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An Extended Generalized Frechet (EGFr) Distribution: Properties and It Applications to Reliability Data

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Abstract: This paper introduces a novel Extended Generalized Frechet (EGFr) distribution, a flexible extension of the Frechet distribution. The EGFr incorporates additional parameters that provide enhanced flexibility for modeling diverse data sets, especially those with complex patterns or extreme values. The probability density function of the EGFr is derived from the T-X family of distributions and can be expressed as a linear combination of Frechet densities. We investigate the statistical properties of the EGFr, including moments, quantiles, hazard functions and order statistics. Maximum likelihood estimation is used to estimate the model parameters. Extensive simulations demonstrate the consistency and efficiency of the EGFr in parameter estimation. Real-life applications to reliability datasets demonstrate the superior performance of the EGFr over existing Frechet-based distributions. The EGFr's ability to accurately capture complex data patterns and provide reliable estimates makes it a valuable tool for researchers and practitioners in the fields of reliability engineering and sciences.

Keywords: akaike information criteria; bayesian information criteria; log-likelihood; simulation; reliability function; flexibility

1. Introduction

The use of generalized distributions to model research data in the applied sciences has grown tremendously in recent decades. Many researchers have developed different probability distributions to model such experimental data because the classical model is not adequate to fit such experimental data sets. This has led researchers in distribution theory to generalize or modify existing distributions by adding additional parameters to the existing distributions to make them more flexible and also more robust in handling different data sets. In modeling extreme events such as supermarket queues, wind speeds, accelerated life tests, ocean currents, earthquakes, horse races, floods, track records, and rainfall, among others, the Frechet (Fr) distribution is one of the most important distributions to use, as pointed out in [1–3]. The Frechet distribution is also commonly used in engineering material properties [4], as well as in advanced mathematical reasoning about point processes and regularly varying functions [5–8].

Thus, various researchers have modified and generalized the Frechet distribution, such as [9] introduced a new lifetime model called the gamma-extended Frechet distribution, Nadarajah and Kotz introduced the exponentiated Frechet distribution as cited by [10,11] developed a transmuted Exponentiated Moment Pareto distribution [12] developed a paper on the application of Nadarajah Haghighi Gompertz distribution, ref. [13] introduced the beta-exponential Fr distribution. Ref. [14] introduced the transmuted Fr distribution, ref. [15] studied some properties on the FrW distribution with application to real data, the Exponentiated Frechet generator of distributions with applications was studied by [16]. The Exponentiated Fréchet distribution has been a fertile ground for statistical research, leading to a family of versatile models. One significant step in its evolution was its development as a generator of distributions by [17]. This laid the groundwork for creating more complex and adaptable statistical models based on the fundamental properties of the Exponentiated Fréchet. Building on this foundation, ref. [18] introduced the Marshall-Olkin Exponentiated Fréchet distribution. This extension incorporates the Marshall-Olkin generator, enhancing the flexibility of the original Exponentiated Fréchet model. Such advancements are crucial for accurately modeling diverse datasets, especially in fields like reliability and survival analysis where the distribution's shape and tail behavior are critical.

Beyond theoretical development, the practical utility of the Exponentiated Fréchet distribution has been thoroughly explored. Reference [19] specifically delves into its applications as a loss distribution, examining its use in actuarial measures and regression analysis. This highlights the distribution's relevance in risk assessment and modeling financial losses, providing valuable tools for actuaries and risk managers.

Limitation of the Classical and Existing Frechet Distribution and Motivation for Developing EGFr Distribution

The existing Generalized Fréchet distribution, while valuable, often falls short when modeling complex real-world data. Its limitations primarily stem from an inherent lack of flexibility, which can lead to inaccurate data fitting and unreliable statistical inferences for certain datasets. Specifically, it struggles to precisely capture the nuances of data exhibiting diverse shapes, especially those with extreme values or heavy tails.

This is precisely why we developed the Extended Generalized Fréchet (EGFr) distribution. By thoughtfully introducing additional parameters, the EGFr overcomes these limitations, offering significantly enhanced flexibility. This allows it to: accurately model intricate data patterns, including various forms of skewness, heavy tails, and even multimodal characteristics that elude its predecessors; significantly improve model fit, leading to more precise parameter estimates and, consequently, more robust and reliable conclusions from data analysis; and broaden its applicability across a wider spectrum of datasets, particularly those challenging ones characterized by extreme observations or pronounced heavy tails.

In summary, the EGFr has been developed to fill the gaps in existing models by providing a more flexible and adaptable tool for modeling reliability data sets that are difficult to fit using traditional or other generalized Frechet distributions. This increased flexibility and applicability makes the EGFr a valuable contribution to the field of statistical modeling.

The rest of the paper is structured as follows. In Section 2, under Methodology, we define the pdf and cdf of the basic distributions; one-parameter Frechet and T-X family of distributions. In Section 3, we present the pdf and cdf of the new model. Section 4 provides functional expansions for the pdf and cdf of the new model, with the graphical representations. Section 5 provides statistical properties including moment, mean, moment generating functions, reliability or survival function, hazard function, reverse or inverse hazard function, mill-ratio, quantile functions, median and asymptotic behaviour of the distribution. Section 6 discusses the distribution of order statistics. Maximum likelihood estimates (MLEs) are demonstrated in Section 7. In Section 8, a simulation analysis is performed to demonstrate the practicality of the proposed distribution by applying it to reliability datasets. Finally, concluding remarks and conclusions are given in Section 9.

2. Methodology

The one-parameter Frechet distribution has a cumulative distribution function (cdf) and a probability density function (pdf) (for $x \geq 0$), is given as:

$$H(x;\alpha) = e^{-x^{-\alpha}} \tag{1}$$

$$h(x;\alpha) = \alpha x^{-(\alpha+1)} e^{-x^{-\alpha}} \tag{2}$$

where $\alpha > 0$ is the scale parameter.

The use of proposed families of distributions is an attractive concept for extending the current distributions in the literature. As most of these classical distributions have only one scale parameter, such as the exponential distribution, they can provide constant hazard rate forms. In line with this, many researchers have proposed families of distributions to generalize these classical distributions by adding flexibility to the basic distribution.

