. A., and Singh, S. (2019). Calibrated estimator of population mean using two- stage ranked set sampling. Ranked Set Sampling, Academic Press, 219-237.

Shukla, G. K. (1965). Multivariate regression estimate. Journal of the Indian Statistical Association, 3, 202–211.

Singh, S., and Arnab, R. (2006). A bridge between the GREG and linear regression estimators. Proceedings of ASA Section on Survey Research Methods, Seattle, 3689–3693.

Singh, S., and Arnab, R. (2011). On calibration of design weights. Metron International Journal of Statistics, 2, 185-205.

Singh, S. (2012). On the calibration of design weights using a displacement function. Metrika, 75, 85–107.

Singh, S. (2013). A dual Problem of calibration of design weights. Statistics, 47, 566–574.

Singh, S., and Sedory, S. A. (2016). Two-step calibration of design weights in survey sampling. Communication in Statistics.–Theory and Methods, 45 (12), 3510–3523.

Singh, D., Sisodia, B. V. S., Nidhi, and Pundir, S. (2019). Some calibration estimators for finite population mean in two-stage stratified random sampling. Communications in Statistics-Theory and Methods, 1-14.

Sisodia, B. V. S., and Singh, D. (2020). Calibration approach-based estimators for finite population mean in multistage stratified random sampling. Statistical Methods and Applications in Forestry and Environmental Sciences, 105-123.

Tracy, D. S., Singh, S., and Arnab, R. (2003). Note on calibration in stratified and double sampling. Survey Methodology, 29 (1), 99–104.

Wu, C., and Thompson, M. E. (2020). Calibration weighting and estimation. Sampling Theory and Practice, 115-137.