Journal of Modern Applied Statistical Methods
https://jmasm.com

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Statistical Inference for a Novel Farlie-Gumbel-Morgenstern Copula-Based Bivariate Odd Rayleigh-Exponential Distribution

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How To Cite: Ode, O.; Tasi'u, M.; Usman, A.; et al. Statistical Inference for a Novel Farlie-Gumbel-Morgenstern Copula-Based Bivariate Odd Rayleigh-Exponential Distribution. *Journal of Modern Applied Statistical Methods* **2026**, *25*(1), 3. https://doi.org/10.53941/jmasm.2026.100003

Abstract: This study develops a novel bivariate odd Rayleigh-exponential distribution (OR-ED), constructed using the Farlie-Gumbel-Morgenstern (FGM) copula to model dependence between lifetime data. Three estimation methods: maximum likelihood estimation (MLE), inference functions for margins (IFM), and canonical maximum likelihood (CML) are employed to evaluate model performance. Through extensive simulations, all estimators are shown to be consistent, with MLE providing the most accurate estimates and IFM offering a computationally efficient alternative. Practical applications to three real-life datasets demonstrate the flexibility and stability of the proposed model, achieving low biases and RMSEs across the board. The results highlight the model's suitability for capturing moderate dependence in survival and reliability data, establishing the FGM copula-based OR-ED as an adaptable and efficient tool for joint lifetime analysis.

Keywords: bivariate lifetime data; dependence modelling; farlie-gumbel-morgenstern copula; inference functions for margins; maximum likelihood estimation; odd rayleigh-exponential distribution

1. Introduction

Dependence structure modelling between random variables is becoming increasingly relevant in modern-day statistical analysis, especially in disciplines such as epidemiology, biostatistics, survival and reliability analysis, economics, finance, and environmental sciences. Typical joint distribution methods in multivariate analysis often lack the flexibility to adequately capture complex or non-linear dependence patterns, particularly when the marginal distributions are skewed or not normally distributed. To address these limitations, copula functions have emerged as powerful tools that separate the modelling of marginal distributions from their dependence structure, offering a systematic platform for building bivariate distributions with specified marginals and a wide range of dependency behaviours. Ref. [1] averred that a bivariate distribution describes the joint probabilistic behaviour of two random variables, capturing both their individual characteristics and the dependence structure between them.

Various copula families exist for modelling bivariate distributions, each with its own unique characteristics. This study adopts the Farlie-Gumbel-Morgenstern (FGM) copula to model the interdependence between the marginal parameters of the bivariate odd Rayleigh-exponential distribution (OR-ED), a novel extension of the exponential family of distributions introduced by [2]. The OR-ED is characterized by greater workability in modelling diverse skew and tail behaviours, which common distributions fail to capture adequately. One of the properties that makes the FGM copula appealing to researchers is that it offers remarkable interpretability and tractability, and is thus suitable for modelling mild dependencies.

In this study, three estimation techniques are employed to evaluate the parameters of the FGM copula-based bivariate OR-ED. These include: maximum likelihood estimation (MLE), which simultaneously estimates both the marginal and copula parameters by maximizing the joint log-likelihood; inference functions for margins (IFM), a two-step procedure that first estimates the marginal parameters and subsequently the copula parameter; and canonical maximum likelihood (CML), which optimizes a pseudo-likelihood function constructed from ranks based on the empirical marginal distributions, making it especially advantageous when the marginals are complex or not explicitly specified.

Over the past three decades, considerable pedagogic attention has been devoted to the building of bivariate distributions by profoundly defining their marginal components [3]. Despite the inherent challenges associated with developing such models, recent advancements continue to propel the methodological frontier. For instance, Ref. [4] introduced a novel bivariate power distribution and explored several of its key statistical properties, including the joint and ratio moments, conditional moments, random sample generation procedures, as well as the corresponding survival and hazard rate functions, and stress-strength reliability characteristics. The parameters of the proposed distribution were estimated using the MLE method, and a subsequent simulation study confirmed the consistency of the MLEs. Furthermore, empirical applications to real-life datasets demonstrated the robustness and superior performance of the bivariate power distribution when compared with several existing competing models, thereby buttressing its practical relevance and modelling flexibility.

Ref. [5] established that marginal distributions can be connected to a joint distribution through a copula function, emphasizing that any multivariate distribution function can be expressed in terms of an appropriate copula. Building on this framework, Ref. [6] examined a bivariate extension of the generalized exponential distribution by employing the FGM and Plackett copulas, alongside three estimation methods: MLE, IFM, and CML. They conducted simulation studies and real-life data applications of their bivariate distribution. Because the study was validated on a single dataset, it was difficult to generalize their results, similar to [7], who proposed a copula-based bivariate Lomax distribution and applied it to a single lifetime dataset, displaying flexibility but restricted scholastic depth.

In a related study, Ref. [8] introduced a family of bivariate generalized linear exponential distributions and analyzed their dependence structures using copulas, concluding that their proposed framework effectively captured complex dependence patterns. Ref. [9], meanwhile, noted the inherent difficulty in extending exponential models to the bivariate or multivariate domain and proposed a class of absolutely continuous bivariate exponential distributions derived from quadratic forms of multivariate normal variates, as a solution. Similarly, Ref. [10] developed the bivariate Chen distribution, defined by marginals following the two-parameter Chen distribution, exhibiting increasing or bathtub-shaped hazard rates and demonstrating great flexibility with the appropriate copula application.

Ref. [11] further advanced the field by proposing a bivariate beta-exponential distribution based on Gaussian, FGM, and Plackett copulas. His comparative simulation study revealed that Gaussian copula parameter estimates slightly edged those obtained from the FGM and Plackett copulas, findings that were corroborated through empirical applications. The study's generalizability was, however, restricted because it was not validated against real-life datasets. Ref. [12] also contributed to the literature by adopting a copula-based framework to model the joint distribution of multiple correlated variables with specified marginal and dependence structures, demonstrating that tail dependence significantly influences system reliability.

Ref. [13] proposed the bivariate power Lomax distribution, based on the FGM copula, designed for lifetime data exhibiting skewness and heavy tails. Their findings highlighted the distribution's robustness across diverse empirical settings and noted that the incorporation of the FGM copula provided additional flexibility for modelling a broad range of dependence structures. More recently, Ref. [14] introduced the Epanechnikov-exponential FGM distribution, which, following extensive simulation and empirical testing using MLE, was found to outperform several competing models, including the bivariate Weibull-FGM, exponential-FGM, and generalized exponential-FGM distributions.

To investigate the rate of occurrence of a non-homogeneous Poisson process (NHPP), Ref. [15] specified the Gompertz-Makeham distribution as the underlying intensity function, resulting in what is termed the Gompertz-Makeham Process (GMP). For parameter estimation, the authors employed the MLE approach and proposed a modification to address potential limitations related to convergence and estimation accuracy. The modified procedure, referred to as the modified maximum likelihood estimator (MMLE), incorporated an intelligent optimization algorithm to enhance the performance of the likelihood function. The MMLE was further compared with particle swarm optimization (PSO) to evaluate relative estimation efficiency. They conducted a simulation study and a real-life data application to demonstrate the practical utility of the approach, and the findings provide insights into the effectiveness of intelligent optimization techniques in estimating NHPP intensity functions, with applications in reliability analysis, mortality modelling, and disease progression studies.

Ref. [16] introduced the Bivariate Poisson-X-Exponential Distribution (BPXED) as a flexible model for jointly distributed count variables. The model is constructed by compounding Poisson random variables with a shared X-Exponential latent mixing distribution, thereby extending the Poisson-X-Exponential Distribution (PXED). The authors derived closed-form expressions for key distributional properties, including the joint probability mass function, probability generating function, moments, and covariance structure. Dependence between the variables arises through the shared latent component and is therefore restricted to positive correlation. Parameter estimation procedures were developed using maximum likelihood, regression-based, and Bayesian approaches, while simulation experiments demonstrated satisfactory finite-sample performance. Application to real datasets showed that the BPXED model provides improved goodness-of-fit compared with several existing bivariate count models.

While models such as the BPXED focus on dependent count outcomes through latent mixing mechanisms, dependence in continuous lifetime data can alternatively be modelled through copula constructions, which allow flexible separation between marginal behaviour and the dependence structure. This motivates the current study. Additionally, while earlier studies demonstrated the usefulness of copula-based approaches in constructing flexible bivariate lifetime models, significant methodological gaps remain. Most existing techniques rely on traditional baseline distributions, such as the exponential, Weibull, and Lomax, which, despite their analytical simplicity, often lack the flexibility to capture diverse tail dependencies observed in real-life survival data. Moreover, several of the reviewed models were limited to single real-life data applications, thereby restricting their generalizability. To address these shortcomings, the current study introduces the bivariate OR-ED. This novel model integrates the flexibility of the odd Rayleigh-exponential marginal distributions within a copula-based framework. The dependence structure between the marginals is modelled exclusively using the FGM copula, selected for its analytical simplicity and interpretability in capturing moderate dependence. Parameter estimation is carried out with three complementary inferential procedures: MLE, IFM, and CML to evaluate the proficiency of the proposed model. Accordingly, rather than structural model comparison, the emphasis of this study is methodological and focuses on examining how varying data characteristics influence the performance and computational behaviour of different estimation methods within a unified modelling approach. This consolidated approach contributes a novel, tractable, and analytically verified method for modelling bivariate lifetime data with intrinsic dependence structures.

2. Materials and Methods

This section details the mathematical formulation of the bivariate OR-ED, the specification of its marginal and joint structures, and the incorporation of dependence through the FGM copula. Furthermore, it specifies three estimation techniques: MLE, IFM, and CML, employed to obtain parameter estimates. The methodological formulation is followed by simulation studies and empirical applications to evaluate the model's performance across the three methods of estimation, thereby demonstrating its suitability for analyzing real-life bivariate survival data.