Based on the idea of the T-X family of distributions introduced by refs. [20,21] introduced the Exponentiated Generalized Topp Leone G Family of distributions with cdf and pdf given as follows.

$$F(x) = [1 - [1 - [1 - [1 - H(x)^{\sigma}]^{\omega}]^{\rho}]^{2}]^{\psi} \tag{3}$$

and

$$f(x)=2\sigma\omega\rho\psi h(x)\left[H(x)\right]^{\sigma-1}\left[1-H(x)^{\sigma}\right]^{\omega-1}\left[1-\left[1-H(x)^{\sigma}\right]^{\omega}\right]^{\rho-1}$$
$$\times\left[1-\left[1-\left[1-H(x)^{\sigma}\right]^{\omega}\right]^{\rho}\right]\left[1-\left[1-\left[1-\left[1-H(x)^{\sigma}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi-1} \tag{4}$$

where $x \geq 0,\ \sigma,\omega,\rho,\psi > 0$ are the shape parameters

3. The Proposed Extended Generalised Frechet (EGFr) Distribution

The proposed Extended Generalized Frechet (EGFr) distribution is derived using the cdf and pdf of the one-parameter Frechet distribution shown in Equations (1) and (2). By inserting these into Equations (3) and (4) respectively, the cdf and pdf of the EGFr distribution are expressed as:

$$F(x)=\left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi} \tag{5}$$

and

$$f(x)=2\sigma\omega\rho\psi\alpha x^{-(\alpha+1)}e^{-\sigma x^{-\alpha}}\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega-1}\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho-1}$$
$$\times\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]\left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi-1} \tag{6}$$

where $x \geq 0,\ \alpha > 0$ is the scale parameter and $\sigma,\omega,\rho,\psi > 0$ are the shape parameters respectively.

Figure 1 displays the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) for the Exponentiated Generalized Frechet (EGFr) distribution across various parameter settings.

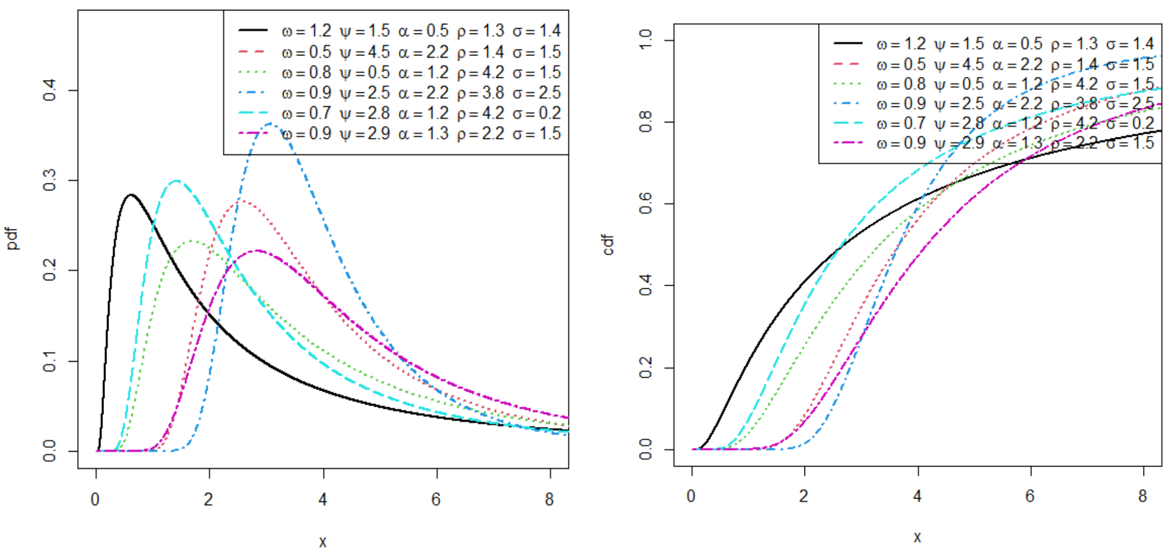

Figure 1. Plots of pdf and cdf of EGFr distribution with different parameter value.

The plots depicted in Figure 1 illustrate how changing the values of $\omega, \psi, \alpha,$ and σ affects the distribution's shape, peak height, and the rate at which the cumulative probability approaches one.

4. Expansion of Density

Using modified binomial expansion, the pdf in (6) is expanded. This can be obtained as follows:

$$(1-\mathrm{x})^{\mathrm{p}-1}=\sum_{\mathrm{i}=1}^{\infty}(-1)^{\mathrm{i}}\binom{\mathrm{p}-1}{\mathrm{i}}\mathrm{x}^{\mathrm{i}} \tag{7}$$

Using Equation (7) on the last term in Equation (6), we have

$$\left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\psi-1}{i}\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2i}$$

$$\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2i+1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{2i+1}{j}\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho j}$$

$$\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho(j+1)-1}=\sum_{k=0}^{\infty}(-1)^{k}\binom{\rho(j+1)-1}{k}\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega k}$$

$$\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega(k+1)-1}=\sum_{m=0}^{\infty}(-1)^{m}\binom{\omega(k+1)-1}{m}\left[e^{-\sigma x^{-\alpha}}\right]^{m}$$

$$f(x)=2\sigma\omega\rho\psi\alpha\sum_{i,j,k,m=0}^{\infty}(-1)^{i+j+k+m}\binom{\psi-1}{i}\binom{2i+1}{j}\binom{\rho(j+1)-1}{k}\binom{\omega(k+1)-1}{m}x^{-(\alpha+1)}\left[e^{-\sigma x^{-\alpha}}\right]^{m+1} \tag{8}$$

Therefore

$$f(x)=\sum_{i,j,k,m=0}^{\infty}\Omega x^{-(\alpha+1)}\left[e^{-\sigma x^{-\alpha}}\right]^{m+1}$$

where

$$\Omega=2\sigma\omega\rho\psi\alpha(-1)^{i+j+k+m}\binom{\psi-1}{i}\binom{2i+1}{j}\binom{\rho(j+1)-1}{k}\binom{\omega(k+1)-1}{m}$$

