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A Monotone and Non-Monotone Hazard Rate Model: Its Application in Real Scenario

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Abstract: In this article, alpha power new logarithmic transformation (APNLT) is proposed to get the new lifetime distributions using some baseline distribution in order to get flexible and superior distributions in terms of fitting to the real data. For the application point of view, we have considered exponential distribution as an appropriate baseline distribution which has constant hazard rate function and thus we have obtained alpha power new logarithmic transformed exponential (APNLTE) distribution. It has increasing, decreasing and upside-down bathtub shapes of failure rate function. Several statistical properties of APNLTE distribution have also been studied. A real dataset is taken to compare this new distribution with some existing distributions.

Keywords: APNLTE-distribution; hazard rate function; Renyi entropy; maximum likelihood estimator

1. Introduction

Exponential distribution plays an active role in reliability theory due to its constant hazard rate but constant hazard rate is rarely available in real situations. In most of the situations it is monotone non-decreasing, monotone non-increasing, bathtub, upside down bathtub etc. Keeping this point in mind, several authors have introduced some new generalized distributions in order to have flexible and superior distributions to cover most of the real situations. For example, Mudholkar and Srivastava (1993) [1] proposed a new distribution by introducing an additional shape parameter to the Weibull distribution (WD) with two parameters one is shape parameter and other is scale parameter and they called it as Exponentiated Weibull Distribution (EWD), and thus having a much flexible distribution as compared to the baseline distribution. Further, Gupta et al. (1998) [2] introduced a technique to generalize some available distribution. EWD introduced by Mudholkar and Srivastava (1993) [1] as a particular case of it, by considering WD with two parameters as a baseline distribution. Later on various new distributions such as generalized exponential, generalized inverse exponential distributions were introduced using the work of Gupta et al. (1998) [2]. Eugene (2002) [3] proposed Beta-Binomial distribution. Jones (2004) [4] introduced a family of distributions arising from distribution of order statistics. Shaw and Buckley (2009) [5] introduced a remarkable technique of generalizing any available distribution and it is called quadratic rank transmuted map (QRTM) and discussed the normality of the distributions obtained from it by considering uniform, exponential and normal distributions as baseline distributions. Cordeiro and de Castro (2011) [6] introduced a family of distributions and called it as Kumaraswamy (KW)—Gamma Distribution, particularly KW-Normal, KW-Weibull, KW-Gamma, KW-Gumbel and KW-inverse Gaussian distributions.

Recently, Kumar et al. (2015) [7] advocated the use of transformation instead of generalizing any available baseline distribution in order to restrict increase in the number of additional parameters as compared to the parameters involved in the baseline distribution. They called it as DUS-transformation and the distribution is obtained by using it corresponding to the exponential distribution as a baseline distribution has increasing failure rate. Further, Mahdavi and Kundu (2017) [8] generalized DUS-transformation by introducing an additional shape parameter and they named the generalization of alpha power transformation (APT). The main aim to generalize DUS- transformation is to add flexibility in the new distribution, thus obtained. For instance, they have taken exponential distribution as a baseline distribution and shown that APT of exponential distribution has increasing as well as decreasing failure rate for $\alpha > 1$ and $\alpha \leq 1$ respectively. Next, Kumar et al. (2015) [9] introduced a new transformation in terms of sine function and called it as SS transformation that uses a baseline distribution, to check its applications in the real scenario they have considered exponential distribution as a baseline distribution and studied its application to real data. Recently, Chesnagy et al. (2019) [10] generalized SS transformation using sine and cosine functions and adding an additional shape parameter in order to add flexibility in terms of fitting and application to wide range of data. A remarkable transformation is proposed by Maurya et al. (2016) [11] using logarithmic function and they called it as logarithmic transformation (LT) and it defined as follows

$$F(x) = \frac{\ln(2 - G(x))}{\ln 2}$$

where, $G(x)$ is a CDF of some baseline distribution. As an application they considered exponential distribution as a suitable baseline distribution and the distribution obtained using LT, viz. logarithmic transformed exponential (LTE)-distribution has increasing failure rate, and they also considered a lifetime data of 50 devices (see, Aarset (1987) [12]) and showed that it fits better as compared to some available distributions. In continuation to it, Nassar et al. (2018) [13] generalized LT using the idea of APT, a work of Madhavi and Kundu (2017) [8] and introduced alpha power logarithmic transformation (APLT), and develop alpha power logarithmic transformation Wiebull (APLTW) distribution using Weibull distribution as a baseline distribution and showed that this new distribution has monotonic, non-monotonic and bathtub shaped hazard rate function. In order to develop parsimonious model, Maurya et al. (2018) [14] introduced another form of LT and called it as new logarithmic transformation (NLT), which is defined as

$$F(x) = \frac{\ln(1 + G(x))}{\ln 2} \tag{1}$$

where, $G(x)$ is a CDF of some baseline distribution.