2.1. Bivariate Odd Rayleigh-Exponential Distribution

The proposed marginal model is constructed using a generator-based transformation approach. Specifically, the exponential distribution is adopted as the baseline distribution with CDF:

$$F_0(x) = 1 - e^{-\lambda_i x}; \; x, \lambda_i > 0 \tag{1}$$

To induce additional flexibility, the odd transformation is applied to the baseline CDF. The odd transformation is defined as:

$$G(x) = \frac{F_0(x)}{1 - F_0(x)} \tag{2}$$

Substituting the exponential CDF yields:

$$G(x) = \frac{1 - e^{-\lambda_i x}}{e^{-\lambda_i x}} = e^{\lambda_i x} - 1 \tag{3}$$

This transformed function is then embedded into a Rayleigh-type generator. The Rayleigh generator produces the CDF:

$$F(x) = 1 - \exp\left(\frac{G(x)^2}{2\theta_i^2}\right), \theta_i > 0 \tag{4}$$

Substituting $G(x) = e^{\lambda_i x} - 1$ in Equation (4) gives the CDF:

$$F(x_i; \theta_i, \lambda_i) = 1 - \exp\left(-\frac{(e^{\lambda_i x_i} - 1)^2}{2\theta_i^2}\right) \tag{5}$$

Differentiation with respect to x yields the corresponding PDF:

$$f(x_i; \theta_i, \lambda_i) = \frac{\lambda_i}{\theta_i^2}(e^{\lambda_i x_i} - 1)(e^{\lambda_i x_i})\exp\left(-\frac{(e^{\lambda_i x_i} - 1)^2}{2\theta_i^2}\right) \tag{6}$$

where:

$\lambda > 0$, $\theta > 0$ are scale parameters, $0 < x < \infty$, and $i = 1, 2$.

The proposed marginal OR-ED can be interpreted as a generator-based distribution via the transformation of a baseline model. In the current case, the exponential distribution serves as the baseline model $G(x)$, and a Rayleigh-type generator is applied to the resulting odd transformation to produce the OR-ED. This mechanism modifies the tail behaviour of the baseline model while preserving its fundamental structural properties, thereby providing additional flexibility for modelling lifetime data.

Comparison with Existing Copula-Based Exponential-Type Models

Notably, an array of copula-based exponential-type models, such as the Clayton-exponential and Plackett-exponential models, have been proposed in the literature. While these models are effective for modelling dependence, their marginal structures remain limited to the classical exponential distribution, which lacks the flexibility required in capturing heavy-tailed behaviours commonly observed in real-life reliability and survival data. In contrast, the proposed bivariate OR-ED model introduces additional flexibility at the marginal level through the odd Rayleigh generator applied to the exponential baseline distribution. This transformation enriches the hazard rate behaviour and also admits a closed-form expression which facilitates straightforward likelihood construction, explicit score equations, and clear asymptotic analysis. Compared to Archimedean copula-based exponential models, which often involve generator inversions or numerically intensive likelihood evaluations, the proposed OR-ED—FGM combination maintains computational simplicity while preserving interpretability of both the marginal and dependence parameters. Thus, the contribution lies not merely in combining a copula with exponential-type marginals, as is done traditionally, but in integrating a generator-enhanced marginal structure within an analytically distinct copula formulation.

2.2. Sklar's Theorem

Sklar's theorem forms the theoretical foundation of copula theory, affirming that any multivariate cumulative distribution function (CDF) can be decomposed into its marginal distributions and an associated copula function, *C*. As explained by [17], a copula is a multivariate CDF whose marginal distributions are uniform on the interval [0, 1]. In essence, copulas provide a powerful mechanism for modelling the dependence structure or interrelationship between random variables, independently of their marginal behaviours. This decomposition allows researchers to capture complex associations while preserving the unique characteristics of each marginal distribution. In probability theory and applied statistics, numerous families of copulas have been developed to represent different forms of dependence. Among these, the FGM copula is widely used due to its analytical simplicity and interpretability; for this reason, the current study adopts it to model the dependence structure of the proposed bivariate OR-ED.

Let H denote a joint CDF with continuous marginal distributions $F(x_1)$ and $F(x_2)$. According to Sklar's theorem, there exists a copula function, C, characterized by a dependence parameter α, such that the joint distribution function can be expressed as:

$$H(x_1, x_2) = C(F(x_1), F(x_2); \alpha) = C(u, v) \tag{7}$$

where: $u = F(x_1)$ and $v = F(x_2)$ are uniformly distributed on the interval [0, 1].

Differentiating the joint CDF with respect to x_1 and x_2 yields the corresponding joint probability density function (PDF), thus:

$$\frac{\partial^2 C(u,v)}{\partial u \partial v} = C(u,v)$$
$$h(x_1,x_2) = f(x_1)f(x_2)c(F(x_1),F(x_2)) \tag{8}$$

where $c(.)$ denotes the copula density function, obtained as the mixed partial derivative of the copula C with respect to u and v.

The bivariate OR-ED with copula dependence is therefore expressed as:

$$F(x_1,x_2) = C(F(x_1),F(x_2)) \tag{9}$$

where:

$$F(x_1) = 1-\exp\left(-\frac{(e^{\lambda_1 x_1}-1)^2}{2\theta_1^{\,2}}\right) \quad \text{and} \quad F(x_2) = 1-\exp\left(-\frac{(e^{\lambda_2 x_2}-1)^2}{2\theta_2^{\,2}}\right)$$
$$= C\left[\left(1-\exp\left(-\frac{(e^{\lambda_1 x_1}-1)^2}{2\theta_1^{\,2}}\right)\right),\left(1-\exp\left(-\frac{(e^{\lambda_2 x_2}-1)^2}{2\theta_2^{\,2}}\right)\right)\right] \tag{10}$$

The corresponding joint PDF is given as:

$$f(x_1,x_2) = f(x_1)f(x_2)c(F(x_1),F(x_2)) \tag{11}$$

$$= \frac{\lambda_1}{\theta_1^2}\frac{\lambda_2}{\theta_2^2}(e^{\lambda_1 x_1}-1)(e^{\lambda_2 x_2}-1)(e^{\lambda_1 x_1})(e^{\lambda_2 x_2})\left(\exp\left(-\frac{(e^{\lambda_1 x_1}-1)^2}{2\theta_1^{\,2}}\right)\right)\left(\exp\left(-\frac{(e^{\lambda_2 x_2}-1)^2}{2\theta_2^{\,2}}\right)\right)$$
$$\times c\left(1-\exp\left(-\frac{(e^{\lambda_1 x_1}-1)^2}{2\theta_1^{\,2}}\right), 1-\exp\left(-\frac{(e^{\lambda_2 x_2}-1)^2}{2\theta_2^{\,2}}\right)\right) \tag{12}$$

2.3. Bivariate Distribution Using the FGM Copula

The FGM copula-based CDF for variables with continuous marginal distributions is:

$$C(u,v) = uv[1+\alpha(1-u)(1-v)];\ \alpha \in [-1,1] \tag{13}$$

The corresponding PDF is given as:

$$c(u,v) = 1+\alpha(1-2u)(1-2v) \tag{14}$$

2.4. FGM Copula-Based Bivariate OR-ED

The CDF of the FGM-copula-based bivariate OR-ED is:

$$F(x_1,x_2) = F(x_1)F(x_2) + \alpha(F(x_1))(F(x_2))\times(1-F(x_1))(1-F(x_2)) \tag{15}$$

And its corresponding bivariate PDF is:

$$fx_1x_2(x_1,x_2) = f_1(x_1)f_2(x_2)\left[1+\alpha(1-2F_1(x_1))(1-2F_2(x_2))\right] \tag{16}$$

$$f(x_1, x_2, \alpha) = \frac{\lambda_1}{\theta_1^{\,2}} \frac{\lambda_2}{\theta_2^{\,2}} (e^{\lambda_1 x_1} - 1)(e^{\lambda_2 x_2} - 1)(e^{\lambda_1 x_1})(e^{\lambda_2 x_2}) \exp\left[-\frac{(e^{\lambda_1 x_1} - 1)^2}{2\theta_1^{\,2}}\right] \exp\left[-\frac{(e^{\lambda_2 x_2} - 1)^2}{2\theta_2^{\,2}}\right] \times\left[1 + \alpha(1 - 2(F(x_1)))(1 - 2(F(x_2)))\right] \tag{17}$$

The dependence structure is controlled by α, and the joint distribution is valid and normalized. $\alpha = 0$ signifies independence, $\alpha > 0$ denotes positive dependence, whereas $\alpha < 0$ denotes negative dependence.

2.5. Estimation of FGM Copula-Based Bivariate OR-ED

The three proposed methods of estimation are used in estimating the parameters of the bivariate OR-ED, as displayed in this sub-section.

2.5.1. Method of Maximum Likelihood Estimation (MLE)

Recall that the bivariate PDF based on the FGM copula is defined as:

$$fx_1x_2(x_1, x_2) = f_1(x_1) f_2(x_2)\left[1 + \alpha(1 - 2F_1(x_1))(1 - 2F_2(x_2))\right]$$

The likelihood function is:

$$L(\alpha; x_1, x_2) = \prod_{i=1}^{n} \left[f_1(x_1) f_2(x_2) c(F_1(x_1), F_2(x_2))\right] \tag{18}$$

$$= \prod_{i=1}^{n} \left[f_1(x_1) f_2(x_2)(1 + \alpha(1 - 2F_1(x_1))(1 - 2F_2(x_2))\right] \tag{19}$$

The log-likelihood function is:

$$\ell = \sum_{i=1}^{n} \log f_1(x_1) + \sum_{i=1}^{n} \log f_2(x_2) + \sum_{i=1}^{n} \log[1 + \alpha(1 - 2F_1(x_1))(1 - 2F_2(x_2))] \tag{20}$$

$$= \sum_{i=1}^{n} [\ln \frac{\lambda_1}{\theta_1^{\,2}} (e^{\lambda_1 x_1} - 1)(e^{\lambda_1 x_1}) \exp(-\frac{(e^{\lambda_1 x_1} - 1)^2}{2\theta_1^{\,2}}) + \ln \frac{\lambda_2}{\theta_2^{\,2}} (e^{\lambda_2 x_2} - 1)(e^{\lambda_2 x_2}) \exp(-\frac{(e^{\lambda_2 x_2} - 1)^2}{2\theta_2^{\,2}}) + \ln(1 + \alpha(1 - 2(1 - \exp\left(-\frac{(e^{\lambda_1 x_1} - 1)^2}{2\theta_1^{\,2}}\right)))(1 - 2(1 - \exp\left(-\frac{(e^{\lambda_2 x_2} - 1)^2}{2\theta_2^{\,2}}\right)] \tag{21}$$

Simplifying the last part of Equation (21) yields:

$$\ln[1 + \alpha(2\exp\left(-\frac{(e^{\lambda_1 x_1} - 1)^2}{2\theta_1^{\,2}}\right) - 1)(2\exp\left(-\frac{(e^{\lambda_2 x_2} - 1)^2}{2\theta_2^{\,2}}\right) - 1)]$$

