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Abstract

We propose and develop the four-parameter Harris Extended Fréchet distribution. It is obtained by inserting the two-parameter Frechet distribution as the baseline in the Harris family and may be a useful alternative method to model income distribution and could be applied to other areas. We demonstrate that the new distribution can have decreasing, increasing and upside-down-bathtub hazard functions and that its probability density function is an infinite linear combination of Frechet densities. Some standard mathematical properties of the proposed distribution are derived, such as the quantile function, ordinary and incomplete moments, incomplete moments, Lorenz and Bonferroni curves, Gini index, Renyi and β-entropies, mean residual life and mean inactivity time, probability weighted moments, stress-strength reliability, and order statistics. We also obtain the maximum likelihood estimators of the model. The potentiality/flexibility of the new distribution is illustrated by means two applications to failure and waiting time data sets

Keywords: Gine index, Bonferroni curve, Probability Weighted Moments, Strength-Stress Reliability.

1.0 Introduction

Statistical distributions are useful tool in describing and predicting real life data. Although many flexible distributions have been developed and studied widely. However, there are always techniques for developing a more robust and flexible distributions which are adaptable for fitting specific real-life data. This technique involves the addition of one or more additional shape parameters to an existing standard probability (say, a baseline model). This study focused on extending the Fréchet distribution which has been widely applied in extreme value theory. It is an important distribution in extreme value theory and found applications in many fields of applied statistics and this includes: Seismography, life testing, insurance, earthquakes, flood, sea waves, rainfall, medicine, and wind speeds. Some extensions of the Fréchet distribution have been studied in literature to model various types of data and this include: the beta Fréchet distribution by Nadarajah and Gupta (2004), Exponentiated Fréchet Distribution by Badr (2010), The Gamma Extended Fréchet Distribution by Ronaldo et al. (2013), Exponentiated Generalized Fréchet Distribution by Abd-Elfattah et al. (2016), Beta Exponential Fréchet Distribution studied by Mead et (2017), Beta generalized exponentiated Fréchet distribution by Majdah (2019), Badr (2019) proposed and study the Beta generalized exponentiated Fréchet distribution. Fréchet Weibull distribution and Fréchet Weibull mixture distributions are, respectively, proposed and studied by Teamah et al. (2020a, 2020b), Lehman Fréchet Poisson distribution by Ogunde et al. (2021) and Type II Half logistic Fréchet distribution by Ogunde et al. (2023). In this article, we introduce and study an extension of the Fréchet model called the Harris Extended Fréchet (HEF) distribution which is flexible and adaptable to modeling lifetime data of varying degree of skewness.

1.1 Motivation of study

We are motivated to extend the Frechet distribution to a more flexible generalized form called the Harris extended Frechet distribution (HEF) distribution based on the following:

(i) When standard probability distribution is extended by the addition of shape parameter(s), it performs better by providing a good fit when used to model extremely skewed data as compared to baseline (Precim et al. 2012).

(ii) The goodness of fit can be improved upon with the addition of shape parameter(s).

(iii) To further analyse extensively, the tail properties of a distribution one can extend the underlying baseline distribution by the addition of a shape parameter.

(iv) Complex statistical programmes are available such as R program, python, MATLAB, Mapple, etc. to handle the complexity of the analysis.

The rest of the paper is organized as follows: In Section 2, we define the HEF distribution, give some plots for its probability density function(pdf), cumulative density function (cdf), survival function (rf) and hazard rate function (hrf). We derived the, quantile function, examine the asymptotic properties of HEF distribution, determine its nature of skewness and kurtosis, obtain an expression for the r^{th} ordinary moment and incomplete moments, mean deviations, Lorenz and Bonferroni curves, mean residual and mean inactivity time, moment generating function, and Gini index in Section 3. The entropies (Rényi and $\beta -$ entropies), probability weighted moments, stress-strength reliability and the order statistics are derived in Section 4. In Section 5, simulation study was carried out to evaluate the performance of the maximum likelihood estimates. The maximum likelihood estimates of the HEF distribution parameters are the Fisher information matrix is discussed. In Section 6, the usefulness and the flexibility of the HEF distribution is demonstrated by means of two real data sets. Finally, in Section 7 we make concluding remarks.

2.0 Harris Extended Fréchet distribution

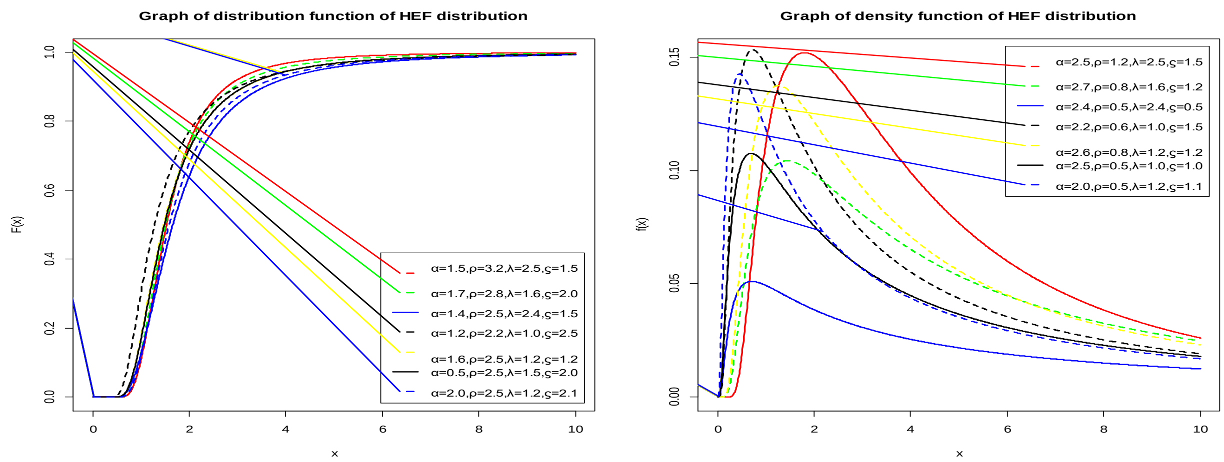
Based on the branching process developed by Harris (1948), a random variable X follows a Harris Extended Fréchet distribution if the Probability density function (pdf) is given by

$$f(x) = \varsigma^{1/\lambda}\alpha\rho x^{-(\rho+1)}e^{-\alpha x^{-\rho}}\left(1-\bar{\varsigma}\left[1-e^{-\alpha x^{-\rho}}\right]^{\lambda}\right)^{-\left(1+1/\lambda\right)}, \qquad (2.1)$$

And the corresponding cumulative density function (cdf) to (2.1) is given by

$$F(x) = 1-\varsigma^{1/\lambda}\left(1-e^{-\alpha x^{-\rho}}\right)\left(1-\bar{\varsigma}\left[1-e^{-\alpha x^{-\rho}}\right]^{\lambda}\right)^{-1/\lambda}. \qquad (2.2)$$

Where λ, ς, and ρ are positive shape parameters and α is a positive scale parameter. Figure 2.1 represent the plot of pdf of HEF model with different values of the parameters.



(a) Plot of the cdf of HEF distribution (b) plot of the pdf of HEF distribution

Figure 2.0. Plots of the cdf and the pdf of the HEF distribution.

- Figure 2.0 indicates that the HEF distribution has proper pdf and that the pdf of the HEF model is non-monotonic.