In this paper, we propose a generalization of NLT by introducing a new shape parameter $\alpha > 0$ is given by the following formula

$$F(x;\alpha) = \frac{\ln(1 + G^{\alpha}(x))}{\ln 2} \tag{2}$$

where, $G(x)$ is a CDF of some baseline distribution.

The PDF $f(x;\alpha)$ corresponding to CDF $F(x;\alpha)$ is given by

$$f(x;\alpha) = \frac{\alpha\, g(x) G^{\alpha-1}(x)}{\ln 2\,(1 + G^{\alpha}(x))} \tag{3}$$

where, $g(x)$ is the PDF corresponding to CDF $G(x)$ of the chosen baseline distribution.

Here, we consider exponential distribution with the parameter θ as a baseline distribution. Its CDF and PDF are respectively given by

$$G(x;\theta) = 1 - e^{-\theta x}\,;\ x > 0,\ \theta > 0 \tag{4}$$

and

$$g(x;\theta) = \theta e^{-\theta x}\,;\ x > 0,\ \theta > 0 \tag{5}$$

Using (4) and (5) in (2) and (3) the CDF and PDF of proposed distribution as follows

$$F(x;\alpha,\theta)=\frac{\ln\left[1+\left(1-\theta e^{-\theta x}\right)^{\alpha}\right]}{\ln 2};\ x>0,\ \theta>0,\ \alpha>0 \tag{6}$$

and

$$f(x;\alpha,\theta)=\frac{\alpha\theta e^{-\theta x}\left(1-\theta e^{-\theta x}\right)^{\alpha-1}}{\ln 2\left[1+\left(1-\theta e^{-\theta x}\right)^{\alpha}\right]};\ x>0,\ \theta>0,\ \alpha>0 \tag{7}$$

where, θ and α are scale and shape parameters of the new distribution having PDF (7) is called alpha power new logarithmic transformed exponential (APNLTE)-distribution.

If we put $\alpha=1$ in (2) then (2) is reduces to (1). This shows that NLT is a special case of the proposed transformation (2).

A series representation of (6) and (7) are as follows:

$$F(x;\alpha,\theta)=\frac{1}{\ln 2}\sum_{i=1}^{\infty}\sum_{j=0}^{i\alpha}(-1)^{i+j-1}\binom{i\alpha}{j}\frac{e^{-j\theta x}}{i!\,j!} \tag{8}$$

and

$$f(x;\alpha,\theta)=\frac{\alpha\theta}{\ln 2}\sum_{i=1}^{\infty}\sum_{j=0}^{i\alpha}(-1)^{i+j}\binom{\omega}{j}e^{-(j+1)\theta x} \tag{9}$$

where, $\omega=i\alpha+\alpha-1$, respectively.

The hazard rate and survival function of APNLTE distribution having PDF (7) are obtained as

$$h(x;\alpha,\ \theta)=\frac{\alpha\theta e^{-\theta x}\left(1-e^{-\theta x}\right)^{\alpha-1}}{\ln 2-[1+\left(1-\theta e^{-\theta x}\right)^{\alpha}]\ \ln[1+\left(1-\theta e^{-\theta x}\right)^{\alpha}]} \tag{10}$$

and

$$S(x;\alpha,\theta)=1-\frac{\ln[1+\left(1-e^{-\theta x}\right)^{\alpha}]}{\ln 2} \tag{11}$$

respectively.

The plots of PDF, CDF and hazard rate function of APNLTE distribution are shown in Figures 1–3 respectively. Figure 3 gives a clear evidence of flexibility of our proposed lifetime distribution as its hazard rate function takes monotone (decreasing and increasing) as well as non-monotone (in particular inverted bathtub) shape depending on the different choice of the values of parameters θ and α.