Also, the expansion for the cumulative distribution function (cdf) in Equation (5) is given as:

$$[F(x)]^{h}=\left[\left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi}\right]^{h} \tag{9}$$

$$\left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi h}=\sum_{w=0}^{h}(-1)^{w}\binom{\psi h}{w}\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2w}$$

$$\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2w}=\sum_{q=0}^{\infty}(-1)^{q}\binom{2w}{q}\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho q}$$

$$\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho q}=\sum_{s=0}^{\infty}(-1)^{s}\binom{\rho q}{s}\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega s}$$

$$\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega s}=\sum_{z=0}^{\infty}(-1)^{z}\binom{\omega s}{z}\left[e^{-\sigma x^{-\alpha}}\right]^{z}$$

$$F(x)=\sum_{w=0}^{h}\sum_{q,s,z=0}^{\infty}(-1)^{w+q+s+z}\binom{\psi h}{w}\binom{2w}{q}\binom{\rho q}{s}\binom{\omega s}{z}\left[e^{-\sigma x^{-\alpha}}\right]^{z} \tag{10}$$

All these extensions are used to obtain some of the statistical and mathematical properties of the new model.

Equations (9) and (10) show that Extended Generalised Frechet (EGFr) densities can be expressed as an infinite linear combination of Frechet (Fr) densities.

5. Properties of EGFr Distribution

This section derives some of the statistical properties including survival function, hazard function, inverse hazard rate, moments, mean, moment generating function, quantile function, median and order statistics of the EGFr distribution.

5.1. Moments and Mean

The following relation can be used to determine the rth moment of X if it has the pdf in Equation (6).

$$E(X^r) = \int_0^\infty x^r f(x) dx \tag{11}$$

$$E(X^r) = \sum_{i,j,k,m=0}^{\infty} \Omega \int_0^\infty x^r \, x^{-(\alpha+1)} \left[e^{-\sigma x^{-\alpha}}\right]^{m+1} dx$$

$$\text{let } y = \sigma(m+1)x^{-\alpha} \Rightarrow x = \left[\frac{y}{\sigma(m+1)}\right]^{\frac{-1}{\alpha}} \Rightarrow dx = \frac{dy}{\sigma\alpha(m+1)x^{-\alpha-1}}$$

$$\int_0^\infty \left[\frac{y}{\sigma(m+1)}\right]^{\frac{-r}{\alpha}} e^{-y} \frac{dy}{\sigma\alpha(m+1)x^{-\alpha-1}} = \int_0^\infty y^{-\frac{r}{\alpha}} e^{-y} dy = \Gamma\left(1-\frac{r}{\alpha}\right)$$

$$E(X^r) = (m+1)^{\frac{r}{\alpha}-1} \sum_{i,j,k,m=0}^{\infty} \Omega\Gamma\left(1-\frac{r}{\alpha}\right) \tag{12}$$

The mean of EGFr distribution is obtained by equating γ = 1 in Equation (11) and it is given as:

$$E(X) = \frac{1}{\alpha}[\sigma(m+1)]^{\frac{1}{\alpha}-1} \sum_{i,j,k,m=0}^{\infty} \Omega\Gamma\left(1-\frac{1}{\alpha}\right) \tag{13}$$

where $\Omega = 2\omega\rho\psi(-1)^{i+j+k+m} \binom{\psi-1}{i}\binom{2i+1}{j}\binom{\rho(j+1)-1}{k}\binom{\omega(k+1)-1}{m}$.

5.2. Moment Generating Function

The moment generating function is defined as:

$$M_{(x)}(t) = \int_0^\infty e^{tx} f(x) dx \tag{24}$$

Since the series expansion for e^{tx} is given as:

$$e^{tx} = \sum_{w=0}^{\infty} \frac{(tx)^w}{w!}$$

$$M_x(t) = \sum_{i,j,k,m=0}^{\infty} \sum_{w=0}^{\infty} \frac{t^w \frac{1}{\alpha}[\sigma(m+1)]^{\frac{w}{\alpha}-1}(-1)^i \Omega\Gamma\left(1-\frac{w}{\alpha}\right)}{w!} \tag{35}$$

5.3. Reliability or Survival Function

The reliability or survival function is given by the following equation

$$R(x) = 1 - \left[1 - \left[1 - \left[1 - \left[1 - e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi} \tag{46}$$

5.4. Hazard Function

As Δ (instantaneous failure rate) approaches 0, the hazard function represents the instantaneous failure rate of a product or item. It can be expressed as:

$$S(x) = \frac{2\sigma\omega\rho\psi\alpha x^{-(\alpha+1)} e^{-\sigma x^{-\alpha}} \left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega-1} \left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho-1} \left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right] \left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi-1}}{1-\left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi}} \tag{57}$$

Figure 2 presents the survival function and hazard function of the Exponentiated Generalized Frechet (EGFr) distribution for six different sets of parameter values. The plots demonstrate how the distribution's reliability characteristics vary.

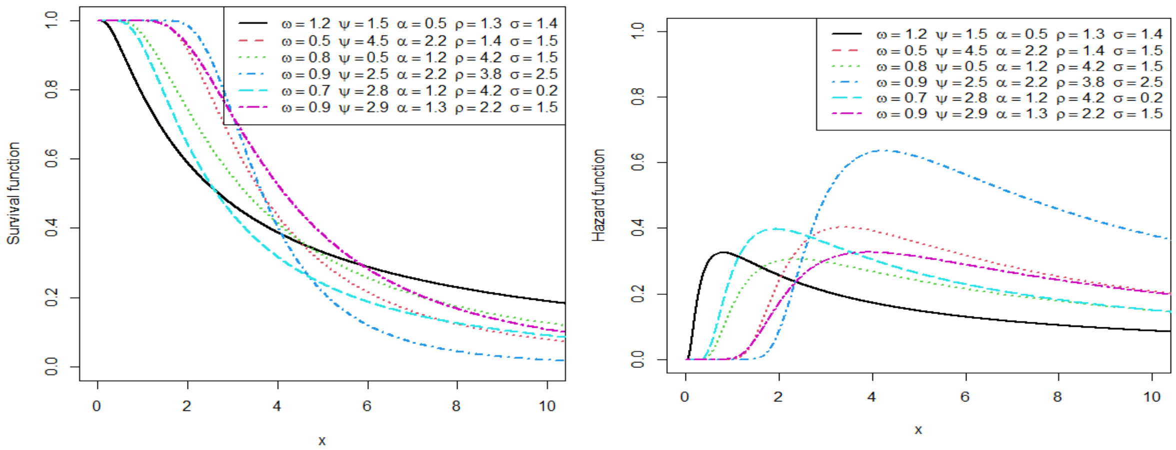

Figure 2. Plots of survival and hazard functions of EGFr distribution with different parameter value.