An expression for the survival and the hazard functions are, respectively, given as

$$s(x) = 1 - F(x) = \varsigma^{1/\lambda}\left(1 - e^{-\alpha x^{-\rho}}\right)\left(1 - \bar{\varsigma}\left[1 - e^{-\alpha x^{-\rho}}\right]^{\lambda}\right)^{-1/\lambda}, \tag{2.3}$$

and

$$h(x) = \frac{\alpha\rho x^{-(\rho+1)} e^{-\alpha x^{-\rho}}\left(1 - \bar{\varsigma}\left[1 - e^{-\alpha x^{-\rho}}\right]^{\lambda}\right)}{\left(1 - e^{-\alpha x^{-\rho}}\right)}. \tag{2.4}$$

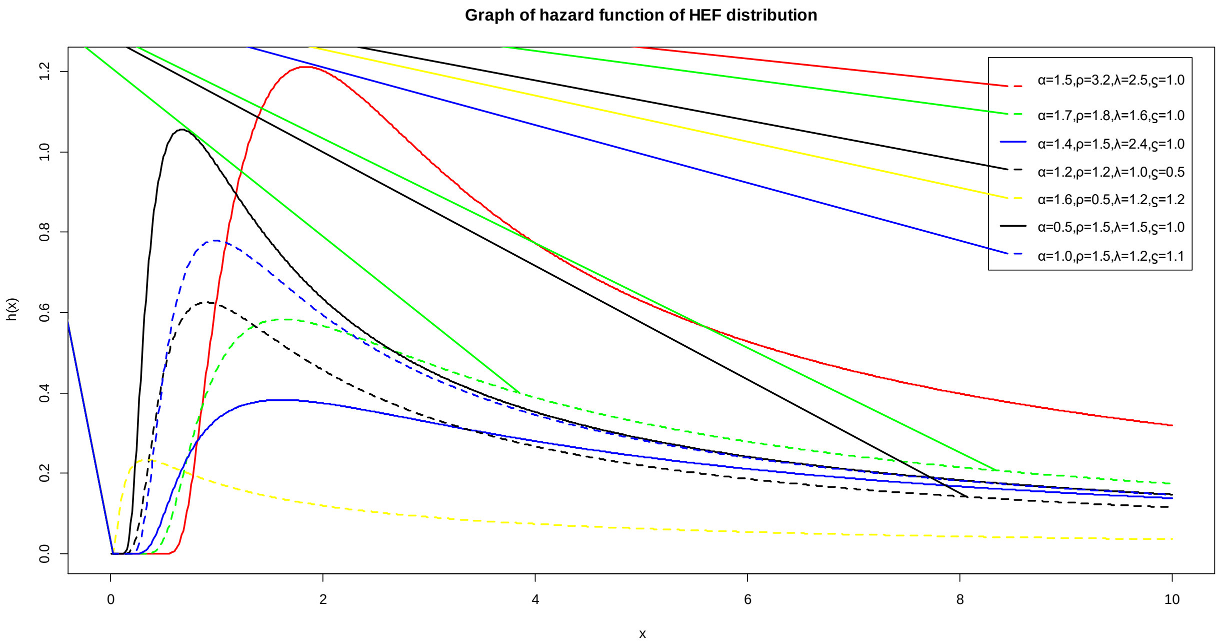


Figure 2.1. Plots of the hazard function HEF distribution

- Figure 2.1 shows that the shape of the hazard function of HEF model can be increasing, decreasing, and inverted-bathtub failure rates.

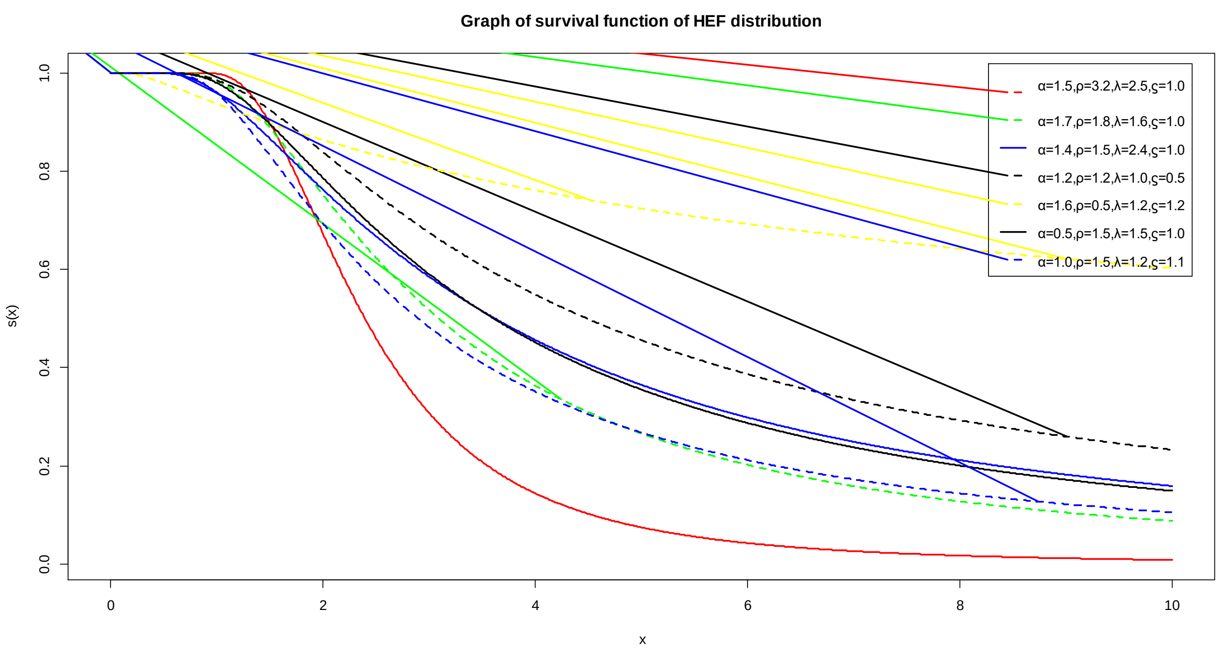


Figure 2.3. Plots of the survival function HEF distribution.

- Figure 2.3 shows that as time increases the survival probability of the HEF distribution approaches zero.

Similarly, the reversed hazard function of the HEF model is

$$\hbar(x) = \frac{\varsigma^{1/\lambda}\alpha\rho x^{-(\rho+1)}e^{-\alpha x^{-\rho}}\left(1-\bar{\varsigma}\left[1-e^{-\alpha x^{-\rho}}\right]^{\lambda}\right)^{-\left(1+1/\lambda\right)}}{1-\varsigma^{1/\lambda}(1-e^{-\alpha x^{-\rho}})(1-\bar{\varsigma}[1-e^{-\alpha x^{-\rho}}]^{\lambda})^{-1/\lambda}} \tag{2.5}$$

it should be noted that cumulative hazard function cannot be described as a probability function but take cognizance of the risk measurement. The higher the value of the risk, the more the failure that will be observed over time x. The cumulative hazard rate function can be represented mathematically as

$$\mathcal{H}(x) = \ell n(1) - {}^{1}/_{\lambda}\,\ell n(\varsigma) + \ell n\left(1-e^{-\alpha x^{-\rho}}\right) - {}^{1}/_{\lambda}\,\ell n\left(1-\bar{\varsigma}\left[1-e^{-\alpha x^{-\rho}}\right]^{\lambda}\right). \tag{2.6}$$

3.0 Statistical properties of the HEF model

The HEF model can be re-written to a reduced a model using generalized binomial series.

$$(1-h)^{z} = \sum_{k=0}^{\infty}(-1)^{k}\binom{z}{k}h^{k}, \tag{3.1}$$

where, $|h| < 1, k > 0$. Now using the binomial series given in (3.1), The pdf of HEF model can be written as a mixture model as

$$f(x) = \rho\sum_{i=0}^{\infty}\delta_{i,j}\,x^{-(\rho+1)}e^{-\alpha(1+j)x^{-\rho}}. \tag{3.2}$$

where,

$$\delta_{i,j} = \varsigma^{1/\lambda}\alpha\sum_{j=0}^{\lambda i}\binom{1/\lambda + i}{i}\binom{\lambda i}{j}(\bar{\bar{\varsigma}})^{i}(-1)^{j}$$