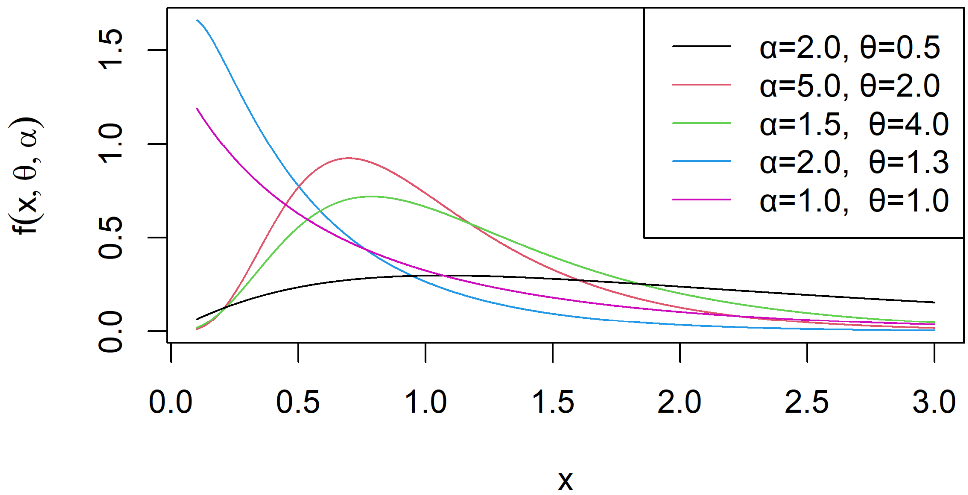

Figure 1. The plot for PDF of the APNLTE for different values of parameters α and θ.

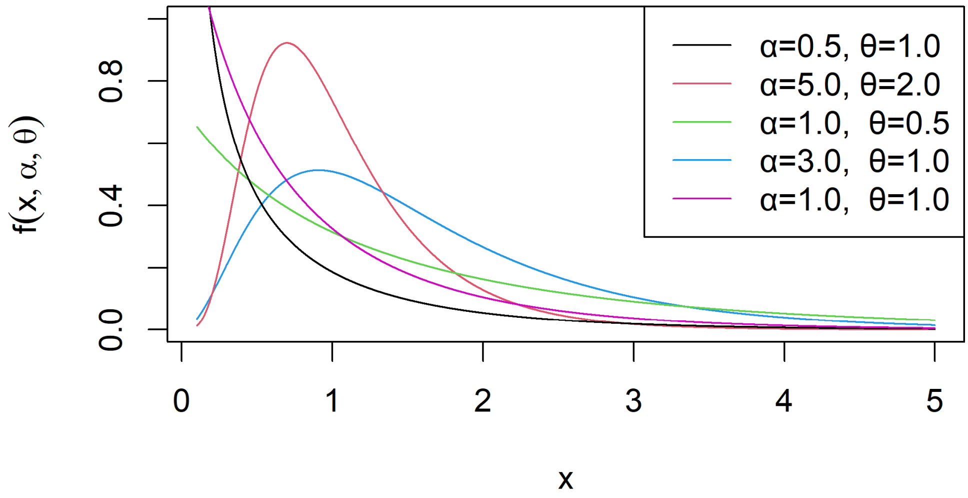

Figure 2. The plot for CDF of the APNLTE for different values of parameters α and θ.