From Figure 2, the survival Function (Left) shows the probability of a unit surviving beyond a specific time x, illustrating different rates of decay as parameters like $\omega, \psi, \alpha,$ and σ change. The hazard Function (Right) displays the instantaneous failure rate over time, showcasing various shapes such as unimodal (upside-down bathtub) profiles that peak and then gradually decline.

5.5. Reverse Hazard Function

The inverse hazard rate is the ratio between the probability density function and its distribution function. The inverse hazard function of the EGFr distribution is given by:

$$r(x)=\frac{2\sigma\omega\rho\psi h(x)\left[e^{-x^{-\alpha}}\right]^{\sigma-1}\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega-1}\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho-1}\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]\left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi-1}}{\left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi}} \tag{68}$$

5.6. Mills Ratio

The mills ratio is the ratio of the survival function to the pdf. The Mills ratio of the EGFr distribution is given by:

$$M(x)=\frac{1-\left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi}}{2\sigma\omega\rho\psi h(x)\left[e^{-x^{-\alpha}}\right]^{\sigma-1}\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega-1}\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho-1}\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]\left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi-1}} \tag{79}$$

5.7. Quantile Function and Median of EGFr Distribution

Using the inverse of Equation (5), the quantile function of the EGFr distribution is given by:

$$x=Q(q)=\left[\frac{-1}{\sigma}\log\left[1-\left[1-\left[1-\left[1-q^{\frac{1}{\psi}}\right]^{\frac{1}{2}}\right]^{\frac{1}{\rho}}\right]^{\frac{1}{\omega}}\right]\right]^{\frac{-1}{\alpha}} \tag{20}$$

The median of the EGFr distribution is obtained by substituting $q = 0.5$ in Equation (20) and is given by

$$x_{median} = Q(0.5) = \left[\frac{-1}{\sigma}\log\left[1-\left[1-\left[1-\left[1-0.5^{\frac{1}{\psi}}\right]^{\frac{1}{2}}\right]^{\frac{1}{\rho}}\right]^{\frac{1}{\omega}}\right]\right]^{\frac{-1}{\alpha}} \tag{28}$$

5.8. Asymptotic Behavior of EGFr Distribution

The asymptotic behavior of EGFr distribution for $x \to 0$ and $x \to \infty$ are

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\frac{2\sigma\omega\rho\psi h(x)\left[e^{-x^{-\alpha}}\right]^{\sigma-1}\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega-1}\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho-1}}{\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]\left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi-1}}\right] = 0 \tag{22}$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left[\frac{2\sigma\omega\rho\psi h(x)\left[e^{-x^{-\alpha}}\right]^{\sigma-1}\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega-1}\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho-1}}{\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]\left[1-\left[1-\left[1-\left[1-e^{-\sigma x^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]^{\psi-1}}\right] = 0 \tag{23}$$

These two results confirm that the EGFr distribution has a mode.

6. Distribution of Order Statistics

Let $X_1, X_2, X_3, ..., X_n$ be n independent random variables from the EGFr distributions and let $X_1 \leq X_2 \leq X_3 \leq, \ldots, \leq X_n$ be their corresponding order statistics. Let $F_{r:n}(X)$ and $f_{r:n}(X)$, $r = 1, 2, 3, ..., n$ denote the cdf and pdf of the rth order statisticX_{rn} respectively. The pdf of the rth order statistics of X_{rn} is given by

$$f_{r:n}(x) = \frac{f(x)}{B(r, n-r+1)}\sum_{v=0}^{n-r}(-1)^{v}\binom{n-r}{v}F(x)^{v+r-1} \tag{24}$$

The pdf of the r^{th} order statistic for EGFr distribution is obtained also by replacing h with $v+r-1$ in Equation (9), then we have

$$f_{r:n}(x) = \frac{\left[e^{-\sigma x^{-\alpha}}\right]^{m+1}}{B(r, n-r+1)}\sum_{v=0}^{n-r}\sum_{i,j,k,m=0}^{\infty}\Omega\sum_{w=0}^{v+r-1}\Lambda\binom{n-r}{v}\left[\left[e^{-\sigma x^{-\alpha}}\right]^{z}\right]^{v+r-1} \tag{25}$$

The pdf of the minimum order statistic of the distribution is obtained by setting $r = 1$ in Equation (25) as

$$f_{1:n}(x) = n\sum_{v=0}^{n-1}\sum_{i,j,k,m=0}^{\infty}\Omega\sum_{w=0}^{v}\Lambda\binom{n-1}{v}\left[e^{-\sigma x^{-\alpha}}\right]^{zv+m+1} \tag{26}$$

where $\Omega = 2\sigma\omega\rho\psi\alpha(-1)^{i+j+k+m}\binom{\psi-1}{i}\binom{2i+1}{j}\binom{\rho(j+1)-1}{k}\binom{\omega(k+1)-1}{m}x^{-(\alpha+1)}$, and

$$\Lambda = \sum_{q,s,z=0}^{\infty}(-1)^{w+q+s+z}\binom{\psi v}{w}\binom{2w}{q}\binom{\rho q}{s}\binom{\omega s}{z}$$

Also, the pdf of the maximum order statistic of the distribution is obtained by setting $r = n$ in Equation (25) as

$$f_{r:n}(x) = n\sum_{i,j,k,m=0}^{\infty}\Omega\sum_{w=0}^{v+n-1}\Delta\left[e^{-\sigma x^{-\alpha}}\right]^{z(v+n-1)+m+1} \tag{27}$$

where

$$\Delta = \sum_{q,s,z=0}^{\infty}(-1)^{w+q+s+z}\binom{\psi(v+n-1)}{w}\binom{2w}{q}\binom{\rho q}{s}\binom{\omega s}{z}$$