3.1 Quantile function

To investigate the theoretical aspects of the probability distribution, we can employ the use of the quantile function. Mathematically, the quantile function can be expressed in form of $Q(p) = F^{-1}(p)$. Correspondingly, the quantile function of HEF model is

$$Q(p) = \left(-\frac{1}{\alpha} log\left[1 - \frac{(1-p)}{\varsigma(1-\varsigma)(1-p)^{\lambda}}\right]\right)^{-1/\rho}, \quad 0 < p < 1. \tag{3.3}$$

If we take $p = 0.25, 0.5, 0.75$, then will derive an expression for the lower (q_1), middle (q_2), and the upper quartiles (q_3), respectively, as

$$q_1 = \left(-\frac{1}{\alpha} log\left[1 - \frac{0.75}{\varsigma(1-\varsigma)(0.75)^{\lambda}}\right]\right)^{-1/\rho}, \tag{3.4}$$

$$q_2 = \left(-\frac{1}{\alpha} log\left[1 - \frac{0.5}{\varsigma(1-\varsigma)0.5^{\lambda}}\right]\right)^{-1/\rho}, \tag{3.5}$$

and

$$q_3 = \left(-\frac{1}{\alpha} log\left[1 - \frac{0.25}{\varsigma(1-\varsigma)0.25^{\lambda}}\right]\right)^{-1/\rho}. \tag{3.6}$$

3.2 Asymptotic Behaviour of HEF model

To determine the asymptotic behaviour, we have to examine, $\lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x)$.

$$\lim_{x\to 0} \frac{\varsigma^{1/\lambda}\alpha\rho x^{-(\rho+1)}e^{-\alpha x^{-\rho}}}{(1-\bar{\varsigma}[1-e^{-\alpha x^{-\rho}}]^{\lambda})^{(1+1/\lambda)}} = \lim_{x\to\infty} \frac{\varsigma^{1/\lambda}\alpha\rho x^{-(\rho+1)}e^{-\alpha x^{-\rho}}}{(1-\bar{\varsigma}[1-e^{-\alpha x^{-\rho}}]^{\lambda})^{(1+1/\lambda)}} = 0. \tag{3.7}$$

The result indicates that both limits are existing, hence the HEF model has conformed with the unimodal distribution. However, since $f(x) > 0$ and $\frac{\partial f(x)}{\partial x} = 0$ then (2.1) becomes

$$-\frac{(\rho+1)}{x} + \alpha\rho x^{-(\rho+1)} - \left(1 + 1/\lambda\right)\frac{\alpha\rho x^{-(\rho+1)}e^{-\alpha x^{-\rho}}\left[1-e^{-\alpha x^{-\rho}}\right]}{(1-\bar{\varsigma}[1-e^{-\alpha x^{-\rho}}]^{\lambda})} = 0 \tag{3.8}$$

Solution to (3.9) cannot be obtained analytically because it is non-linear equation. The solution can only be obtained using numerical process such as the Newton-Raphson iteration procedure.

3.3 Skewness and Kurtosis

The skewness and Kurtosis employed in statistical analysis to measure some desirable characteristics of a distribution. Bowley's skewness was developed using quartile by Bowley (1920) and is of the form

$$\mathcal{B}_{sk} = \frac{Q(0.75) - 2Q(0.5) + Q(0.25)}{Q(0.75) - Q(0.25)}.$$

Moore (1988) developed the Moore kurtosis using octiles and it can be estimated as

$$\mathcal{M}_k = \frac{Q(0.875) - Q(0.625) + Q(0.375) - (0.125)}{Q(0.75) - Q(0.25)}.$$

where, $Q(.)$ is the quantile function defined in (2.7). Table 3.0 gives the q_1, q_2, q_3, $\mathcal{B}_{sk}$, and $\mathcal{M}_k$ of the HEF model for fixed values of $\alpha = 0.5$, $\rho = 1.2$ and various values of ς, and λ.

Table 3.0: Values of q_1, q_2, q_3, $\mathcal{B}_{sk}$, and $\mathcal{M}_k$ of the HEF model

ς, λ	q_1	q_2	q_3	$\mathcal{B}_{sk}$	$\mathcal{M}_k$
0.1, 0.1	1.0592	3.8174	21.1937	0.7260	0.0240
0.2,0.2	1.3166	4.4899	24.2670	0.7235	4.4714
0.3,0.3	1.4906	4.9405	26.3122	0.7249	4.4587
0.4,0.4	1.5618	5.1243	27.1432	0.7215	4.4540
0.5,0.5	1.5271	5.0348	26.7385	0.7217	4.4563
0.6,0.6	1.4010	4.7088	25.262	0.7227	4.4650
0.7,0.7	1.2145	4.2237	23.0543	0.7244	4.4797
0.8,0.8	1.0097	3.6870	20.5946	0.7266	4.4990
0.9,0.9	0.8349	3.2249	18.4690	0.7289	4.5165

3.4 Moments of HEF model

Moments are very properties for any statistical investigation, most especially in application areas. Suppose $X \sim HEF(\alpha, \varsigma, \rho, \lambda)$,then many important features such as dispersion, skewness, measures of central tendency, and kurtosis of the HEF model can be derived by using ordinary moments. The r^{th} raw moment of the HEF model is obtained as

$$E(X)^r = \mu'_r = \int_{-\infty}^{\infty} x^r f(x)dx = \rho \sum_{i=0}^{\infty} \delta_{i,j} \int_{-\infty}^{\infty} x^{r-(\rho+1)} e^{-\alpha(1+j)x^{-\rho}} dx, \qquad (3.9)$$

Letting $y = \alpha(1+j)x^{-\rho},\ x = [\alpha(1+j)]^{1/\rho} y^{-1/\rho},\ dx = \rho^{-1}[\alpha(1+j)]^{1/\rho} y^{-1/\rho-1} dy$, putting in (11), we have

$$\mu'_r = \sum_{i=0}^{\infty} \delta_{i,j}\, [\alpha(1+j)]^{(r-\rho)/\rho} \int_{0}^{\infty} y^{\frac{-r}{\rho}} e^{-y} dx, \qquad (3.10)$$

Finally, we have an expression for the r^{th} raw moment of HEF distribution as

$$\mu'_r = \sum_{i=0}^{\infty} \delta_{i,j}\, [\alpha(1+j)]^{(r-\rho)/\rho} \Gamma\left(1 - {r}/{\rho}\right); \qquad \rho > r. \qquad (3.11)$$

Where $\Gamma\left(1 - {r}/{\rho}\right)$ is a gamma function. In the like manner, incomplete moments play a vital role in measuring inequality. Hence, we can determine the lower incomplete moments, say, $\gamma_s(t)$ is given by

$$\gamma_s(t) = \int_{0}^{t} x^s f(x)dx = \rho \sum_{i=0}^{\infty} \delta_{i,j} \int_{-\infty}^{t} x^{s-(\rho+1)} e^{-\alpha(1+j)x^{-\rho}} dx \qquad (3.12)$$

Letting $y = \alpha(1+j)x^{-\rho},\ x = [\alpha(1+j)]^{1/\rho} y^{-1/\rho},\ dx = -\rho^{-1}[\alpha(1+j)]^{1/\rho} y^{-1/\rho-1} dy$, putting in (3.12), we have

$$\gamma_s(t) = \sum_{i=0}^{\infty} \delta_{i,j}\, [\alpha(1+j)]^{(r-\rho)/\rho} \int_{-\infty}^{t} y^{\frac{-s}{\rho}} e^{-y} dx, \qquad (3.13)$$

Finally, we have an expression for the r^{th} incomplete moment of HEF distribution as

$$\gamma_s(t) = \sum_{i=0}^{\infty} \delta_{i,j} \, [\alpha(1+j)]^{(s-\rho)/\rho} \Gamma\left(1 - {}^{s}/_{\rho}, \alpha(1+j)t^{-\rho}\right); \qquad \rho > s. \tag{3.14}$$