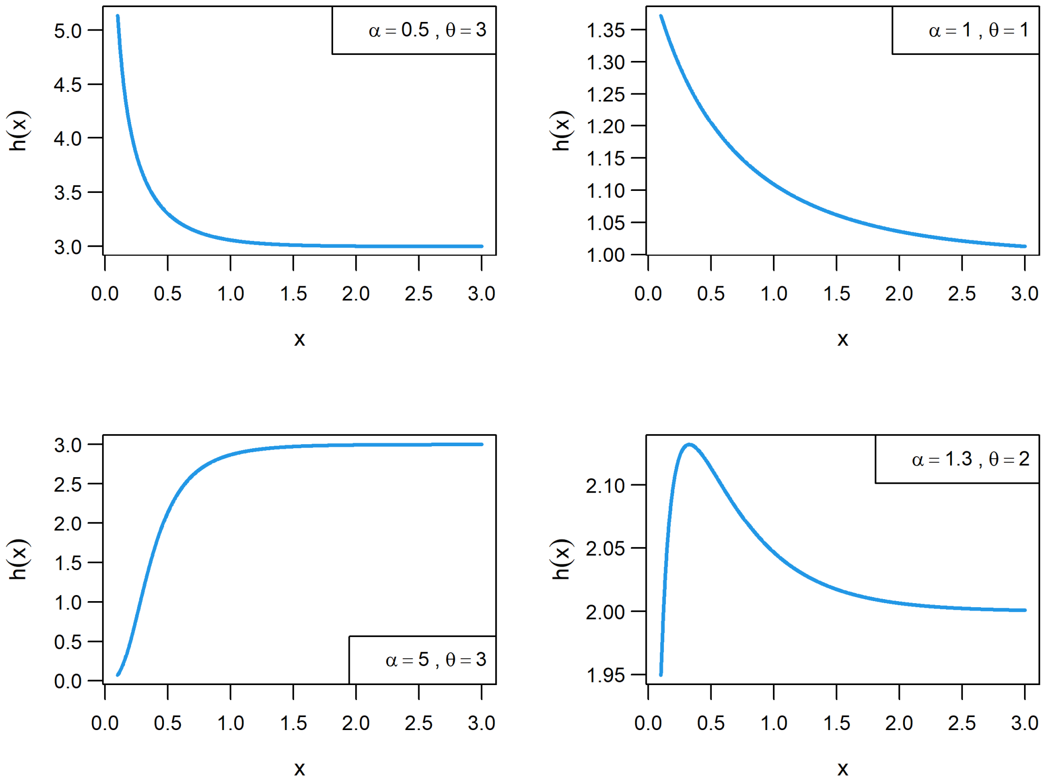

Figure 3. The plot for hazard rate of the APNLTE for different values of parameters α and θ.

2. Sample Generation

If u be a uniform variate over (0,1), then

$$F(x;\alpha)=\frac{\ln\left(1+G^{\alpha-1}(x)\right)}{\ln 2}=u$$
$$\Rightarrow x=G^{-1}\left(\left(2^{u}-1\right)^{1/\alpha}\right) \tag{12}$$

Using (4) in (12), we can generate a random sample of size n form the proposed distribution as follows

$$\Rightarrow x_i=-\frac{1}{\theta}\ln\left(1-\left(2^{u_i}-1\right)^{1/\alpha}\right),\ \forall i=1,2,\ldots,n \tag{13}$$

3. Statistical Properties

In this section, we discuss some mathematical properties of the proposed APNLTE distribution having PDF (7).

3.1. Moments

The r^{th} moment about origin of APNLTE distribution with PDF (7) is obtained as follows

$$\mu_r^{'} = \int_0^\infty x^r f(x;\theta,\alpha)dx \tag{14}$$

Using (9) in (14), we get

$$\mu_r^{'} = \frac{\alpha\theta}{\ln(2)}\sum_{i=1}^{\infty}\sum_{j=0}^{\omega}(-1)^{i+j}\binom{\omega}{j}\frac{e^{-(j+1)\theta x}\,\Gamma(r+1)}{(\theta(j+1))^{r+1}} \tag{15}$$

The first four moments about origin are given below;

$$\mu_1^{'} = \frac{\alpha\theta}{\ln(2)}\sum_{i=1}^{\infty}\sum_{j=0}^{\omega}(-1)^{i+j}\binom{\omega}{j}\frac{1}{(\theta(j+1))^{2}}$$

$$\mu_2^{'} = \frac{\alpha\theta}{\ln(2)}\sum_{i=1}^{\infty}\sum_{j=0}^{\omega}(-1)^{i+j}\binom{\omega}{j}\frac{2}{(\theta(j+1))^{3}}$$

$$\mu_3^{'} = \frac{\alpha\theta}{\ln(2)}\sum_{i=1}^{\infty}\sum_{j=0}^{\omega}(-1)^{i+j}\binom{\omega}{j}\frac{6}{(\theta(j+1))^{4}}$$

$$\mu_4^{'} = \frac{\alpha\theta}{\ln(2)}\sum_{i=1}^{\infty}\sum_{j=0}^{\omega}(-1)^{i+j}\binom{\omega}{j}\frac{24}{(\theta(j+1))^{5}}$$

and the first four central moments are given below

$$\mu_2 = \mu_2^{'} - \mu_1^{'2}$$

$$\mu_3 = \mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\mu_1^{'3}$$

$$\mu_4 = \mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'2} - 3\mu_1^{'4}$$

Using above series representation of the moments, we can obtain the measures skewness and kurtosis, viz. β_1, γ_1, β_2 and γ_2 as follows

$$\beta_1 = \frac{\mu_3^{2}}{\mu_2^{3}} \Rightarrow \gamma_1 = \sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^{2}} \Rightarrow \gamma_2 = \beta_2 - 3 = \frac{\mu_4}{\mu_2^{2}} - 3$$