7. Maximum Likelihood Estimation (MLE)

In this section, the parameters of the EGFr distribution were estimated using theMaximum Likelihood Estimation (MLE) method. Let $X_1, X_2, X_3, ..., X_n$ be random sample of size n from the EGFr $(\alpha, \psi, \omega, \rho, \sigma)$ distribution. Then the sample log-likelihood function of the EGFr $(\alpha, \psi, \omega, \rho, \sigma)$ distribution is obtained as:

$$\begin{aligned}\log L = & n\log(2)+n\log(\sigma)+n\log(\omega)+n\log(\rho)+n\log(\psi)+n\log(\alpha)-(\alpha+1)\sum_{i=1}^{n}\log(x_i)\\ & -\sigma\sum_{i=1}^{n}x_i^{-\alpha}+(\omega-1)\sum_{i=1}^{n}\log\left[1-e^{-\sigma x_i^{-\alpha}}\right]+(\rho-1)\sum_{i=1}^{n}\log\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]\\ & +\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]+(\psi-1)\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\partial L}{\partial \omega} = & \frac{n}{\omega}+\sum_{i=1}^{n}\log\sum\left[1-e^{-\sigma x_i^{-\alpha}}\right]-(\rho-1)\sum_{i=1}^{n}\frac{\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\log\left[1-e^{-\sigma x_i^{-\alpha}}\right]}{\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]}\\ & +\sum_{i=1}^{n}\frac{\rho\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho-1}\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\log\left[1-e^{-\sigma x_i^{-\alpha}}\right]}{\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]}\\ & -(\psi-1)\sum_{i=1}^{n}\frac{2\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]\rho\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho-1}\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\log\left[1-e^{-\sigma x_i^{-\alpha}}\right]}{\left[1-\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]}\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\partial L}{\partial \rho} = & \frac{n}{\rho}+\sum_{i=1}^{n}\log\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]-\sum_{i=1}^{n}\frac{\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\log\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]}{\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]}\\ & +(\psi-1)\sum_{i=1}^{n}\frac{2\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\log\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]}{\left[1-\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]}\end{aligned} \tag{30}$$

$$\frac{\partial L}{\partial \psi} = \frac{n}{\psi}+\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right] \tag{31}$$

$$\begin{aligned}\frac{\partial L}{\partial \sigma} = & \frac{n}{\sigma}-\sum_{i=1}^{n}x_i^{-\alpha}+(\omega-1)\sum_{i=1}^{n}\frac{e^{-\sigma x_i^{-\alpha}}\sigma x_i^{-\alpha}\log(x_i^{-\alpha})}{\left[1-e^{-\sigma x_i^{-\alpha}}\right]}+(\rho-1)\sum_{i=1}^{n}\frac{\omega\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega-1}e^{-\sigma x_i^{-\alpha}}\sigma x_i^{-\alpha}\log(x_i^{-\alpha})}{\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]}\\ & +\sum_{i=1}^{n}\frac{\rho\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho-1}\omega\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega-1}e^{-\sigma x_i^{-\alpha}}\sigma x_i^{-\alpha}\log(x_i^{-\alpha})}{\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]}\\ & -(\psi-1)\sum_{i=1}^{n}\frac{2\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]\rho\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho-1}\omega\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega-1}e^{-\sigma x_i^{-\alpha}}\sigma x_i^{-\alpha}\log(x_i^{-\alpha})}{\left[1-\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]}\end{aligned} \tag{32}$$

$$\begin{aligned}\frac{\partial L}{\partial \alpha} = & \frac{n}{\alpha}-\sum_{i=1}^{n}\log(x_i)+\sigma\sum_{i=1}^{n}x_i^{-\alpha}\log(x_i)+(\omega-1)\sum_{i=1}^{n}\frac{e^{-\sigma x_i^{-\alpha}}\sigma x_i^{-\alpha}\log(x_i)}{\left[1-e^{-\sigma x_i^{-\alpha}}\right]}\\ & +(\rho-1)\sum_{i=1}^{n}\frac{\omega\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega-1}e^{-\sigma x_i^{-\alpha}}\sigma x_i^{-\alpha}\log(x_i)}{\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]}+\sum_{i=1}^{n}\frac{\rho\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho-1}\omega\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega-1}e^{-\sigma x_i^{-\alpha}}\sigma x_i^{-\alpha}\log(x_i)}{\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]}\\ & -(\psi-1)\sum_{i=1}^{n}\frac{2\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]\rho\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho-1}\omega\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega-1}e^{-\sigma x_i^{-\alpha}}\sigma x_i^{-\alpha}\log(x_i)}{\left[1-\left[1-\left[1-\left[1-e^{-\sigma x_i^{-\alpha}}\right]^{\omega}\right]^{\rho}\right]^{2}\right]}\end{aligned} \tag{33}$$

These log-likelihood Equations (30)–(33) are non-linear and cannot be solved analytically. Therefore, statistical software such as R with iterative numerical techniques is required to obtain the value of the unknown parameters.

8. Simulation Study and Applications

8.1. Simulation Study

Using a statistical software (R version 4.5.1), a simulation study is carried out to evaluate the efficiency of the MLE. The precisionof the MLE is investigated by the bias and the root mean square error (RMSE) for taking different samples considering different parameter values. 1000 repetitions of theEGFr distribution are performed to quantify the bias and the RMSE by taking samplesof sizes n = 10, 20, 50, 100, 150, 200, 500 and the initial values of the model parameters α, σ, ω, ρ and ψ are chosen as (0.5, 1, 0.6, 0.7, 0.6) in the first case and (1.5, 0.5, 2.1, 2, 3) in the second case. The initial values of the model parameters are also the true values of the model parameter because these are the parameters that the estimated values will mimic (complete details in Supplementary Material).

As the sample size increases, the bias of the estimates decreases, as can be seen from the output in Table 1, resulting in a significant decrease in the mean square errors of the estimates. Furthermore, it can be seen that the estimates get closer to the actual values of the parameters as the sample size increases. Therefore, it can be said that the distribution is consistent.