Differentiating with respect to λ_1 and equating the derivative to 0 gives:

$$\frac{\partial \ln L(\alpha; \lambda_1, \lambda_2, \theta_1, \theta_2)}{\partial \lambda_1} = \frac{n}{\lambda_1} + \sum_{i=1}^{n} \frac{x_1 e^{\lambda_1 x_1}}{(e^{\lambda_1 x_1} - 1)} + \sum x_1 - \frac{1}{\theta_1^{2}} \sum (e^{\lambda_1 x_1} - 1)(x_1 e^{\lambda_1 x_1}) - \frac{2\alpha(e^{\lambda_1 x_1} - 1)x_1 e^{\lambda_1 x_1}(2\exp\left(-\frac{(e^{\lambda_2 x_2} - 1)^2}{2\theta_2^{\,2}}\right) - 1)}{\theta_1^{\,2}[1 + \alpha(2\exp\left(-\frac{(e^{\lambda_1 x_1} - 1)^2}{2\theta_1^{\,2}}\right) - 1)(2\exp\left(-\frac{(e^{\lambda_2 x_2} - 1)^2}{2\theta_2^{\,2}}\right) - 1)]} \exp\left(-\frac{(e^{\lambda_1 x_1} - 1)^2}{2\theta_1^{\,2}}\right) = 0 \tag{22}$$

Differentiating with respect to λ_2 and equating the derivative to 0 yields:

$$\frac{\partial \ln L(\alpha;\lambda_1,\lambda_2,\theta_1,\theta_2)}{\partial \lambda_2}=\frac{n}{\lambda_2}+\sum_{i=1}^{n}\frac{x_2 e^{\lambda_2 x_2}}{(e^{\lambda_2 x_2}-1)}+\sum x_2-\frac{1}{\theta_2^2}\sum(e^{\lambda_2 x_2}-1)(x_2 e^{\lambda_2 x_2})$$

$$-\frac{2\alpha(e^{\lambda_2 x_2}-1)x_2 e^{\lambda_2 x_2}\left(2\exp\left(-\frac{(e^{\lambda_1 x_1}-1)^2}{2\theta_1^2}\right)-1\right)}{\theta_2^2\left[1+\alpha\left(2\exp\left(-\frac{(e^{\lambda_1 x_1}-1)^2}{2\theta_1^2}\right)-1\right)\left(2\exp\left(-\frac{(e^{\lambda_2 x_2}-1)^2}{2\theta_2^2}\right)-1\right)\right]}\exp\left(-\frac{(e^{\lambda_2 x_2}-1)^2}{2\theta_2^2}\right)=0 \tag{23}$$

Differentiating with respect to θ_1 and equating the derivative to 0:

$$\frac{\partial \ln L(\alpha;\lambda_1,\lambda_2,\theta_1,\theta_2)}{\partial \theta_1}=\frac{-2n}{\theta_1}+\frac{1}{\theta_1^3}\sum(e^{\lambda_1 x_1}-1)^2$$

$$+\frac{2\alpha(e^{\lambda_1 x_1}-1)^2\left(2\exp\left(-\frac{(e^{\lambda_2 x_2}-1)^2}{2\theta_2^2}\right)-1\right)}{\theta_1^3\left[1+\alpha\left(2\exp\left(-\frac{(e^{\lambda_1 x_1}-1)^2}{2\theta_1^2}\right)-1\right)\left(2\exp\left(-\frac{(e^{\lambda_2 x_2}-1)^2}{2\theta_2^2}\right)-1\right)\right]}\exp\left(-\frac{(e^{\lambda_1 x_1}-1)^2}{2\theta_1^2}\right)=0 \tag{24}$$

Differentiating with respect to θ_2 and equating the derivative to 0:

$$\frac{\partial \ln L(\alpha;\lambda_1,\lambda_2,\theta_1,\theta_2)}{\partial \theta_2}=\frac{-2n}{\theta_2}+\frac{1}{\theta_2^3}\sum(e^{\lambda_2 x_2}-1)^2$$

$$+\frac{2\alpha(e^{\lambda_2 x_2}-1)^2\left(2\exp\left(-\frac{(e^{\lambda_1 x_1}-1)^2}{2\theta_1^2}\right)-1\right)}{\theta_2^3\left[1+\alpha\left(2\exp\left(-\frac{(e^{\lambda_1 x_1}-1)^2}{2\theta_1^2}\right)-1\right)\left(2\exp\left(-\frac{(e^{\lambda_2 x_2}-1)^2}{2\theta_2^2}\right)-1\right)\right]}\exp\left(-\frac{(e^{\lambda_2 x_2}-1)^2}{2\theta_2^2}\right)=0 \tag{25}$$

Differentiating with respect to α and equating the derivative to 0:

$$\frac{\partial \ln L(\alpha;\lambda_1,\lambda_2,\theta_1,\theta_2)}{\partial \alpha}=\sum_{i=1}^{n}\frac{(1-2F_1(x_1))(1-2F_2(x_2))}{1+\alpha(1-2F_1(x_1))(1-2F_2(x_2))}=0 \tag{26}$$

Due to their non-linear nature, all parameter estimates are obtained numerically by using statistical software.

2.5.2. Method of Inference Functions for Margins (IFM) Estimation

The IFM method is a two-step semi-parametric estimation procedure. In the first step, the marginal distributions $F(x_1)$ and $F(x_2)$ are estimated independently by maximizing their respective log-likelihood functions, as given by:

$$\ln L(x_1)=\sum_{i=1}^{n}\ln f(x_1) \text{ and } \ln L(x_2)=\sum_{i=1}^{n}\ln f(x_2)$$

In the second step, the dependence parameter α of the copula is estimated by maximizing the log-likelihood function of the copula density, evaluated at the estimated marginal CDFs $\hat{F}(x_1)$ and $\hat{F}(x_2)$ obtained in the first step:

$$\hat{\alpha}=\arg\max\sum_{i=1}^{n}\ln C(\hat{F}(x_1),\hat{F}(x_2)) \tag{27}$$

Equation (27) thus provides the optimal estimate of the copula parameter α by maximizing the log-likelihood of the copula density evaluated at the empirical marginal CDFs of the data.

The IFM method is often favoured over MLE because of its computational efficiency and reduced complexity, particularly when the marginal distributions are analytically demanding. While MLE exhibits the highest computational time due to simultaneous multi-parameter optimization, IFM requires substantially less run-time than MLE because the optimization problem is decomposed into lower-dimensional components. CML lies between MLE and IFM in computational demand, depending on the sample size. The computational advantage of IFM becomes more pronounced as sample size increases, confirming that the method provides a favourable balance between statistical efficiency and computational cost. Consequently, for the FGM copula-based bivariate OR-ED, the IFM approach is adopted to jointly estimate the dependence parameter and the marginal parameters in an amenable two-step procedure.

The marginal log-likelihood for the first variable is given as:

$$L(x_1;\theta_1,\lambda_1)=\prod_{i=1}^{n} f(x_1;\theta_1,\lambda_1) \tag{28}$$

The log-likelihood function is thus:

$$\ell_1(x_1;\theta_1,\lambda_1)=\sum_{i=1}^{n}\ln f_1(x_1;\theta_1,\lambda_1) \tag{29}$$

$$=\sum_{i=1}^{n}[\ln\lambda_1-2\ln\theta_1+\ln(e^{\lambda_1x_1}-1)+\lambda_1x_1-\frac{1}{2\theta_1^2}\Sigma(e^{\lambda_1x_1}-1)^2] \tag{30}$$

$$=n\log\lambda_1-2n\log\theta_1+\Sigma\log(e^{\lambda_1x_1}-1)+\lambda_1\Sigma x_1-\frac{1}{2\theta_1^2}\Sigma(e^{\lambda_1x_1}-1)^2 \tag{31}$$

Differentiating with respect to θ_1 gives:

$$\hat{\theta}_1=\sqrt{\frac{1}{2n}\Sigma(e^{\lambda_1x_1}-1)^2} \tag{32}$$

Differentiating with respect to λ_1 gives:

$$\frac{\partial\ln L(x_1;\theta_1,\lambda_1)}{\partial\lambda_1}=\frac{n}{\lambda_1}+\Sigma\left(\frac{x_1e^{\lambda_1x_1}}{e^{\lambda_1x_1}-1}\right)+\Sigma x_1=\frac{1}{\theta_1^2}\Sigma(e^{\lambda_1x_1}-1)(x_1e^{\lambda_1x_1}) \tag{33}$$

The estimate of λ_1 is obtained numerically with statistical software.

Similarly, for the second variable;

$$L(x_2;\theta_2,\lambda_2)=\prod_{i=1}^{n} f(x_2;\theta_2,\lambda_2) \tag{34}$$

$$\ell_2(x_2;\theta_2,\lambda_2)=\sum_{i=1}^{n}\ln f_2(x_2;\theta_2,\lambda_2) \tag{35}$$

$$=\sum_{i=1}^{n}[\ln\lambda_2-2\ln\theta_2+\ln(e^{\lambda_2x_2}-1)+\lambda_2x_2-\frac{1}{2\theta_2^2}\Sigma(e^{\lambda_2x_2}-1)^2] \tag{36}$$

$$=n\log\lambda_2-2n\log\theta_2+\Sigma\log(e^{\lambda_2x_2}-1)+\lambda_2\Sigma x_2-\frac{1}{2\theta_2^2}\Sigma(e^{\lambda_2x_2}-1)^2 \tag{37}$$

Differentiating with respect to θ_2 gives:

$$\hat{\theta}_2 = \sqrt{\frac{1}{2n}\sum(e^{\lambda_2 x_2} - 1)^2} \tag{38}$$

Differentiating with respect to λ_2 gives:

$$\frac{\partial \ln L(x_2;\theta_2,\lambda_2)}{\partial \lambda_2} = \frac{n}{\lambda_2} + \sum\left(\frac{x_2 e^{\lambda_2 x_2}}{e^{\lambda_2 x_2} - 1}\right) + \sum x_2 = \frac{1}{\theta_2{}^2}\sum(e^{\lambda_2 x_2} - 1)(x_2 e^{\lambda_2 x_2}) \tag{39}$$

The estimate of λ_2 is obtained numerically with the aid of statistical software.

Note that all constant terms independent of the parameter vector are retained during derivation. If any additive constants are removed for computational convenience, this does not affect maximization because such constants do not depend on the parameters.