An expression for the first incomplete moment of HEF distribution is obtained by taken $s = 1$ in (3.14) and given as

$$\gamma_1(t) = \sum_{i=0}^{\infty} \delta_{i,j} \, [\alpha(1+j)]^{(1-\rho)/\rho} \Gamma\left(1 - {}^{1}/_{\rho}, \alpha(1+j)t^{-\rho}\right). \tag{3.15}$$

Where $\Gamma\left(1 - {}^{1}/_{\rho}, \alpha(1+j)t^{-\rho}\right)$ is an incomplete gamma function. Consequently, an expression for the variance, skewness and the kurtosis can respectively, be obtained as follows $\sigma^2 = \mu_2' - [\mu_1']^2$, $\boldsymbol{s_k} = \mu_3^2(\sqrt{\mu_2})^{-3}$ and $k_u = \mu_4(\mu_2)^{-2}$ respectively. Where $\mu_r = E[(x - \mu_1')^r]$, $\mu_3 = -3\mu_2'\mu_1' + \mu_3' + 2(\mu_1')^3$ and $\mu_4 = 6(\mu_1')^2\mu_2' - 3(\mu_1')^4 - 4\mu_3'\mu_1' + \mu_4'$

Tables 3.1, 3.2 gives the values for the first six moments, variance (σ^2), skewness, and kurtosis of HEF model for fixed values of α and ρ and varying the values of ς and λ.

Table 3.1. First six moments, σ^2, $\boldsymbol{s_k}$, and k_u of the HEF model

$\alpha = 5.5, \rho = 10.5$					
Moments	$\lambda = 1.2$, $\varsigma = 0.1$	$\lambda = 1.4$, $\varsigma = 0.3$	$\lambda = 1.8$, $\varsigma = 0.5$	$\lambda = 2.5$, $\varsigma = 0.8$	$\lambda = 5.5$, $\varsigma = 0.8$
μ_1'	1.1109	1.1741	1.2108	1.2411	1.2459
μ_2'	1.2454	1.3977	1.490	1.5673	1.5799
μ_3'	1.4120	1.6925	1.8699	2.0204	2.0455
μ_4'	1.6242	2.0935	2.4042	2.6706	2.7163
μ_5'	1.9054	2.6625	3.1881	3.6440	3.7241
μ_6'	2.298	3.5173	4.4065	5.1862	5.3263
σ	0.0113	0.0192	0.0240	0.0270	0.0276
$\boldsymbol{s_k}$	2.8102	2.4078	2.0968	1.8687	1.7952
k_u	19.7087	12.0421	11.2802	10.4996	10.8794

Table 3.2. Moments, σ^2, $\boldsymbol{s_k}$, and k_u of the HEF model

$\alpha = 2.1, \rho = 7.5$					
$moments$	$\lambda = 0.2,$ $\varsigma = 0.1$	$\lambda = 0.4,$ $\varsigma = 0.3$	$\lambda = 0.8,$ $\varsigma = 0.5$	$\lambda = 1.5,$ $\varsigma = 0.8$	$\lambda = 2.5,$ $\varsigma = 0.8$
μ_1'	0.9759	1.0621	1.1306	1.1903	1.1957
μ_2'	0.9564	1.1446	1.3144	1.4680	1.4828
μ_3'	0.9415	1.2555	1.5855	1.8967	1.9281
μ_4'	0.9311	1.4101	2.0165	2.6153	2.6788
μ_5'	0.9254	1.6408	2.7942	3.9919	4.1248
μ_6'	0.9245	2.0454	4.5971	7.3938	7.7160
σ^2	0.0040	0.0165	0.0361	0.0512	0.0531
$\boldsymbol{s_k}$	1.1954	2.1987	2.5775	2.3735	2.2988
k_u	-5.2608	20.9734	19.3344	16.0091	15.8564

3.5 Mean deviation

The amount of variability observed in a distribution can be measured using the deviations about the mean and about the median. The mean deviation about the mean and the mean deviation about the median can be estimated respectively, USING

$$\varDelta_1(x) = 2\left[\mu F(\mu) - \int_0^{\mu} xf(x)dx\right] \tag{3.16}$$

and

$$\varDelta_2(x) = E(x) + 2MF(M) - M - 2\int_0^{M} xf(x)dx \tag{3.17}$$

Where μ is the mean of HEF obtained from (3.11), taking $r = 1$, M is the median (q_2). Hence, we obtain

$$\Delta_1(x) = 2\left[\mu F(\mu) - \sum_{i=0}^{\infty} \delta_{i,j} \left[\alpha(1+j)\right]^{(1-\rho)/\rho} \Gamma\left(1 - {}^{1}/_{\rho}, \alpha(1+j)\mu^{-\rho}\right)\right],$$

and

$$\Delta_2(x) = E(x) + 2MF(M) - M - 2\sum_{i=0}^{\infty} \delta_{i,j} \left[\alpha(1+j)\right]^{(1-\rho)/\rho} \Gamma\left(1 - {}^{1}/_{\rho}, \alpha(1+j)M^{-\rho}\right)$$

3.6 Lorenz and Bonferroni curves

Lorenz index (l_p) was introduced by American economist Lorenz (1905). He developed a graphical diagram that illustrates wealth distribution called Lorenz curve. The Lorenz index is defined as

$$\boldsymbol{l_p} = \frac{\mathbf{1}}{\mu}\int_0^x xf(x)dx. \tag{3.18}$$

The Lorenz plot is described as a plot of Lorenz index, $\boldsymbol{l_p}$, versus x. Given below is an expression for the Lorenz index for HEF distribution

$$\boldsymbol{l_p} = \frac{\sum_{i=0}^{\infty} \delta_{i,j} \left[\alpha(1+j)\right]^{(1-\rho)/\rho} \Gamma\left(1 - {}^{1}/_{\rho}, \alpha(1+j)M^{-\rho}\right)}{\sum_{i=0}^{\infty} \delta_{i,j} \left[\alpha(1+j)\right]^{(1-\rho)/\rho} \Gamma\left(1 - {}^{1}/_{\rho}\right).} \tag{3.19}$$

Bonferroni curve was developed by Bonferroni (1930) as a measure of income inequality and was founded on partial means, required when the source of income inequality is the occurrence of units whose income is much lower when compared to others. The Bonferroni index, $\boldsymbol{B_p}$**,** can be obtained using

$$B_p = \frac{l_p}{F(x)}. \tag{3.20}$$

This is a plot of Bonferroni index versus x, and this index for HEF model is given by

$$B_p = \frac{\sum_{i=0}^{\infty} \delta_{i,j} \left[\alpha(1+j)\right]^{(1-\rho)/\rho} \Gamma\left(1 - {}^{1}/_{\rho}, \alpha(1+j)M^{-\rho}\right)}{1 - \varsigma^{1/\lambda}(1 - e^{-\alpha x^{-\rho}})(1 - \bar{\varsigma}[1 - e^{-\alpha x^{-\rho}}]^{\lambda})^{-1/\lambda}\left(\sum_{i=0}^{\infty} \delta_{i,j} \left[\alpha(1+j)\right]^{(1-\rho)/\rho} \Gamma\left(1 - {}^{1}/_{\rho}\right)\right)}.$$

3.7 Mean residual life and mean inactivity time

The mean Residual life (MRL) can be used to describe the additional expected life span of a unit, which happens to be alive at age x. It can be applied in many areas of life which includes; product quality control, biomedical sciences, life insurance, demography among many others. The MRL is defined as

$$MRL(t) = E(X - t/x > t) = \frac{1 - \gamma_1(t)}{S(t)} - t, \qquad t > 0, \tag{3.21}$$