Table 1. Simulation Output of MLEs, Biases and RMSE for Some Parameters' Values.

		(0.5, 1, 0.6)	**(0.7, 0.6)**		**(1.5, 0.5, 2.1, 2, 3)**		
n	**Parameters**	**Estimated Values**	**Bias**	**RMSE**	**Estimated Values**	**Bias**	**RMSE**
20	σ	0.5059	0.0059	0.2035	1.4983	−0.0017	0.2768
	ρ	1.0091	0.0091	0.1912	0.5046	0.0046	0.4707
	α	0.6595	0.0595	0.1566	2.2933	0.1933	0.1361
	ψ	0.6914	−0.0086	0.1605	1.9997	−0.0003	0.4649
	ω	0.6426	0.0426	0.2457	3.126	0.126	0.5426
50	σ	0.5028	0.0028	0.1559	1.5278	0.0278	0.1969
	ρ	1.0128	0.0128	0.1662	0.5028	0.0028	0.0883
	α	0.6247	0.0247	0.1014	2.176	0.076	0.2794
	ψ	0.7098	0.0098	0.1329	2.0646	0.0646	0.3870
	ω	0.6251	0.0251	0.1842	3.0828	0.0828	0.4676
100	σ	0.5025	0.0025	0.1165	1.5285	0.0285	0.1494
	ρ	1.0146	0.0146	0.1183	0.5005	0.0005	0.0765
	α	0.6112	0.0112	0.0699	2.1431	0.0431	0.1961
	ψ	0.7216	0.0216	0.1065	2.0842	0.0842	0.3068
	ω	0.6165	0.0165	0.1337	3.0413	0.0413	0.4124
150	σ	0.5013	0.0013	0.0953	1.532	0.032	0.1248
	ρ	1.015	0.015	0.1158	0.5003	0.0003	0.0684
	α	0.6043	0.0043	0.0633	2.1228	0.0228	0.1715
	ψ	0.7189	0.0189	0.0948	2.0903	0.0903	0.2427
	ω	0.6181	0.0181	0.1272	3.0359	0.0359	0.3552
200	σ	0.5011	0.0011	0.0785	1.5339	0.0339	0.1084
	ρ	1.0172	0.0172	0.1008	0.5001	0.0001	0.0642
	α	0.6004	0.0004	0.0534	2.111	0.011	0.1443
	ψ	0.7151	0.0151	0.0815	2.0912	0.0912	0.2212
	ω	0.6205	0.0205	0.1117	3.0533	0.0533	0.3327
250	σ	0.5001	0.0001	0.0688	1.5323	0.0323	0.1009
	ρ	1.0139	0.0139	0.0996	0.5	0	0.0598
	α	0.6	0	0.0479	2.1047	0.0047	0.1266
	ψ	0.7165	0.0165	0.0809	2.092	0.0920	0.2027
	ω	0.6159	0.0159	0.1008	3.0494	0.0494	0.3132

8.2. Applications

In this section, the flexibility and robustness of the new distribution is tested using real data sets. The new distribution is compared with four other existing distributions associated with the baseline Fr distribution, viz: FrW distribution $\left(f(x) = \alpha\varphi\sigma^{\alpha}\rho^{\alpha\varphi}x^{-1-\alpha\varphi}e^{-\sigma^{\alpha}\left(\frac{\rho}{x}\right)^{\alpha\varphi}}\right)$, two parameter Fr (2Fr) distribution $\left(f(x) = \alpha\varphi^{\alpha}\rho^{\alpha\varphi}(x)^{\alpha-1}e^{-\left(\frac{\varphi}{x}\right)^{\alpha}}\right)$, one parameter Fr (1Fr) distribution $\left(f(x) = \frac{1}{\alpha}\left(\frac{x}{\alpha}\right)^{-1-\frac{1}{k}}e^{-\left(\frac{x}{\alpha}\right)^{k}}\right)$ and Gompertz inverse Rayleigh (GoIRa) distribution $\left(f(x) = 2\alpha^2\varphi x^{-3}e^{-\left(\frac{\alpha}{x}\right)^2}\left(1-e^{-\left(\frac{\alpha}{x}\right)^2}\right)^{-\sigma-1}e^{\frac{\varphi}{\sigma}\left[1-\left[1-e^{-\left(\frac{\alpha}{x}\right)^2}\right]^{-\sigma}\right]}\right)$.

The MLE method is used to obtain the numerical estimates of the unknown model parameters. Dataset 1 provides the breaking strength measurements in GPa for carbon fibers with a length of 50 mm as shown in Table 2. Similarly, Dataset 2 includes 63 observations regarding 10 mm rod lengths which are presented in Table 3 [22]. The estimates of the unknown parameters and the performance metrics of the models for data set 1 and 2 are presented in Tables 4 and 5, respectively. The plots of the pdf, cdf, Q-Q and P-P plots in Figures 3 and 4 shows that; the EGFr model is a good fit to the two data sets as the models follow the pattern of the data sets. The performance metrics from the numerical results of the analytical measures presented show that the new model is the best fit and gives satisfactory results among the four competing models considered, as seen in Figures 5 and 6.

Table 2. Breaking Strength of Carbon Fibres of 50 mm Length (GPa).

0.85	0.39	1.25	1.08	1.47	1.57	1.61	1.69	1.61	1.8
1.87	1.84	2.03	1.89	2.03	2.05	2.35	2.41	2.12	2.43
2.5	2.48	2.55	2.53	2.55	2.56	2.67	2.73	2.59	2.74
2.81	2.79	2.85	2.82	2.87	2.88	2.95	2.96	2.93	2.97
3.11	3.09	3.15	3.11	3.15	3.19	3.22	3.27	3.22	3.28
3.31	3.31	3.39	3.33	3.39	3.56	3.65	3.68	3.6	3.7
4.2	3.75								

Table 3. Lenghts of 63 Observations of 10 mm Rods.