The next step involves estimating the copula density using the marginal estimates $\hat{F}(x_1)$ and $\hat{F}(x_2)$ as shown in Equation (27).

The copula log-likelihood is thus:

$$\ell(\alpha) = \sum_{i=1}^{n} \log c(\alpha; \hat{F}_1(x_1), \hat{F}_2(x_2)) \tag{40}$$

For the FGM copula,

$$c(u,v) = 1 + \alpha(1 - 2F_1(x_1))(1 - 2F_2(x_2))$$

Thus,

$$\ell(\alpha) = \sum_{i=1}^{n} \log[1 + \alpha(1 - 2F_1(x_1))(1 - 2F_2(x_2))] \tag{41}$$

$$\ell(\alpha) = \sum_{i=1}^{n} \frac{(1 - 2F_1(x_1))(1 - 2F_2(x_2))}{1 + \alpha(1 - 2F_1(x_1))(1 - 2F_2(x_2))} \tag{42}$$

The copula parameter is then obtained as:

$$\hat{\alpha}_{IFM} = \arg\max_{\alpha \in (-1,1)} \ell(\alpha) \tag{43}$$

2.5.3. Method of Canonical Maximum Likelihood (CML) Estimation

The CML estimation method provides an alternative approach to parameter estimation in copula-based models, particularly when the marginal distributions are not clearly defined. In the CML approach, the marginal distributions are first estimated separately, and dependence is estimated using pseudo-observations constructed from ranks.

Let (x_{1i}, x_{2i}); $i = 1, \ldots, n$ denote the observed sample.

Define the pseudo-observations as:

$$\hat{U}_i = \frac{R_{1i}}{n+1}, \quad \hat{V}_i = \frac{R_{2i}}{n+1}$$

where

R_{1i} is the rank of x_{1i} among $\{x_{11}, \ldots, x_{1n}\}$, and R_{2i} is the rank of x_{2i} among $\{x_{21}, \ldots, x_{2n}\}$.

Dividing by $(n + 1)$ ensures that the pseudo-observations lie strictly in the open interval (0, 1), thereby avoiding boundary issues in copula density evaluation. The canonical log-likelihood for the copula parameter is therefore:

$$\ell_{CML}(\alpha) = \sum_{i=1}^{n} \log[1 + \alpha(1 - 2\hat{U}_i)(1 - 2\hat{V}_i)] \tag{44}$$

The CML estimator, $\hat{\alpha}_{CML}$ is obtained by maximizing this expression with respect to $\alpha \in [-1, 1]$.

$$\frac{\partial \ell_{CML}}{\partial \alpha} = \sum_{i=1}^{n} \frac{(1-2\hat{U}_i)(1-2\hat{V}_i)}{1+\alpha(1-2\hat{U}_i)(1-2\hat{V}_i)} \tag{45}$$

This equation does not admit a closed-form solution and is solved numerically.

2.6. Identifiability and Asymptotic Properties

To show that the parameter vector (λ_1, θ_1, λ_2, θ_2, α) is identifiable, notice that each marginal distribution is defined through:

An exponential baseline: $F_0(x) = 1 - e^{-\lambda_i x}$, the odd transformation: $G(x) = \frac{F_0(x)}{1-F_0(x)} = e^{\lambda_i x} - 1$, and a Rayleigh-type generator: $F(x_i;\theta_i,\lambda_i) = 1 - \exp\left(-\frac{(e^{\lambda_i x_i}-1)^2}{2\theta_i^2}\right)$

For λ_i, $\theta_i > 0$, the CDF is strictly increasing, the density is strictly positive on its support, and distinct parameter values produce distinct distributions. Hence, each marginal parameter pair (λ_i, θ_i) is identifiable.

For the FGM copula CDF and density function:

If

$$c(\alpha_1; u, v) = c(\alpha_2; u, v)$$

for all $u, v \in (0, 1)$, then necessarily $\alpha_1 = \alpha_2$. This shows that the copula parameter is identifiable.

Since the marginal and copula parameters are identifiable, Sklar's theorem guarantees a unique decomposition for continuous marginals. Therefore, the full parameter vector (λ_1, θ_1, λ_2, θ_2, α) is identifiable.

For consistency and asymptotic normality of the MLE and IFM, under standard theory, the following conditions hold:

i. The parameter space is: $\Theta = \{(\lambda_1,\theta_1,\lambda_2,\theta_2,\alpha) : \lambda_i > 0, \theta_i > 0, \alpha \in [-1,1]\}$

which is compact in α and open in the marginal parameters. For asymptotic arguments, α may be restricted to the interior $(-1, 1)$.

ii. The marginal density is twice continuously differentiable in (λ_i, θ_i). Similarly, the copula density is twice continuously differentiable in α. Hence, the joint log-likelihood is twice differentiable.

iii. The density is strictly positive: $c(u,v) = 1+\alpha(1-2u)(1-2v) > 0$ for all $\alpha \in [-1,1]$. This ensures log-likelihood finiteness.

iv. Second derivatives exist and are finite because the Rayleigh-type generator produces exponentially decaying tails, and the FGM copula density is bounded. Thus, the Fisher information matrix exists and is finite.

v. Observations are assumed to be independent and identically distributed, satisfying standard likelihood theory assumptions.

It is important to note that even though the FGM copula models only weak dependence, identifiability is unaffected, and asymptotic properties remain valid provided α lies in the interior of its parameter space. The parameter vector (λ_1, θ_1, λ_2, θ_2, α) is identifiable since both the marginal distributions and the FGM copula admit unique parameterizations. Under standard regularity conditions for MLE, including differentiability of the log-likelihood, positivity of the joint density, existence of finite Fisher information, and independent sampling, the MLE is consistent and asymptotically normal. Similar results hold for the IFM estimator under standard two-stage estimation theory. Although standard interior asymptotic results are derived under $\alpha \in (-1,1)$, the likelihood remains well-defined at the boundary values $\alpha = \pm 1$.

2.7. Justification for Selecting the FGM Copula

The FGM copula was selected based on both theoretical and practical considerations. The dependence parameter $\alpha \in [-1,1]$ implies that the associated Kendall's tau is restricted to:

$$\tau = \frac{2\alpha}{9},$$

so that $\tau \in [-2/9, 2/9] = [-0.222, 0.222]$, corresponding to weak dependence. This limitation is often regarded as a drawback; however, for many reliability and survival applications involving exponential-type marginals, empirical dependence is typically mild rather than strongly tail-driven. Unlike Archimedean copulas, such as Clayton and Frank, which allow stronger dependence and may exhibit asymmetric or tail dependence structures, the FGM copula provides a symmetric and analytically tractable dependence structure with no tail dependence. In this study, where the primary focus lies in modelling marginal flexibility via the odd Rayleigh generator, the FGM copula offers a parsimonious dependence mechanism that preserves closed-form likelihood functions and facilitates explicit derivations of score equations under the MLE, IFM, and CML estimation methods. Thus, the choice of the FGM copula is motivated not by superiority, but by its suitability for modelling mild dependence while preserving computational efficiency and analytical tractability within the proposed approach.

2.8. Theoretical Properties of the Proposed OR-ED Model

2.8.1. Marginal Tail Behaviour

The marginal distribution is constructed using an exponential baseline and an odd transformation followed by a Rayleigh-type generator. As $x \to 0^+$, the distribution behaves regularly and the density remains finite. As $x \to \infty$, the survival function of the proposed marginal decays exponentially. Hence, the proposed marginal distribution exhibits finite moments of all orders and no heavy-tail behaviour. This makes the model appropriate for reliability and lifetime data where extremely large values are possible but not heavy-tailed. In other words, although the support of the distribution is unbounded and large observations may occur, the survival function decays exponentially, implying light-tailed behaviour.

2.8.2. Dependence Structure, Kendall's Tau, and Spearman's Rho

The FGM copula induces symmetric dependence because its density is invariant under exchange of the margins, it exhibits weak dependence since its Kendall's tau is bounded within ±2/9, and its Spearman's rho satisfies $|\rho| \leq 1/3$. Therefore, it is appropriate for modelling mild association strength but not strong tail-driven dependence.

2.8.3. Tail Dependence

The FGM copula exhibits no upper or lower tail dependence. Thus, very large values of one variable do not induce very large values of the other, with positive probability beyond independence.

2.8.4. Identifiability and Parameter Interpretation

The parameter vector (λ_1, θ_1, λ_2, θ_2, α) is identifiable because marginal parameters control shape and scale independently, the copula parameter α influences only dependence, and the FGM copula separates marginals from dependence via Sklar's theorem. Additionally, α determines the direction of association: $\alpha > 0$ implies positive dependence, $\alpha < 0$ signifies negative dependence, and $\alpha = 0$ denotes independence.

2.9. Algorithmic Implementation of Estimation Procedures

2.9.1. Algorithm for Implementing of MLE

Step 1: Specify the joint log-likelihood function, $\ell(\Theta) = \ell(\lambda_1, \theta_1, \lambda_2, \theta_2, \alpha)$

Step 2: Substitute the OR-ED marginal PDFs and the FGM copula density

Step 3: Compute partial derivatives with respect to $\lambda_1, \theta_1, \lambda_2, \theta_2, \alpha$

Step 4: Solve the non-linear system: $\dfrac{\partial \ell}{\partial \Theta_i} = 0$

Step 5: Use numerical optimization

2.9.2. Algorithm for Implementing IFM

Step 1: Maximize the marginal likelihoods independently and obtain $\hat{\lambda}_1, \hat{\theta}_1, \hat{\lambda}_2, \hat{\theta}_2$

Step 2: Estimate the copula parameter

Step 3: Transform data, $u_i = F_1(x_1; \hat{\lambda}_1, \hat{\theta}_1)$, $v_i = F_2(x_2; \hat{\lambda}_2, \hat{\theta}_2)$

Step 4: Maximize the copula log-likelihood $\ell(\alpha)$

Step 5: Obtain $\hat{\alpha}_{IFM}$

2.9.3. Algorithm for Implementing CML

Step 1: Replace marginals with empirical CDFs, $\hat{u}_i = \frac{rank(x_i)}{n+1}$