Consequently, we have

$$MRL(t) = \frac{1 - \sum_{i=0}^{\infty} \delta_{i,j} \left[\alpha(1+j)\right]^{(1-\rho)/\rho} \Gamma\left(1 - {}^{1}/_{\rho}, \alpha(1+j)t^{-\rho}\right)}{\varsigma^{1/\lambda}(1 - e^{-\alpha t^{-\rho}})(1 - \bar{\varsigma}[1 - e^{-\alpha t^{-\rho}}]^{\lambda})^{-1/\lambda}} - t \tag{3.22}$$

The mean inactivity time (MIT) is used to describe the waiting time elapsed since the failure of an item occurred on the condition that this failure occurred between an interval $(0, t)$. Mathematically, it is represented as

$$MIT(x) = t - \frac{\gamma_1(t)}{F(t)}, \tag{3.23}$$

$$= t - \frac{\sum_{i=0}^{\infty} \delta_{i,j} \left[\alpha(1+j)\right]^{(1-\rho)/\rho} \Gamma\left(1 - {}^{1}/_{\rho}, \alpha(1+j)t^{-\rho}\right)}{1 - \varsigma^{1/\lambda}(1 - e^{-\alpha t^{-\rho}})(1 - \bar{\varsigma}[1 - e^{-\alpha t^{-\rho}}]^{\lambda})^{-1/\lambda}}.$$

3.8 Moment generating function of HEF model

The moment generating function (MGF) of a random variable X provides an alternative method that can be used in describing the characteristics of a distribution. Mathematically, the MGF is defined as

$$\mathcal{M}_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r). \tag{3.24}$$

Putting (3.11) in (3.24) for $E(X^r)$ for HEF model, we obtain

$$\mathcal{M}_X(t) = \sum_{i=r=0}^{\infty} \frac{t^r}{r!} \delta_{i,j} \left[\alpha(1+j)\right]^{(r-\rho)/\rho} \Gamma\left(1 - {}^{r}/_{\rho}\right); \qquad \rho > r. \tag{3.25}$$

By setting $t = it$ in (3.25), we derived an expression for the characteristics function of the HEF model.

3.9 Gini Index

One of the well-known inequality indexes is the Gini index developed by Gini (1914), is defined as

$$G = \frac{1}{E(x)} \int_0^\infty [F(x) - F(x)^2] dx, \tag{3.26}$$

Inserting (2.2) in (3.26), we have

$$G = \frac{1}{E(x)} \left\{ x - V_{ij} \int_0^\infty e^{-j\alpha x^{-\rho}} dx - V_{klp} \int_0^\infty e^{-p\alpha x^{-\rho}} dx \right\}, \tag{3.27}$$

Consequently, taking $z = j\alpha x^{-\rho}$and further $u = p\alpha x^{-\rho}$, we have

$$G = \frac{1}{E(x)} \left\{ x - \alpha^{1/\rho} \left[M_{ij} \frac{j^{1/\rho}}{\rho} \int_0^\infty z^{-1/\rho - 1} e^{zu} dz - M_{kl} \frac{p^{1/\rho}}{\rho} \int_0^\infty u^{-1/\rho - 1} e^{-u} \right] \right\}, \tag{3.28}$$

Finally, we obtain an expression for the coefficient of Gini index as

$$G = \frac{\left[x - \alpha^{1/\rho} \Gamma\left(-1/\rho\right) \left\{ M_{ij} j^{1/\rho} - M_{klp} p^{1/\rho} \right\} \right]}{\rho \sum_{i=0}^{\infty} \delta_{i,j} \left[\alpha(1+j)\right]^{(r-\rho)/\rho} \Gamma\left(1 - r/\rho\right)} \tag{3.29}$$

where

$$V_{ij} = \frac{\varsigma^{1/\alpha}}{\rho} \sum_{i=j=0}^{\infty} \binom{1/\lambda + i - 1}{i} \binom{\lambda i + 1}{j} (-1)^j \, (\bar{\varsigma})^i,$$

and

$$V_{klp} = \sum_{k=l=p=0}^{\infty} \binom{2}{k} \binom{k/\lambda + l - 1}{l} \binom{\lambda l + 1}{p} (-1)^{k+p} \, \varsigma^{k/\lambda} (\bar{\varsigma})^l$$

4.0 Entropies

The Rényi entropy of a random variable X with density function $f(x)$ can be described as a measure of variation off uncertainty or randomness and its defined (for $\zeta > 0$ and $\zeta \neq 1$) as;

$$I_R(\zeta) = \frac{1}{1-\zeta} log[M(\zeta)], \tag{4.1}$$

where

$$M(\zeta) = \int_{-\infty}^{\infty} f^{\zeta}(x)dx \tag{4.2}$$

Inserting (2.1) in (4.2), we have

$$M(\zeta) = \int_{-\infty}^{\infty} \left[\varsigma^{1/\lambda} \alpha\rho x^{-(\rho+1)} e^{-\alpha x^{-\rho}} \left(1 - \bar{\varsigma}\left[1 - e^{-\alpha x^{-\rho}}\right]^{\lambda}\right)^{-\left(1+1/\lambda\right)} \right]^{\zeta} dx \tag{4.3}$$

Upon simplification, we obtain

$$M(\zeta) = n_{i,j}\Gamma\left(\frac{\zeta(\rho+1)-1}{\rho}\right), \tag{4.4}$$

where

$$n_{i,j} = \varsigma^{\zeta/\lambda}\alpha^{\zeta}\rho^{\zeta-1}\sum_{i,j}^{\infty}\binom{\zeta\left(1/\lambda+1\right)+i-1}{i}\binom{i\lambda}{j}(-1)^{i}\bar{\varsigma}^{i}\left(\frac{1}{\alpha(i+1)}\right)$$

Putting (4.4) in (4.1), we generate an expression for the Rényi entropy of HEF model as

$$I_R(\zeta) = \frac{1}{1-\zeta} log\left[n_{i,j}\Gamma\left(\frac{\zeta(\rho+1)-1}{\rho}\right)\right]. \tag{4.5}$$

The β −entropy, $\mathcal{H}_{\beta}$, is defined by

$$\mathcal{H}_{\beta} = \frac{1}{\beta-1} log\{1-(1-\beta)I_R(\zeta)\}.$$

Using, $I_R(\zeta)$, we obtain an expression for the β −entropy of HEF model as

$$\mathcal{H}_{\beta} = \frac{1}{\beta - 1} log \left\{1 - (1-\beta)\frac{1}{1-\zeta} log \left[n_{i,j}\Gamma\left(\frac{\zeta(\rho+1)-1}{\rho}\right)\right]\right\}. \tag{4.6}$$

4.1 Probability Weighted Moment (PWM)

The PWM is a better alternative to the existing moment for estimation of parameter, most especially in a situation when maximum likelihood estimation procedure is difficult to apply or completely unavailable. It is defined as

$$Л_{r,v} = E[X^r F(X)^r] = \int_{-\infty}^{\infty} x^r f(x) F(x)^v d(x). \tag{4.7}$$

If X ~ HEF (Л), then $Л_{r,v}$ is given by

$$Л_{r,v} = \alpha \sum_{i=j=k=0}^{\infty} \mathcal{H}^{*} \Gamma\left(\frac{r+\rho+1}{\rho} + 1\right). \tag{4.8}$$

where, $\mathcal{H}^{*} = \binom{1/\lambda + i/\lambda + j - 1}{j}\binom{v}{i}\binom{\lambda j + i}{k}\varsigma^{\frac{i+1}{\lambda}}\bar{\varsigma}^{j}\left(\frac{1}{\alpha(1+k)}\right)^{\frac{r+1}{\rho}}(-1)^{k+i}$