1.901	2.132	2.203	2.228	2.257	2.35	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618
2.624	2.659	2.675	2.738	2.74	2.856	2.917	2.928	2.937	2.937
2.977	2.996	3.03	3.125	3.139	3.145	3.22	3.223	3.235	3.243
3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501
3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	5.395	5.02							

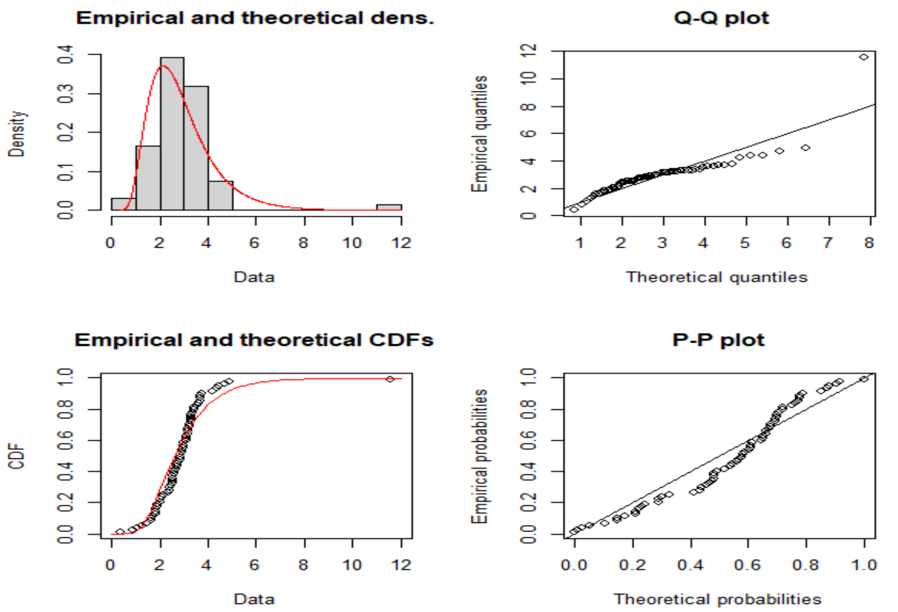

Figure 3. Fitted plots of pdf, cdf, Q-Q and P-P plots for data set 1.

Figure 3 provide a visual assessment of the Exponentiated Generalized Frechet (EGFr) distribution, evaluating both its theoretical flexibility and its practical goodness-of-fit to real-world data set 1. This figure uses four diagnostic plots to show how well the theoretical EGFr model aligns with an empirical data set.

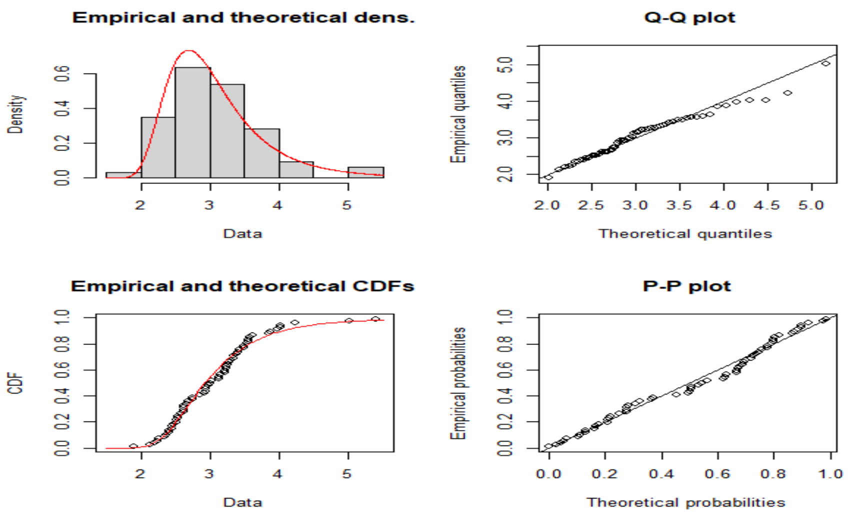

Figure 4. Fitted plots of pdf, cdf, Q-Q and P-P plots for data set 2.

Figure 4 also provide a visual assessment of the Exponentiated Generalized Frechet (EGFr) distribution, evaluating both its theoretical flexibility and its practical goodness-of-fit to real-world data set 2. This figure uses four diagnostic plots to show how well the theoretical EGFr model aligns with an empirical data set.

Table 4 and 5 also shows that the EGFr distribution outperforms the other four distributions.

Table 4. The models' MLEs and performance requirements based on data set 1.

Models	$\widehat{\omega}$	$\widehat{\psi}$	$\widehat{\alpha}$	$\widehat{\rho}$	$\widehat{\sigma}$	ll	AIC	BIC
EGFr	14.960	2.984	0.550	0.644	5.827	−104.667	219.333	230.281
FrW	-	4.142	0.392	1.996	1.148	−124.446	256.893	265.651
2Fr	-	2.064	1.623	-	-	−124.446	252.893	257.272
1Fr	-	-	0.396	-		−248.945	499.891	502.080
GoIRa	-	0.639	0.038	-	1.016	−109.576	225.153	231.723

Table 4 shows that the proposed EGFr distribution has a higher log-likelihood value and lower AIC and BIC values than the remaining four competing distributions, making it more suitable for modeling data set 1. As a result, the proposed EGFr distribution outperformed the other distribution in predicting the reliability dataset.

Table 5. The models' MLEs and performance requirements based on data set 2.

Models	$\widehat{\omega}$	$\widehat{\psi}$	$\widehat{\alpha}$	$\widehat{\rho}$	$\widehat{\sigma}$	ll	AIC	BIC
EGFr	2.297	2.329	1.982	3.135	5.922	−58.084	125.768	136.003
FrW	-	5.864	0.921	2.433	1.934	−61.734	127.668	138.241
2Fr	-	2.723	5.400	-	-	−64.734	133.468	137.754
1Fr	-	-	0.361	-		−252.537	507.017	509.217
GoIRa	-	−4.926	9.720	-	0.386	−62.534	131.068	137.479

According to Table 5, the EGFr distribution is the most suitable model for this dataset because it achieved the highest log-likelihood value alongside the lowest AIC and BIC values compared to the four competing models. Consequently, the EGFr distribution demonstrated superior predictive performance for reliability dataset 2, outperforming all other distributions in the study.