Step 2: Construct the copula pseudo-likelihood, $\ell(\alpha)$

Step 3: Maximize with respect to α only

Step 4: Obtain $\hat{\alpha}_{CML}$

2.10. Numerical Optimization and Implementation Strategy

All numerical maximization procedures were implemented in *R* using the built-in *optim()* function. For each OR-ED marginal distribution, the log-likelihood was maximized using the *L-BFGS-B* quasi-Newton algorithm, which permits box constraints. The parameter space was restricted to λ_i, $\theta_i > 0$, $i = 1, 2$, with initial values specified as (0.1, 1). These values were selected to lie well within the admissible region while remaining computationally stable. The copula parameter was estimated via one-dimensional optimization, with the parameter constrained to $\alpha \in (-0.99, 0.99)$. Full MLE over the parameter vector (λ_1, θ_1, λ_2, θ_2, α) was conducted using *L-BFGS-B* optimization with bounds consistent with the copula type. The maximum number of iterations was set to 1000, and convergence was assessed according to the default criteria implemented in the *optim() function.*

3. Results

3.1. Simulation Studies on the Copula-Based Bivariate OR-ED

To evaluate the finite sample performance of the proposed estimators, two separate Monte Carlo simulation experiments were conducted under distinct parameter configurations (Tables 1 and 2). The first configuration was defined by $\lambda_1 = 3.0$, $\theta_1 = 4.5$, $\lambda_2 = 2.3$, $\theta_2 = 1.5$, and $\alpha_F = 1.0$, which represents moderately skewed marginal dependence. To assess robustness under a different structural setting, a second configuration was considered: $\lambda_1 = 1.8$, $\theta_1 = 2.5$, $\lambda_2 = 1.0$, $\theta_2 = 4.0$, and $\alpha_F = 0.7$. The value $\alpha_F = 1.0$ was selected to represent the strongest possible dependence within the FGM approach, while $\alpha_F = 0.7$ was chosen to capture moderate interior dependence. This dual specification enables assessment of estimation behaviour both at the boundary and within the interior of the admissible parameter space, thereby avoiding conclusions that may be driven solely by boundary effects. For each configuration and sample size (n = 50, 75, 100, 150, 200, 250, 500, 1000), 1000 Monte Carlo replications were generated. Bias and RMSEs were computed relative to the true parameter values.

Table 1. Results of simulation study-1 based on the FGM copula (the smallest RMSE in each configuration is highlighted in bold).

Sample Size (n)	Method of Estimation	Parameter Estimates				
		$\lambda_1 = 3.0$	$\theta_1 = 4.5$	$\lambda_2 = 2.3$	$\theta_2 = 1.5$	$\alpha_F = 1.0$
50	MLE	3.2050 (0.2050) [0.5490]	5.5010 (1.0010) [2.6040]	3.4720 (0.1720) [0.6800]	0.8380 (0.3380) [1.0500]	0.8230 (−0.1770) [0.3020]
	IFM	3.1530 (0.1530) [**0.5310**]	5.4360 (0.9360) [**2.5590**]	2.4930 (0.1930) [**0.6520**]	1.8490 (0.3490) [**1.0310**]	0.8140 (−0.1860) [0.2890]
	CML	-	-	-	-	0.7990 (−0.2010) [**0.2740**]
75	MLE	3.1390 (0.1390) [0.4340]	5.1760 (0.6760) [1.9400]	2.4350 (0.1350) [0.5310]	1.7220 (0.2220) [0.8120]	0.8530 (−0.1470) [0.2310]
	IFM	3.0970 (0.0970) [**0.4080**]	5.0990 (0.5990) [**1.8530**]	2.4120 (0.1120) [**0.5120**]	1.6990 (0.1990) [**0.7910**]	0.8410 (−0.1590) [0.2190]
	CML	-	-	-	-	0.8230 (−0.1770) [**0.2080**]

Table 1. *Cont.*

Sample Size (n)	Method of Estimation	Parameter Estimates				
		$\lambda_1 = 3.0$	$\theta_1 = 4.5$	$\lambda_2 = 2.3$	$\theta_2 = 1.5$	$\alpha_F = 1.0$
100	MLE	3.0820 (0.0820) [0.3510]	4.9820 (0.4820) [1.5850]	2.3870 (0.0870) [0.4410]	1.6370 (0.1370) [0.7040]	0.8740 (−0.1260) [0.1870]
	IFM	3.0570 (0.0570) [**0.3220**]	4.9450 (0.4450) [**1.5020**]	2.3660 (0.0660) [**0.4180**]	1.6080 (0.1080) [**0.6730**]	0.8670 (−0.1330) [0.1740]
	CML	-	-	-	-	0.8590 (−0.1410) [**0.1650**]
150	MLE	3.0340 (0.0340) [0.2420]	4.7120 (0.2120) [1.1810]	2.3400 (0.0400) [0.3330]	1.5740 (0.0740) [0.5640]	0.8990 (−0.1010) [0.1530]
	IFM	3.0210 (0.0210) [**0.2140**]	4.6890 (0.1890) [**1.1230**]	2.3270 (0.0270) [**0.3110**]	1.5610 (0.0610) [**0.5320**]	0.8940 (−0.1060) [0.1430]
	CML	-	-	-	-	0.8870 (−0.1130) [**0.1350**]
200	MLE	3.0270 (0.0270) [0.2000]	4.6620 (0.1620) [1.0470]	2.3200 (0.0200) [0.2750]	1.5460 (0.0460) [0.4600]	0.9130 (−0.0087) [0.1260]
	IFM	3.0170 (0.0170) [**0.1770**]	4.6450 (0.1450) [**1.000**]	2.3140 (0.0140) [**0.2540**]	1.5370 (0.0370) [**0.4340**]	0.9070 (−0.0930) [0.1200]
	CML	-	-	-	-	0.9000 (−0.1000) [**0.1150**]
250	MLE	3.0180 (0.0180) [0.1810]	4.6020 (0.1020) [0.9720]	2.3150 (0.0150) [0.2410]	1.5310 (0.0310) [0.4080]	0.9260 (−0.0740) [0.1180]
	IFM	3.0110 (0.0110) [**0.1570**]	4.5910 (0.0910) [**0.9310**]	2.3080 (0.0080) [**0.2220**]	1.5250 (0.0250) [**0.3900**]	0.9210 (−0.0790) [0.1120]
	CML	-	-	-	-	0.9150 (−0.0850) [**0.1060**]
500	MLE	3.0060 (0.0060) [0.1200]	4.5400 (0.0400) [0.5140]	2.3030 (0.0030) [0.1570]	1.5080 (0.0080) [0.2890]	0.9510 (−0.0490) [0.0820]
	IFM	3.0030 (0.0030) [**0.1070**]	4.5350 (0.0350) [**0.4950**]	2.3010 (0.0010) [**0.1440**]	1.5040 (0.0040) [**0.2730**]	0.9470 (−0.0530) [0.0770]
	CML	-	-	-	-	0.9430 (−0.0570) [**0.0720**]
1000	MLE	3.0020 (0.0020) [0.0910]	4.5200 (0.0200) [0.3900]	2.2980 (−0.0020) [0.1120]	1.4970 (−0.0030) [0.1330]	0.9640 (−0.0360) [0.0710]
	IFM	3.0010 (0.0010) [**0.0850**]	4.5160 (0.0160) [**0.3790**]	2.2960 (−0.0040) [**0.1090**]	1.4960 (−0.0040) [**0.1320**]	0.9620 (−0.0380) [0.0680]
	CML	-	-	-	-	0.9600 (−0.0400) [**0.0660**]

Table 2. Results of simulation study-2 based on the FGM copula (the smallest RMSE in each configuration is highlighted in bold).

Sample Size (n)	**Method of Estimation**	**Parameter Estimates**				
		$\lambda_1 = 1.8$	$\theta_1 = 2.5$	$\lambda_2 = 1.0$	$\theta_2 = 4.0$	$\alpha_F = 0.7$
50	MLE	1.9477 (0.1477) [**0.1477**]	2.3742 (−0.1258) [0.1258]	0.9159 (−0.0841) [**0.0841**]	3.6125 (−0.3875) [**0.3875**]	0.7335 (0.0335) [0.0335]
	IFM	1.9989 (0.1989) [0.1989]	2.5278 (0.0278) [**0.0278**]	1.1542 (0.1542) [0.1542]	3.6016 (−0.3984) [0.3984]	0.7866 (0.0866) [0.0866]
	CML	-	-	-	-	0.7334 (0.0334) [**0.0334**]
75	MLE	1.8903 (0.0903) [**0.0903**]	2.4386 (−0.0614) [0.0614]	0.9438 (−0.0562) [**0.0562**]	3.7312 (−0.2688) [**0.2688**]	0.7098 (0.0098) [0.0098]
	IFM	1.9402 (0.1402) [0.1402]	2.5142 (0.0142) [**0.0142**]	1.1088 (0.1088) [0.1088]	3.7215 (−0.2785) [0.2785]	0.7809 (0.0809) [0.0809]
	CML	-	-	-	-	0.7097 (0.0097) [**0.0097**]
100	MLE	1.8539 (0.0539) [**0.0539**]	2.4531 (−0.0469) [0.0469]	0.9679 (−0.0321) [**0.0321**]	3.8154 (−0.1846) [0.1846]	0.7023 (0.0023) [0.0023]
	IFM	1.8899 (0.0899) [0.0899]	2.5051 (0.0051) [**0.0051**]	1.0723 (0.0723) [0.0723]	3.8211 (−0.1789) [**0.1789**]	0.7624 (0.0624) [0.0624]
	CML	-	-	-	-	0.7022 (0.0022) [**0.0022**]
150	MLE	1.8284 (0.0284) [**0.0284**]	2.4795 (−0.0205) [0.0205]	0.9775 (−0.0225) [**0.0225**]	3.8736 (−0.1264) [0.1264]	0.7008 (0.0008) [**0.0008**]
	IFM	1.8605 (0.0605) [0.0605]	2.5002 (0.0002) [**0.0002**]	1.0469 (0.0469) [0.0469]	3.8982 (−0.1018) [**0.1018**]	0.7527 (0.0527) [0.0527]
	CML	-	-	-	-	0.7008 (0.0008) [**0.0008**]
200	MLE	1.8105 (0.0105) [**0.0105**]	2.4889 (−0.0111) [0.0111]	0.9834 (−0.0166) [**0.0166**]	3.9083 (−0.0917) [0.0917]	0.7004 (0.0004) [**0.0004**]
	IFM	1.8481 (0.0481) [0.0481]	2.4951 (−0.0049) [**0.0049**]	1.0346 (0.0346) [0.0346]	3.9291 (−0.0709) [**0.0709**]	0.7472 (0.0472) [0.0472]
	CML	-	-	-	-	0.7004 (0.0004) [**0.0004**]
250	MLE	1.8025 (0.0025) [**0.0025**]	2.4925 (−0.0075) [**0.0075**]	0.9859 (−0.0141) [**0.0141**]	3.9246 (−0.0754) [0.0754]	0.7002 (0.0002) [**0.0002**]
	IFM	1.8439 (0.0439) [0.0439]	2.4922 (−0.0078) [0.0078]	1.0293 (0.0293) [0.0293]	3.9426 (−0.0574) (**0.0574**)	0.7428 (0.0428) [0.0428]
	CML	-	-	-	-	0.7002 (0.0002) [**0.0002**]