4.2 Stress strength Reliability

Here, we derived an expression for the stress-strength parameter of HEF model. Suppose X_1 represents the strength of a structure experiencing stress X_2, and if X_1 follows HEF $(\alpha, \rho, \varsigma_1, \lambda_1)$ and X_2 follows HEF $(\alpha, \rho, \varsigma_2, \lambda_2)$given that $X_1$and X_2 are independent random variables. Then the mathematical expression for the Stress-strength Reliability ($\mathfrak{K}$) of HEF model is obtained as follows:

$$R = P(X_2 < X_1) = \int_{0}^{\infty} f_1(x; \alpha, \rho, \varsigma_1, \lambda_1) F_2(x; \alpha, \rho, \varsigma_2, \lambda_2) dx \tag{4.9}$$

If X ~ HEF (Л), then R is given by

$$R = F_1(x; \alpha, \rho, \varsigma_2, \lambda_2) - \alpha \varsigma_1^{1/\lambda_1} \varsigma_2^{1/2} (C_1 - C_2) \tag{4.10}$$

where

$$C_1 = \sum_{i=j=k=l}^{\infty} \binom{\frac{1}{\lambda_1}+i}{i} \binom{i\lambda_1}{j} \binom{\frac{1}{\lambda_2}+k-1}{k} \binom{k\lambda_2}{l} \bar{\varsigma}_1^{i} \bar{\varsigma}_2^{k} (-1)^{j+l} \left(\frac{1}{\alpha(1+j+l)}\right)$$

And

$$C_2 = \sum_{m=n=p=q}^{\infty} \binom{\frac{1}{\lambda_2}+m}{m} \binom{m\lambda_2}{n} \binom{\frac{1}{\lambda_2}+p-1}{p} \binom{p\lambda_2}{q} \bar{\varsigma}_1^{m} \bar{\varsigma}_2^{p} (-1)^{n+p} \left(\frac{1}{\alpha(1+j+l)}\right)$$

4.3 Order statistics

Given $x_1, x_2, x_3 \ldots, x_n$ as a random sample having CDF $F(x)$. Let $X_{1:1}, X_{2:n}, X_{3:n}, \ldots, x_{n:n}$ is the ordered sample of size n, then the density of j^{th} order statistics is given as

$$g_{j:n}(x) = W^* \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} f(x) F(x)^{i+j-1}, \tag{4.11}$$

Where $W^* = \frac{n!}{(n-r)!r!}$.

Putting (2.1) and (2.2) in (4.11), followed by simply algebraic manipulation gives

$$g_{j:n}(x) = 2\alpha\rho\varsigma^{1/\lambda} W^* \sum_{i=k=l=m=0}^{n-r} (-1)^{i+k+m} \binom{n-r}{i} \binom{i+j-1}{k} \binom{\frac{k}{\lambda}+\frac{1}{\lambda}+l}{l} \binom{\lambda l+k}{m} \varsigma^{k/\lambda}$$
$$\times \bar{\varsigma}^{l} x^{-(\rho+1)} e^{-(m+1)\alpha x^{-\rho}}.$$

5.0 Maximum Likelihood Estimation (MLE)

Given that $x_1, x_2, \ldots, x_n$ are the observed sample values from the HEF distribution. The log-likelihood (l) function is defined as follows:

$$l(\eta, \varphi, \rho) = {}^{n}/_{\lambda} \, log(\varsigma) + nlog(\alpha) + nlog(\rho) - (\rho+1) \sum_{i=1}^{\infty} log(x_i) - \alpha \sum_{i=1}^{\infty} x^{-\rho}$$
$$\times \left(1 + {}^{1}/_{\lambda}\right) \sum_{i=1}^{n} log\left(1 - (1-\varsigma)\left[1 - e^{-\alpha x^{-\rho}}\right]^{\lambda}\right). \tag{5.1}$$

Maximizing $l(\alpha, \varsigma, \rho, \lambda)$ with respect to α, ς, ρ and λ, we derive the following system of nonlinear equations:

$$\frac{n}{\alpha}-\sum_{i=0}^{\infty}x_i^{-\rho}-\left(\frac{\lambda(1+\varsigma)}{\varsigma}\right)\sum_{i=0}^{\infty}\frac{x_i^{-\rho}e^{-\alpha x_i^{-\rho}}y_i^{\lambda}}{[1-e^{-\alpha x_i^{-\rho}}](1-(1-\varsigma)y_i^{\lambda})}=0. \tag{5.2}$$

$$-\frac{n}{\lambda^2}log(\varsigma)+\frac{1}{\lambda^2}\sum_{i=1}^{n}log\left(1-(1-\varsigma)y_i^{\lambda}\right)-\left(1+{}^{1}/_{\lambda}\right)\sum_{i=1}^{n}\frac{y_i log y_i}{[log(1-(1-\varsigma)y_i^{\lambda})]}=0 \tag{5.3}$$

$$\frac{n}{\rho}-\sum_{i=1}^{\infty}log(x_i)+\alpha\sum_{i=1}^{\infty}x_i^{-\rho}\log(x_i)-\alpha(1+\lambda)\sum_{i=1}^{n}\frac{(1-\varsigma)x_i^{-\rho}log(x_i)e^{-\alpha x_i^{-\rho}}y_i}{[log(1-(1-\varsigma)y_i^{\lambda})]}=0 \tag{5.4}$$

and

$$\frac{n}{\lambda\varsigma}-\left(1+{}^{1}/_{\lambda}\right)\sum_{i=1}^{n}\frac{y_i^{\lambda}}{[log(1-(1-\varsigma)y_i^{\lambda})]}=0 \tag{5.5}$$

Where $y_i=\left[1-e^{-\alpha x_i^{-\rho}}\right]$. Solving equations (5.2) - (5.5) simultaneously we produce the MLEs of $\hat{\alpha},\hat{\rho},\hat{\lambda}$, and $\hat{\varsigma}$. To obtain an approximate confidence interval (CIs) of the parameters of HEF model, it is necessary to obtain an estimate of the elements of the variance covariance matrix D of the MLEs. The variance-covariance matrix D is calculated by the observed information matrix $\widehat{D}$, and

$$\widehat{D}=-\begin{bmatrix}S_{11} & S_{12} & S_{13} & S_{14}\\ S_{21} & S_{22} & S_{23} & S_{24}\\ S_{31} & S_{32} & S_{33} & S_{34}\\ S_{41} & S_{42} & S_{43} & S_{44}\end{bmatrix}. \tag{5.6}$$

Where $S_{i,j}$, $i,j=1,2,3,4$ represent the second partial derivatives of (5.1) with respect to α,ρ,λ, and α. This value represents the Fisher's information matrix analogous to α,ρ,λ, and α, respectively. the elements at the diagonal of the matrix given in (5.6) is the values of the variance of the MLEs of to α,ρ,λ, and ς, respectively. an estimated $100(1-f)\%$ confidence interval for ζ_c as

$$\widehat{\zeta_c}\pm Z_{\frac{d}{2}}\sqrt{\widehat{var}\left(\widehat{\zeta_c}\right)},$$

Where $\widehat{\zeta_c} = (\hat{\alpha}, \hat{\rho}, \hat{\lambda}, \hat{\varsigma})$, $Z_{\frac{d}{2}}$ is the upper $\left(\frac{d}{2}\right)$ 100^{th} percentile of normal distribution. The likelihood ratio (LR) test can be used compare the performance of HEF model with its sub-models for any given lifetime data set. For example, say, $\alpha = 1$, the LR statistic is $P = 2\left[ln\left(L(\hat{\alpha}, \hat{\rho}, \hat{\lambda}, \hat{\varsigma})\right) - ln\left(L\left((\hat{1}, \tilde{\rho}, \tilde{\lambda}, \tilde{\varsigma})\right)\right)\right]$, where $\hat{\alpha}, \hat{\rho}, \hat{\lambda}$, and $\hat{\varsigma}$ are the unrestricted estimates and $\tilde{\rho}, \tilde{\lambda}$, and $\tilde{\varsigma}$ are the restricted estimates.