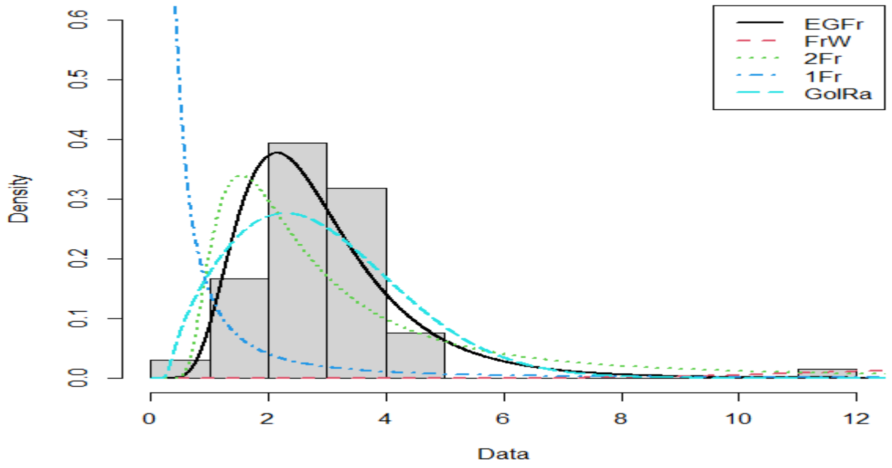

Figure 5. Density plots for EGFr distribution and its comparators for data set 1.

The visual evidence in these Figure 5 supports the statistical findings that the EGFr distribution is the best model for this reliability dataset 1. Predictive Accuracy. It outperformed the competing models (FrW, 2Fr, 1Fr, and GolRa) in modeling the specific skewness and tail behavior of the data.

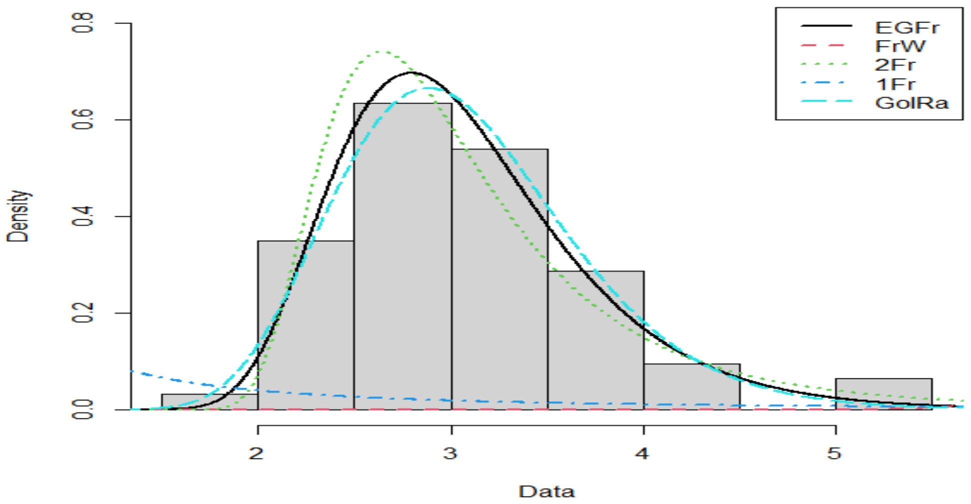

Figure 6. Density plots for EGFr distribution and its comparators for data set 2.

The visual diagnostics in Figure 6 also corroborate the statistical results, confirming that the EGFr distribution is also the best model for reliability dataset 2. By effectively capturing the unique skewness and tail behavior of the data, the EGFr model demonstrates superior predictive accuracy over the competing FrW, 2Fr, 1Fr, and GolRa distributions.

9. Discussion

The Extended Generalized Frechet (EGFr) distribution, as introduced in this study, offers a valuable extension to the existing family of Frechet distributions. Its additional parameters provide enhanced flexibility, allowing it to capture a wider range of data patterns and accurately model complex datasets.

The EGFr's performance was evaluated through extensive simulations and real-life applications. The results consistently demonstrated its superior performance compared to existing Frechet-based distributions. The EGFr's ability to accurately model skewed distributions, heavy-tailed distributions, and multimodal distributions highlights its flexibility and applicability in various fields. Given its demonstrated performance and flexibility, the EGFr distribution is a suitable choice for analyzing reliability data and understanding the probability of failures in engineering systems.

While the study on the Extended Generalized Frechet (EGFr) distribution appears promising, there are a few potential areas where its limitation might exist: Insufficient data could lead to biased or unreliable results, and the EGFr's complexity could lead to increased computational costs, especially for large datasets or complex models.

10. Conclusions

The Extended Generalized Frechet (EGFr) distribution represents a significant improvement in modeling complex data sets. Its increased flexibility, demonstrated accuracy, and flexibility make it a valuable tool for researchers and practitioners in various fields.

Supplementary Materials

The following supporting information can be downloaded at: https://jmasm.com/index.php/jmasm/article/view/1269/s1

Author Contributions

Conceptualization: J.O.B., I.S., O.A.B. and F.M.C.; Data curation: J.O.B., I.S., O.A.B. and F.M.C.; Formal analysis: J.O.B., I.S., O.A.B. and F.M.C.; Validation: J.O.B., I.S., O.A.B. and F.M.C.; Methodology: J.O.B., I.S., O.A.B. and F.M.C.; Software: J.O.B., I.S., O.A.B. and F.M.C.; Visualisation: J.O.B., I.S., O.A.B. and F.M.C.; Supervision: F.M.C.; Writing—original draft: J.O.B., I.S., O.A.B. and F.M.C.; Writing—revision and editing: J.O.B., I.S., O.A.B. and F.M.C. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

No new data were generated or analyzed during this study. This research involved a secondary analysis of existing datasets which are publicly available at https://onlinelibrary.wiley.com/doi/10.1155/2023/4458562.

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Conflicts of Interest

The authors declare no conflicts of interest.

Use of AI and AI-Assisted Technologies

The authors declare that they did not use AI tools for data analysis, image generation, or text drafting in this study.

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