Table 2. *Cont.*

Sample Size (*n*)	Method of Estimation	Parameter Estimates				
		$\lambda_1 = 1.8$	$\theta_1 = 2.5$	$\lambda_2 = 1.0$	$\theta_2 = 4.0$	$\alpha_F = 0.7$
500	MLE	1.7969 (−0.0031) [**0.0031**]	2.4961 (−0.0039) [**0.0039**]	0.9925 (−0.0075) [**0.0075**]	3.9621 (−0.0379) [0.0379]	0.7001 (0.0001) [**0.0001**]
	IFM	1.8266 (0.0266) [0.0266]	2.4847 (−0.0153) [0.0153]	1.0134 (0.0134) [0.0134]	3.9746 (−0.0254) [**0.0254**]	0.7257 (0.0257) [0.0257]
	CML	-	-	-	-	0.7001 (0.0001) [**0.0001**]
1000	MLE	1.7984 (−0.0016) [**0.0016**]	2.4983 (−0.0017) [**0.0017**]	0.9961 (−0.0039) [**0.0039**]	3.9817 (−0.0183) [0.0183]	0.7000 (0.0000) [**0.0000**]
	IFM	1.8168 (0.0168) [0.0168]	2.4749 (−0.0251) [0.0251]	1.0065 (0.0065) [0.0065]	3.9883 (−0.0117) [**0.0117**]	0.7119 (0.0119) [0.0119]
	CML	-	-	-	-	0.7000 (0.0000) [**0.0000**]

3.2. Sensitivity Analysis under Alternative Copula Specification

To examine the robustness of the estimation methods beyond the FGM copula's moderate dependence range, a sensitivity analysis was performed comparing the FGM copula to the Plackett copula, which captures considerably stronger symmetric dependence. Monte Carlo simulations indicate that the relative performance of the MLE, IFM, and CML methods is largely unaffected by the choice of copula, demonstrating that the FGM-based results are robust across a broader range of dependence structures.

Specifically, estimation under the Plackett dependence parameter was performed under a similar modelling framework with the primary FGM-based simulations, with bias and RMSE computed for all parameters. The results indicate that while MLE and IFM exhibit comparable performance for marginal parameters, CML remains competitive for dependence estimation, RMSE differences across copula specifications are minor, and convergence patterns with increasing sample sizes are preserved. A detailed simulation result for the Placket copula is reported in Table 3.

Table 3. Results of simulation study-1 for the Plackett copula (the smallest RMSE in each configuration is highlighted in bold).

Sample Size (*n*)	Method of Estimation	Parameter Estimates				
		$\lambda_1 = 3.0$	$\theta_1 = 4.5$	$\lambda_2 = 2.3$	$\theta_2 = 1.5$	$\alpha_P = 3.0$
50	MLE	3.1432 (0.1432) [**0.5240**]	5.4879 (0.9879) [**2.6064**]	2.5104 (0.2104) [**0.6873**]	1.8786 (0.3786) [**1.0804**]	2.7912 (−0.2088) [**0.3206**]
	IFM	3.1535 (0.1535) [0.5410]	5.5092 (1.0092) [2.6731]	2.5241 (0.2241) [0.7103]	1.9042 (0.4042) [1.1020]	2.7613 (−0.2387) [0.3510]
	CML	-	-	-	-	2.7347 (−0.2653) [0.3721]
75	MLE	3.1010 (0.1010) [**0.4422**]	5.1222 (0.6222) [**2.1045**]	2.4523 (0.1523) [**0.5824**]	1.7455 (0.2455) [**0.8945**]	2.8364 (−0.1636) [**0.2771**]
	IFM	3.1127 (0.1127) [0.4531]	5.1398 (0.6398) [2.1410]	2.4681 (0.1681) [0.5938]	1.7610 (0.2610) [0.9110]	2.8059 (−0.1941) [0.3035]
	CML	-	-	-	-	2.7792 (−0.2208) [0.3324]

Table 3. *Cont.*

Sample Size (n)	**Method of Estimation**	**Parameter Estimates**				
		$\lambda_1 = 3.0$	**$\theta_1 = 4.5$**	**$\lambda_2 = 2.3$**	**$\theta_2 = 1.5$**	**$\alpha_P = 3.0$**
100	MLE	3.0678 (0.0678) [**0.3021**]	4.7535 (0.2535) [**0.9762**]	2.3629 (0.0629) [**0.3821**]	1.6011 (0.1011) [**0.6242**]	2.8824 (−0.1176) [**0.1982**]
	IFM	3.0754 (0.0754) [0.3165]	4.7997 (0.2997) [1.0203]	2.3817 (0.0817) [0.4024]	1.6150 (0.1150) [0.6386]	2.8579 (−0.1421) [0.2153]
	CML	-	-	-	-	2.8411 (−0.1589) [0.2284]
150	MLE	3.0503 (0.0503) [**0.2415**]	4.6058 (0.1058) [**0.7063**]	2.3268 (0.0268) [**0.3002**]	1.5502 (0.0502) [**0.4904**]	2.9177 (−0.0823) [**0.1430**]
	IFM	3.0587 (0.0587) [0.2542]	4.6120 (0.1120) [0.7428]	2.3384 (0.0384) [0.3116]	1.5634 (0.0634) [0.5088]	2.8915 (−0.1085) [0.1639]
	CML	-	-	-	-	2.8689 (−0.1311) [0.1765]
200	MLE	3.0312 (0.0312) [**0.2012**]	4.5210 (0.0210) [**0.5547**]	2.3109 (0.0109) [**0.2519**]	1.5292 (0.0292) [**0.4057**]	2.9454 (−0.0546) [**0.1210**]
	IFM	3.0396 (0.0396) [0.2106]	4.5289 (0.0289) [0.5764]	2.3197 (0.0197) [0.2612]	1.5387 (0.0387) [0.4182]	2.9191 (−0.0809) [0.1408]
	CML	-	-	-	-	2.9012 (−0.0988) [0.1507]
250	MLE	3.0257 (0.0257) [**0.1713**]	4.4952 (−0.0048) [**0.4702**]	2.3011 (0.0011) [**0.2136**]	1.5110 (0.0110) [**0.3599**]	2.9618 (−0.0382) [**0.1082**]
	IFM	3.0304 (0.0304) [0.1821]	4.5063 (0.0063) [0.4890]	2.3075 (0.0075) [0.2241]	1.5195 (0.0195) [0.3701]	2.9350 (−0.0650) [0.1240]
	CML	-	-	-	-	2.9153 (−0.0847) [0.1341]
500	MLE	3.0122 (0.0122) [**0.1321**]	4.4817 (−0.0183) [**0.3561**]	2.2966 (−0.0034) [**0.1680**]	1.5013 (0.0013) [**0.2862**]	2.9805 (−0.0195) [**0.0883**]
	IFM	3.0178 (0.0178) [0.1383]	4.4959 (−0.0041) [0.3725]	2.2991 (−0.0009) [0.1749]	1.5102 (0.0102) [0.2961]	2.9532 (−0.0468) [0.1004]
	CML	-	-	-	-	2.9403 (−0.0597) [0.1095]
1000	MLE	3.0058 (0.0058) [**0.1010**]	4.4710 (−0.0290) [**0.2714**]	2.2927 (−0.0073) [**0.1312**]	1.4991 (−0.0009) [**0.2153**]	2.9922 (−0.0078) [**0.0612**]
	IFM	3.0101 (0.0101) [0.1072]	4.4864 (−0.0136) [0.2833]	2.2952 (−0.0048) [0.1373]	1.5063 (0.0063) [0.2242]	2.9683 (−0.0317) [0.0706]
	CML	-	-	-	-	2.9562 (−0.0438) [0.0798]

3.3. Real-Life Data Applications

Dataset 1: originally analyzed by [18,19], contains a frailty component and represents the recurrence times of infections among patients undergoing kidney dialysis. The dataset comprises thirty-eight patients and a total of seventy-six recorded observations, providing a classic example of clustered survival data with within-patient dependence. The data are presented below:

8, 23, 22, 447, 30, 24, 7, 511. 53, 15, 7, 141, 96, 149, 536, 17, 185, 292, 22, 15, 152, 402, 13, 39, 12, 113, 132, 34, 2, 130, 27, 5, 152, 190, 119, 54, 6, 63

16, 13, 28, 318, 12, 245, 9, 30, 196, 154, 333, 8, 38, 70, 25, 4, 177, 114, 159, 108, 562, 24, 66, 46, 40, 201, 156, 30, 25, 26, 58, 43, 30, 5, 8, 16, 78, 8

Dataset 2: Table 4 presents the reliability data for a parallel system comprising two motors, as reported by [20]. In this setup, the operational load is shared equally between both motors when they are functioning simultaneously. When one motor fails, the entire load is transferred to the remaining operational unit, thereby increasing its stress and likelihood of subsequent failure. The system ultimately ceases to function once both motors fail.

Table 4. Time to failure for two motors.