The LR test do not accept the null hypothesis if $P > \chi^2_{\varepsilon}$, where χ^2_{ε} represent the upper 100% point of the χ^2 distribution with 1 degree of freedom.

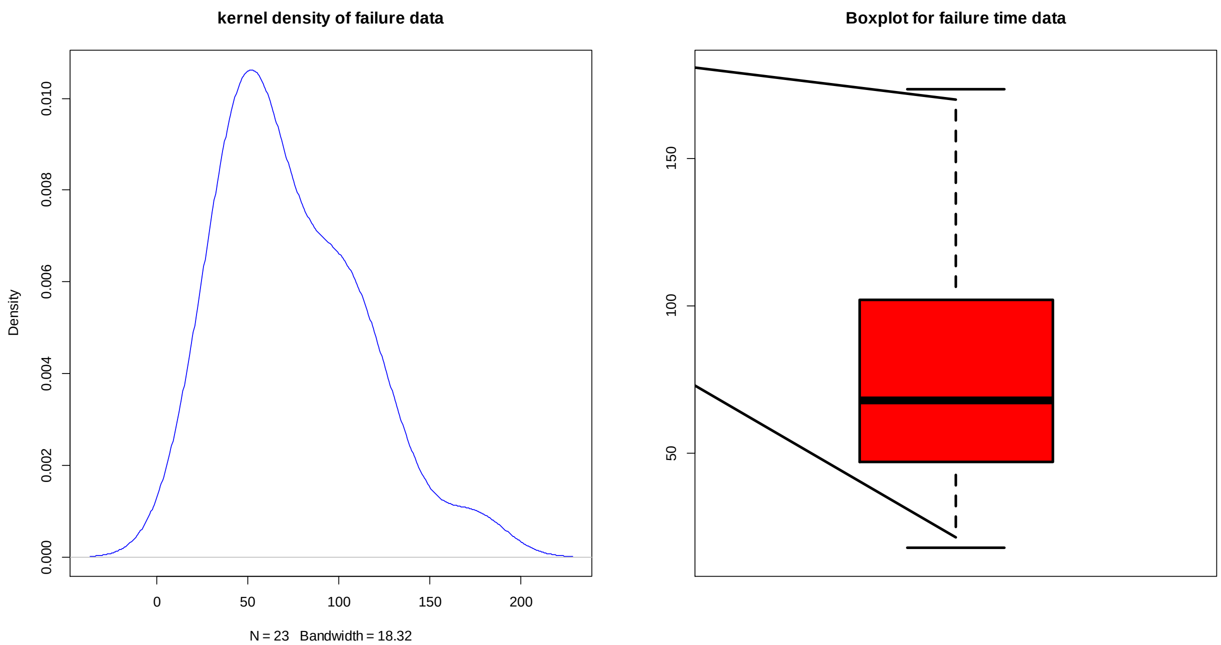
6. Applications of the HEF distribution

A comparison of the newly developed HEF distribution was carried out with the Harris Inverse Exponential (HIE), Harris Inverted Weibull (HIW), Marshall Olkin Inverse Exponential ($MOIE$), Inverted Weibull (IW), Inverse exponential (IE), and the Frechet (F) distributions with the help of two lifetime data sets. We employ the use of five goodness of fit criterion which are: Akaike information criterion ($AICr$), Consistent Akaike Information Criterion ($CAICr$), Kolmogorov Smirnoff (K), Cramer-Von Mises (CrM), and the Probability value (Pv). The model with the smallest value of $AICr$, $CAICr$, K, CrM, and the highest Pv is considered the best model in the class of models considered.

Data Set 1. The first data set is taken from Lawless (2003). The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test. The kernel density and the boxplots for the data shows that the positively skewed. The kernel density and the boxplots for the failure time data is given in Figure 6.0 which shows that the data is positively skewed. Table 6.1 gives the exploratory data analysis of the failure time data which indicates that the data over-dispersed, leptokurtic. Table 6.2 presents the MLEs estimate of the model parameters and the measures of goodness of fit.

Table 6.1 Exploratory data analysis for failure time data

n	q_1	$median$	$mean$	q_3	$Range$	$skewness$	$Kurtosis$	$variance$
23	47.20	67.80	73.85	101.88	155.52	0.79	3.14	1452.81



(a) Kernel density (b) Boxplots

Figure 6.0. Plots Kernel density and the Boxplots for failure time data

Table 6.2. MLEs, standard error (in parenthesis), and measure of the goodness of fit for failure time data.

	$Estimated\ parameter$				$Measures\ of\ goodness\ of\ fit$					
$Model$	α	ρ	λ	ς	$-2l$	$AICr$	$CAICr$	K	CrM	Pv
HF	20.71 (10.31)	1.15 (0.23)	2.62 (1.67)	23.91 (17.75)	239.15	247.25	249.47	0.206	0.045	0.286
HIE	35.58	$-$ $(-)$	9.05	21.64	240.61	246.62	247.88	0.297	0.067	0.034

HIW	$-$ $(-)$	1.12 (0.36)	0.74 (0.32)	29.12 (11.66)	263.16	269.37	270.63	0.398	0.038	0.001
MIE	22.71	$-$ $(-)$	$-$ $(-)$	2.19	251.41	255.42	256.02	0.255	0.041	0.100
MIW	$-$ $(-)$	0.91 (0.11)	$-$ $(-)$	26.39 (10.30)	264.81	268.82	269.42	0.395	0.038	0.002
F	21.21 (7.53)	0.83 (0.09)	$-$ $(-)$	$-$ $(-)$		256.96	257.56	0.277	0.045	0.058
IW	$-$ $(-)$	0.323 (0.06)	$-$ $(-)$	$-$ $(-)$	317.87	319.87	320.06	0.675	0.038	$1.6e-9$
IE	$-$ $(-)$	38.20	$-$ $(-)$	$-$ $(-)$	247.54	249.54	249.73	0.268	0.048	0.073

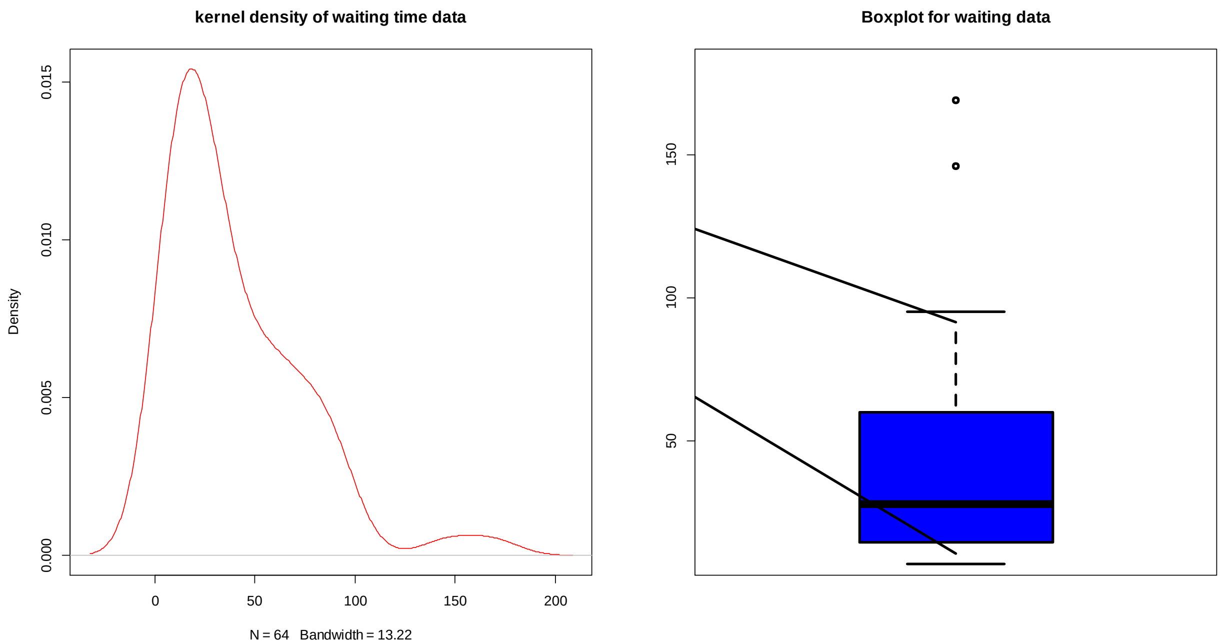
The variance covariance matrix for the failure data is given by