System	Time to Failure for Motor X_1	Time to Failure for Motor X_2	Event Order
1	102	65	X_2 failed first
2	84	148	X_1 failed first
3	88	202	X_1 failed first
4	156	121	X_1 failed first
5	148	123	X_2 failed first
6	139	150	X_2 failed first
7	245	156	X_1 failed first
8	235	172	X_2 failed first
9	220	192	X_2 failed first
10	207	214	X_2 failed first
11	250	212	X_1 failed first
12	212	220	X_2 failed first
13	213	265	X_1 failed first
14	220	275	X_1 failed first
15	243	300	X_1 failed first
16	300	248	X_2 failed first
17	257	330	X_1 failed first
18	263	350	X_1 failed first

Dataset 3: represents the diabetic retinopathy competing risks data, originally analyzed by [10]. The dataset summarizes the results of an experiment conducted on seventy-one diabetic patients aimed at assessing the effectiveness of laser treatment in reducing the risk of blindness. The first variable records the time to onset of blindness following laser treatment, while the second variable serves as an indicator, specifying whether one eye was treated, untreated, or whether blindness eventually occurred in both eyes. The dataset is presented below:

266 91 154 285 583 547 79 622 707 469 93 1313 805 344 790 125 777 306 415 307 637 577 178 517 272 1137 1484 315 287 1252 717 642 141 407 356 1653 427 699 36 667 588 471 126 350 350 663 567 966 203 84 392 1140 901 1247 448 904 276 520 485 248 503 423 285 315 727 210 409 584 355 1302 227

1 2 2 0 1 2 1 0 2 2 1 2 1 1 2 2 2 1 1 2 2 2 1 2 0 0 1 1 2 1 2 1 2 1 1 0 2 1 2 1 2 0 1 2 1 0 2 0 0 1 1 2 1 0 2 2 1 1 2 2 1 2 2 2 2 2 2 1 1 1 2

4. Discussion

Tables 1 and 2 present the simulation results for the copula-based bivariate OR-ED, based on the assigned true parameter values indicated at the head of each table. MLE, IFM, and CML estimation methods were employed across varying sample sizes ranging from 50 to 1000. Each cell in the table reports the parameter estimates, its bias (in parentheses), and the corresponding root-mean-square error (RMSE) in square brackets. The bias measures the deviation of an estimated parameter from its true value, while the RMSE provides an overall measure of estimation accuracy.

Both tables demonstrate that the marginal parameter estimates converge toward their true values as the sample size increases, affirming the consistency of the estimators. For the copula parameter α_F in Table 1, sample estimates exhibited negative bias, with values of $\alpha_F = 0.823$ (MLE), 0.814 (IFM), and 0.799 (CML) at $n = 50$. This bias reduced steadily, as the sample size increased toward $n = 1000$, and the parameter estimates approached the true value ($\alpha_F = 1.0$) with $\alpha_F = 0.964$ (MLE), 0.962 (IFM), and 0.960 (CML), and RMSEs below 0.07, indicating strong convergence.

The second simulation experiment, summarized in Table 2, used true marginal parameter values of $\lambda_1 = 1.8$, $\theta_1 = 2.5$, $\lambda_2 = 1.0$, $\theta_2 = 4.0$, and copula parameter (α_F) = 0.7. Parameter estimation was again conducted using MLE, IFM, and CML for various sample sizes. Across replications, MLE consistently produced better estimates, with biases and RMSEs decreasing markedly as sample size increased. For small samples (n = 50, 75), moderate bias was observed for λ_1, θ_1, and particularly θ_2, though these diminished substantially by n = 150. The copula parameter α_F displayed higher variability at small samples due probably to the limited dependence range of the FGM structure. Nevertheless, RMSEs declined steadily as sample size grew, with θ_1 and λ_2 showing notably lower RMSEs, indicating better estimability. The results further demonstrate that IFM performs well in large samples and is computationally advantageous. Results of the computational runtime comparison of the competing estimation procedures indicate that the IFM procedure is consistently faster than MLE. IFM required between 20–40% less computation time than the full MLE procedure because it separates the estimation of the marginal parameters from the copula parameter, thereby avoiding repeated optimization over the full joint likelihood required by the MLE approach, while CML exhibited the least computational time because it estimates only the copula parameter. Despite these differences, all three methods converged reliably across the sample sizes investigated. While CML remains computationally efficient, it does not offer superior accuracy compared to MLE.

Note that the estimator with the smallest RMSE in each configuration is highlighted in bold in Tables 1 and 2, to improve interpretability. Relative efficiency measures, computed as ratios of RMSEs relative to the MLE, are advanced to provide a quantitative comparison of estimator performance. For example, in Table 1, the relative efficiency of IFM relative to MLE, using (n = 50, λ_1) is $(0.5490^2)/(0.5310^2) \approx 1.07$. This is interpreted to mean that IFM is about 7% more efficient than MLE at n = 50 for λ_1. In other words, a relative efficiency greater than 1 indicates that the estimation method is more efficient than MLE, whereas a relative efficiency less than one shows that the method is less efficient than MLE. Under stronger dependence (α_F = 1.0), the IFM estimator exhibits modest efficiency gains over the MLE for marginal parameters, while CML provides the most accurate estimation of the dependence parameter. In contrast, under moderate dependence (α_F = 0.7), MLE generally outperforms IFM for marginal parameters, and MLE and CML become virtually indistinguishable in estimating the dependence parameter. In both scenarios, efficiency differences diminish as sample size increases, confirming the asymptotic equivalence of the estimators. These findings suggest that no single estimation method consistently dominates across different dependence regimes, thereby underscoring the need for context-specific estimator selection.

The FGM copula dependence parameter satisfies $\alpha \in [-1,1]$, with $\alpha = \pm 1$ representing boundary values corresponding to maximal dependence within the FGM family. Standard asymptotic normality results are derived under interior parameter values $\alpha \in (-1,1)$; thus, these results do not formally apply at the boundary. However, the copula likelihood remains well-defined and differentiable at α = 1, and estimation proceeds without numerical instability. Since the current study focuses on finite-sample performance via simulation rather than solely on asymptotic normality, the inclusion of the boundary case α_F = 1.0 allows examination of estimator behaviour under maximal attainable dependence. For interior values (e.g., α = 0.7), the standard asymptotic theory applies directly. The boundary behaviour of likelihood-based estimators is well understood within the general framework of constrained MLE.

Collectively, the simulation findings indicate that estimation accuracy and convergence speed depend strongly on the estimation method, sample size, and the strength of copula dependence. All three estimation methods are consistent under the FGM copula approach. However, MLE consistently demonstrates the highest accuracy and reliability across all sample sizes, particularly in jointly estimating both marginal and copula parameters. The IFM method performs comparably to MLE and is preferred when computational efficiency is critical, though it may inadequately capture dependence in small samples. On the whole, MLE and IFM are recommended for moderate to large sample sizes. At the same time, MLE remains the most robust and dependable approach for small sample analyses where precise dependence modelling is essential.

Tables 5–7 summarize the empirical performance of the bivariate OR-ED when applied to three real-life datasets, using the MLE, IFM, and CML estimation techniques. For each parameter, both the bias and RMSE were computed to assess efficiency and accuracy.

Table 5. Estimates of copula-based bivariate OR-ED application to dataset 1 (with biases in round braces and RMSEs in square braces).

Copula	Estimation Method	Parameter Estimates				Copula Parameter
		λ_1	θ_1	λ_2	θ_2	A
FGM	MLE	0.6083 (0.0083) [0.0666]	4.7983 (0.2982) [1.3846]	0.4081 (0.0081) [0.0599]	1.9329 (0.1329) [0.5912]	0.2908 (−0.0092) [0.2670]
	IFM	0.6080 (0.0080) [0.0664]	4.7900 (0.2900) [1.3763]	0.4078 (0.0078) [0.0597]	1.9297 (0.1297) [0.5878]	0.2897 (−0.0103) [0.2661]
	CML	-	-	-	-	0.2870 (−0.0130) [0.2681]

Table 6. Estimates of copula-based bivariate OR-ED application to dataset 2 (with biases in round braces and RMSEs in square braces).

Copula	Estimation Method	Parameter Estimates				Copula Parameter
		λ_1	θ_1	λ_2	θ_2	A
FGM	MLE	0.0091 (0.0009) [0.0029]	6.3694 (2.5992) [8.2213]	0.0048 (0.0007) [0.0023]	1.8983 (0.7727) [2.0981]	0.7582 (−0.2418) [0.4672]
	IFM	0.0090 (0.0007) [0.0029]	6.1683 (2.3981) [8.4592]	0.0047 (0.0006) [0.0023]	1.8348 (0.7092) [2.0469]	0.7517 (−0.2483) [0.4696]
	CML	-	-	-	-	0.7535 (−0.2465) [0.4578]

Table 7. Estimates of copula-based bivariate OR-ED application to dataset 3 (with biases in round braces and RMSEs in square braces).

Copula	Estimation Method	Parameter Estimates				Copula Parameter
		λ_1	θ_1	λ_2	θ_2	A
FGM	MLE	0.0003 (0.0003) [0.0013]	7.8596 (7.8596) [63.4875]	0.7266 (0.0200) [0.1013]	4.3528 (0.4321) [1.5605]	−0.3161 (−0.0048) [0.3693]
	IFM	0.0003 (0.0003) [0.0013]	7.8590 (7.8590) [63.4875]	0.7272 (0.0206) [0.1016]	4.3666 (0.4459) [1.5733]	−0.3186 (−0.0072) [0.3659]
	CML	-	-	-	-	−0.3236 (−0.0123) [0.3749]

From Table 5, corresponding to the kidney dialysis dataset, it is immediately evident that the MLE and IFM methods yield similar parameter estimates across all cases. The marginal parameters exhibit extremely small bias (less than 0.01 for λ and below 0.3 for θ) and minimal RMSEs, confirming the consistency and efficiency of both estimators in this context. The copula parameter α shows a slight negative bias (−0.01) and moderate RMSE (0.267), reflecting the weak dependence typically captured by the FGM copula. The CML method, as expected, provides estimates for the copula parameter only, with bias (−0.013) and RMSE (0.268) comparable to those of MLE and IFM, demonstrating that the simplification inherent in CML does not compromise its estimation accuracy in this setting. These findings suggest that the bivariate OR-ED, when modelled using the FGM copula, can be reliably estimated from real-life data using MLE or IFM without appreciable loss of precision. The consistently low biases and RMSEs across parameters highlight excellent finite-sample properties. Among the estimation methods, IFM demonstrated the most consistent performance, with relatively lower biases and RMSEs, followed closely by MLE. In other words, the IFM estimation method emerged as the most reliable estimation architecture for this dataset.

Turning to Table 6, which reports results for the motor failure dataset, interesting variations in estimation accuracy are observed. Both MLE and IFM deliver comparable performance across parameters, with very small marginal parameter biases and low RMSEs. However, θ_1 and θ_2 are remarkably overestimated, with θ_1 exhibiting a notably higher RMSE (>8.2) than θ_2 (approximately 2.1). The copula parameter α reveals moderate negative bias (−0.24 to −0.25) and RMSEs in the range 0.46–0.47. CML estimates for α closely align with those obtained from MLE and IFM, both in terms of bias and RMSE. For this dataset, IFM marginally outperforms MLE, displaying competitive RMSEs across parameters and proving advantageous when both marginal and joint parameters must be estimated simultaneously. Thus, for the motor failure dataset, the IFM procedure provides the best balance between low bias and RMSE.

Results for dataset 3, reported in Table 7, reveal a contrasting pattern. While λ_1 performed similarly for both MLE and IFM estimation methods, MLE slightly edged IFM for both λ_2 and θ_2. The RMSE for θ_1 demonstrated extreme instability (MLE, IFM = 63.4875). The instability observed may be attributed to several factors. The sample size and distributional characteristics of the data, including skewness and heavy-tail behaviour, may have affected the reliability of joint estimation. Additionally, weak copula dependence may result in flat likelihood surfaces, which can affect numerical convergence and increase variability in parameter estimates. To mitigate such issues, the use of carefully selected starting values and alternative estimation strategies may improve stability. Bootstrap-based inference can further provide more reliable measures of uncertainty in the presence of such numerical insensitivity.

In a nutshell, these findings confirm that the bivariate OR-ED, when combined with the FGM copula-based dependence modelling, provides an alluring tool for analyzing real-life survival and reliability data. MLE and IFM methods provide a good balance between computational efficiency and accuracy, particularly for large or moderately-sized samples. The CML method, though computationally simple, is most advantageous when the focus lies solely on dependence estimation.

While the proposed model provides a satisfactory fit to the observed data, certain data-related limitations should be acknowledged. Firstly, the available sample size may restrict estimation precision, particularly for the copula dependence parameter, where variability tends to increase in moderate samples. Secondly, censoring reduces the effective information content of the data, potentially affecting both marginal and joint parameter estimates. Finally, measurement variability inherent in real-life data may introduce additional uncertainty in parameter estimation. These factors may influence the accuracy and stability of the estimates and should therefore be considered when interpreting the empirical results.

While the current study focuses on complete bivariate survival data, the proposed OR-ED FGM copula-based method can naturally be extended to accommodate right-censored observations. In such cases, the likelihood function is constructed using appropriate contributions from the joint density, joint survival function, and mixed partial derivatives depending on the censoring pattern. Specifically, uncensored pairs contribute to the joint density, singly censored observations contribute partial derivatives of the joint survival function, and doubly censored observations contribute the joint survival probability. Since the copula structure is defined in terms of marginal survival functions, incorporation of censoring does not alter the dependence structure but only modifies the likelihood specification.

5. Conclusions

The FGM copula-based bivariate OR-ED, which this study considered, is a novel foundation for modelling joint lifetime data exhibiting moderate dependence structures. By combining the analytical tractability of the FGM copula with the structural properties of the OR-ED marginals, the model offers a useful approach for representing asymmetric lifetime behaviour while accommodating dependence between paired observations. Three estimation methods (MLE, IFM, and CML) were employed to evaluate the model's performance. Monte Carlo simulations across multiple sample sizes indicate that the estimators exhibit desirable finite-sample behaviour, with bias and RMSE decreasing as the sample size increases. In general, the MLE method produced slightly more stable estimates, while the IFM approach yielded comparable performance with reduced computational complexity. The CML method provided a reliable estimation of the dependence parameter when interest was primarily focused on the copula component. The applicability of the model was further illustrated through empirical analyses of three real datasets. The results demonstrate that the proposed framework can effectively capture dependence patterns in paired lifetime and reliability data. The empirical findings also suggest that the relative performance of the estimation methods may vary depending on the characteristics of the dataset, including the strength of dependence and the variability present in the marginal distributions. Consequently, the choice of estimation method should be guided by both statistical and computational considerations in practical applications.

From a methodological perspective, MLE is advantageous when full joint efficiency is desired, and numerical optimization remains stable. The IFM approach provides a computationally attractive alternative because it separates marginal and dependence estimation while maintaining comparable inferential accuracy in many settings. The CML method is especially useful when the primary interest lies in modelling the dependence structure alone. The results generally suggest that the FGM copula-based bivariate OR-ED model constitutes a viable addition to the class of copula-based survival models for moderately dependent lifetime data.

Future research should consider extensions to higher-dimensional copula constructions, alternative dependence structures capable of modelling stronger dependence, and the incorporation of censoring mechanisms to broaden the model's applicability in survival and reliability studies.

Author Contributions

O.O., M.T., and I.A.S.: conceptualization; O.O., M.T., A.U., A.Y., and I.A.S.: methodology; O.O., M.T., and A.Y.: data curation; O.O. and M.T.: writing—original draft preparation; O.O., M.T., A.U., A.Y., and I.A.S.: visualization, investigation; M.T., A.U., A.Y., and I.A.S.: supervision; O.O., M.T., I.A.S., A.U., and A.Y.: software, validation; O.O., M.T., A.U., A.Y., and I.A.S.: writing—reviewing and editing. All authors have read and agreed to the published version of the manuscript.

Funding

This work was supported by the Tertiary Education Trust Fund (TETFund), Nigeria.

Institutional Review Board Statement

Not applicable.

Informed Consent Statement

Not applicable.

Data Availability Statement

Not applicable.

Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

During the preparation of this work, the authors used OpenAI/ChatGPT and Grammarly to improve clarity and language refinement. After using these tools/services, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

References

1. Mahmoud, A.A.M.; Khashab, R.H.; Kalantan, Z.I.; et al. On Bivariate Compound Exponentiated Survival Function of the Beta Distribution: Estimation and Prediction. *J. Radiat. Res. Appl. Sci.* **2024**, *17*, 100886. https://doi.org/10.1016/j.jrras.2024.100886.
2. Ode, O.; Tasi'u, M.; Usman, A.; et al. A Novel Odd Rayleigh-Exponential Distribution (OR-ED) and its Application to Lifetime Datasets. *J. Stat. Sci. Comput. Intell.* **2025**, *1*, 262–282. https://doi.org/10.64497/jssci.91.
3. Mohammed, B.I.; Makumi, N.; Aldallal, R.; et al. A New Model of Discrete-Continuous Bivariate Distribution with Applications to Medical Data. *Comput. Math. Methods Med.* **2022**, *2022*, 1883491. https://doi.org/10.1155/2022/1883491.
4. Darwish, J.A.; Shahbaz, S.A.; Shahbaz, M.Q. A New Class of Bivariate Distributions: Properties, Estimation and Modelling. *Mathematics* **2024**, *12*, 4021. https://doi.org/10.3390/math12244021.
5. Achcar, J.A.; Moala, F.A.; Tarumoto, M.H.; et al. A Bivariate Generalized Exponential Distribution Derived from Copula Functions in the Presence of Censored Data and Covariates. *Pesqui. Oper.* **2015**, *35*, 165–186. https://doi.org/10.1590/0101-7438.2015.035.01.0165.
6. Al Turk, L.I.; Elaal, M.K.A.; Jarwan, R.S. Inference of Bivariate Generalized Exponential Distribution based on Copula Functions. *Appl. Math. Sci.* **2017**, *11*, 1155–1186. https://doi.org/10.12988/ams.2017.7398.
7. Fayomi, A.; Refaey, R.M.; AL-Dayian, G.R.; et al. Bivariate Lomax Distribution based on Clayton Copula: Estimation and Prediction. *J. Stat. Appl. Probab.* **2024**, *13*, 1321–1337. https://doi.org/10.18576/jsap/130415.
8. Pathak, A.K.; Vellaisamy, P. A Bivariate Generalized Linear Exponential Distribution: Properties and Estimation. *Commun. Stat. Simul. Comput.* **2022**, *51*, 5426–5446. https://doi.org/10.1080/03610918.2020.1771591.
9. Regoli, G. A Class of Bivariate Exponential Distributions. *J. Multivar. Anal.* **2009**, *100*, 1261–1269. https://doi.org/10.1016/j.jmva.2008.11.003.

10. Sarhan, A.M.; Apaloo, J.; Kundu, D. A New Bivariate Lifetime Distribution: Properties, Estimations and its Extension. *Commun. Stat. Simul. Comput.* **2024**, *53*, 879–896. https://doi.org/10.1080/03610918.2022.2034866.
11. Elaal, M.K.A. Bivariate Beta Exponential distributions. *IOSR J. Math.* **2017**, *13*, 7–19. https://doi.org/10.9790/5728-1303010719.
12. Tang, X.; Li, D.; Zhou, C.; et al. Bivariate Distribution Models using Copulas for Reliability Analysis. *J. Risk Reliab.* **2013**, *227*, 499–512. https://doi.org/10.1177/1748006X13481928.
13. Qura, M.E.; Fayomi, A.; Kilai, M.; et al. Bivariate Power Lomax Distribution with Medical Applications. *PLoS ONE* **2023**, *18*, e0282581. https://doi.org/10.1371/journal.pone.0282581.
14. Barakat, H.M.; Alawady, M.A.; Husseiny, I.A.; et al. Bivariate Epanechnikov-exponential Distribution: Statistical Properties, Reliability Measures, and Applications to Computer Science Data. *AIMS Math.* **2024**, *9*, 32299–32327. https://doi.org/10.3934/math.20241550.
15. Hussain, A.S.; Mahmood, K.B.; Ibrahim, I.M.; et al. Parameters Estimation of the Gompertz-Makeham Process in Non-Homogeneous Poisson Processes: Using Modified Maximum Likelihood Estimation and Artificial Intelligence Methods. *Math. Stat.* **2025**, *13*, 1–11. https://doi.org/10.13189/ms.2025.130101.
16. Treidi, W.; Zeghdoudi, H. The Bivariate Poisson-X-Exponential Distribution: Theory, Inference, and Multidomain Applications. *Stats* **2026**, *9*, 18. https://doi.org/10.3390/stats9010018.
17. Schmidt, T. Coping with Copulas. 2006. Available online: https://everything.explained.today/Copula_(statistics)/ (accessed on 8 August 2025).
18. McGilchrist, C.; Aisbett, C. Regression with Frailty in Survival Analysis. *Biometrics* **1991**, *47*, 461–466.
19. Menberu, A.A.; Goshu, A.T. The Transformed MG-Extended Exponential Distribution: Properties and Applications. *J. Stat. Theory Appl.* **2024**, *23*, 175–205. https://doi.org/10.1007/s44199-024-00078-8.
20. Raja, V.A. On the Analysis of Bivariate Lifetime Data-some Models and Applications. Ph.D. Thesis, Cochin University of Science and Technology, Kochi, India, 2017; pp. 1–217.