$$\widehat{D} = -\begin{bmatrix} 3.0216\mathrm{e}-01 & 5.6369\mathrm{e}-02 & -3.1886\mathrm{e}-05 & 3.5015\mathrm{e}-07 \\ 5.6369\mathrm{e}-02 & 1.8985\mathrm{e}-01 & 6.5723\mathrm{e}-04 & -7.2181\mathrm{e}-0 \\ -3.1887\mathrm{e}-05 & 6.5723\mathrm{e}-04 & 3.2889\mathrm{e}-05 & -3.6000\mathrm{e}-07 \\ 3.5016\mathrm{e}-07 & -7.2181\mathrm{e}-06 & -3.60\mathrm{e}-07 & 1.2205\mathrm{e}-09 \end{bmatrix}$$

The second dataset consists of the waiting times between 65 consecutive eruptions of the Kiama Blowhole. The data consists of time between eruptions of a 1340 hours period starting from July 12th of 1998 were recorded using a digital watch. For details on this data set, see Pinho (2012). figure 6.1 shows that the failure data is positively skewed. Table 6.3 gives the exploratory data analysis of the failure time data which indicates that the data over-dispersed, leptokurtic. Table 6.4 presents the MLEs estimate of the model parameters and the measures of goodness of fit.

Table 6.3 Exploratory data analysis for waiting time data

n	q_1	$median$	$mean$	q_3	$Range$	$skewness$	$Kurtosis$	$variance$
64	14.75	28.0	39.83	60.0	162	1.54	5.77	1139.10



(a) Kernel density (b) Boxplots

Figure 6.1. Plots Kernel density and the Boxplots for waiting time data

Table 6.4. MLEs, standard error (in parenthesis), and measure of the goodness of fit for waiting time data.

	Estimated parameters				*Measures of goodness of fit*					
Model	α	ρ	λ	ς	$-2l$	$AICr$	$CAICr$	K	CrM	Pv
HF	21.93 (15.83)	1.85 (0.75)	0.91 (0.61)	17.01 (27.05)	590.90	598.90	599.58	0.082	0.128	0.781
HIE	13.69 (3.25)	– (–)	18.86 (12.51)	17.64 (18.89)	594.0	600.01	600.41	0.125	0.200	0.270
HIW	– (–)	1.23 (0.23)	0.72 (0.20)	19.10 (4.82)	627.82	633.83	634.23	0.280	0.116	$8.6e-5$
MIE	25.40 (5.76)	– (–)	– (–)	0.61 (0.27)	595.13	599.22	599.42	0.125	0.184	0.267
MIW	–	1.11	–	27.33	620.90	624.90	625.10	0.241	0.122	0.001

	$(-)$	(0.09)	$(-)$	(7.26)						
F	24.49 (6.89)	1.09 (0.10)	$-$ $(-)$	$-$ $(-)$	594.98	598.98	599.18	0.129	0.162	0.238
IW	$-$ $(-)$	0.40 (0.04)	$-$ $(-)$	$-$ $(-)$	752.29	754.29	754.35	0.632	0.116	$2.2e$ -16
IE	20.41 (2.55)	$-$ $(-)$	$-$ $(-)$	$-$ $(-)$	598.35	600.35	600.42	0.162	0.157	0.069

The variance covariance matrix for the time waiting data is given as

$$\widehat{D} = -\begin{bmatrix} 9.7337e-07 & -8.5025e-07 & -2.7647e-11 & 6.7547e-13 \\ -8.5025e-07 & 2.2269e-02 & 1.5755e-09 & -3.8206e-11 \\ -2.7647e-11 & 1.5755e-09 & 4.4396e-10 & -1.0845e-11 \\ 6.7547e-13 & -3.8206e-11 & -1.0845e-11 & 2.894418e-16 \end{bmatrix}$$

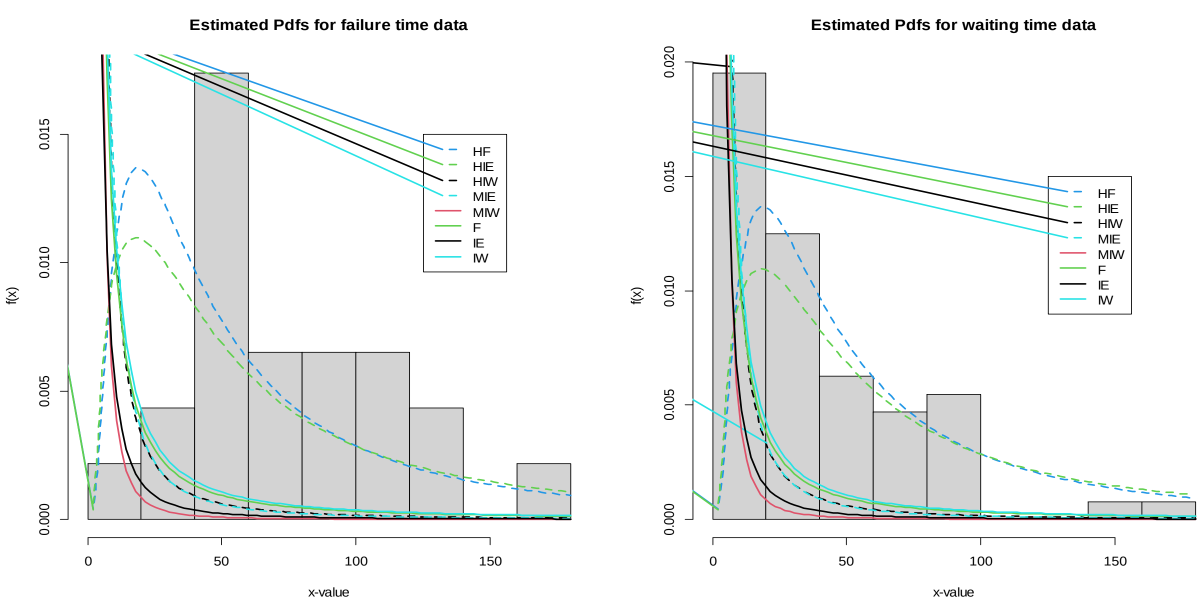


Figure 6.2 Fitted densities for the failure time and the Waiting time data

From Tables 6.2 and 6.4, it can be deduced that the HEF possessed the smallest $AICr, CAICr, CrM, K$, and the largest Pv for the two data sets. Therefore, HEF can be considered the best model that fit the two data sets in the class of model considered in this study and also supported by the fitted densities given in Figure 6.2.

7. Concluding remarks

In this study, we proposed and developed a new generalization of Frechet distribution named as Harris Extended Frechet (HEF) distribution. Statistical properties i.e. quantiles, moments, incomplete moments, moment generating function, mean deviations, mean residual life and mean inactivity time, Probability weighted moments, stress-strength reliability, Lorenz and Bonferroni curves, Rényi and β-entropies, Gine index, and order statistics of the new model are derived. Maximum likelihood method is used to estimate the unknown parameters of the proposed model. Two lifetime data sets are used to demonstrate the flexibility and the competitiveness of proposed model and finally concluded that the new model may be better than other competing models considered in this study.

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