3.2. Quantile Function

The p^{th} quantile function $Q(p)$ of APNLTE distribution having PDF (7) is obtained as the solution of the following expression:

$$\int_0^{Q(p)} f(x;\alpha,\theta)dx = p$$

After simplification, it reduces to

$$Q(p) = -\frac{1}{\theta}\ln\left(1-\left(2^p-1\right)^{1/\alpha}\right) \tag{16}$$

3.3. Median

If M be the median of APNLTE distribution having PDF (7), then it can be obtained by putting $p=\frac{1}{2}$ in (16), the same is obtained as follows

$$M = -\frac{1}{\theta}\ln\left(1-\left(\sqrt{2}-1\right)^{1/\alpha}\right) \tag{17}$$

From Table 1, we observed that the population mean decreases as θ increases for fixed value α, if α increases then the population mean is also increases for fixed value of θ and another case also important that if both the parameters are increases then the population mean is decreases. Same situation arises in case of median and standard deviation of the distribution. Here, we only standard deviation is calculated because the standard deviation is most appropriate in all others dispersion. From the values of γ_1, the proposed lifetime APNLTE distribution is positively skewed for all considered values of θ and α. It is noted that, the value of $\gamma_1 > 0$ as both θ and α are increases. Therefore, the proposed lifetime distribution (7) having positively skewed. The proposed distribution (7) is tending to symmetry for large value of θ and α. In same way, the proposed lifetime distribution having PDF (7) is leptokurtic and platykurtic for the considered value of θ and α. If the values of both the parameters are increases then values of γ_1 and γ_2 are decreases.

Table 1. Mean, Median, Standard Deviation, Skewness and Kurtosis of the APNLTE model for various values of α and θ.

A	*θ*	Mean	Median	S.D.	Skewness (γ_1)	Kurtosis (γ_2)
0.5	0.5	0.9869	0.3765	1.5174	3.0892	10.2465
	1	0.4935	0.1882	0.7587	3.0892	10.2465
	3	0.1646	0.0627	0.2528	3.0892	10.2468
	10	0.0496	0.0188	0.0757	3.0902	10.2522
1	0.5	1.6800	1.0696	1.8379	2.3211	4.3264
	1	0.8400	0.5348	0.9190	2.3211	4.3264
	3	0.2800	0.1783	0.3063	2.3211	4.3264
	10	0.0840	0.0535	0.0919	2.3214	4.3274
2	0.5	2.6198	2.0634	2.0877	1.8265	1.3876
	1	1.3099	1.0317	1.0438	1.8265	1.3876
	3	0.4366	0.3439	0.3479	1.8265	1.3876
	10	0.1310	0.1032	0.1044	1.8260	1.3863
5	0.5	4.1407	3.6451	2.2869	1.4527	−0.3842
	1	2.0704	1.8226	1.1434	1.4527	−0.3842
	3	0.6901	0.6075	0.3811	1.4527	−0.3842
	10	0.2066	0.1823	0.1153	1.4523	−0.3870
10	0.5	5.4148	4.9452	2.3641	1.2979	−1.0048
	1	2.7074	2.4726	1.1820	1.2979	−1.0048
	3	0.9025	0.8242	0.3940	1.2979	−1.0048
	10	0.2724	0.2473	0.1143	1.2960	−1.0060
25	0.5	7.1780	6.7255	2.4133	1.1826	−1.4234
	1	3.5890	3.3627	1.2066	1.1826	−1.4234
	25	1.1964	1.1209	0.4020	1.1826	−1.4234
	10	0.3558	0.3363	0.1311	1.1842	−1.4267

3.4. Mean Deviation

The mean deviation of APNLTE distribution about arbitrary points is given by

$$M_A(x) = \int_0^\infty |x - A| f(x;\ \alpha,\theta) dx \tag{18}$$

If μ and $\overline{\mu}$ are the mean and median of the proposed distribution (7) then the mean deviation about mean μ is

$$\begin{aligned} M_\mu(x) &= \int_0^\infty |x - \mu| f(x;\ \alpha,\theta) dx \\ &= 2\int_\mu^\infty f(x,\alpha,\theta) dx + 2\mu F(\mu) - 2\mu \end{aligned} \tag{19}$$

Using (9) and (6) in (19), then after solving, we get

$$M_\mu(x) = \frac{\alpha\theta}{\ln(2)} \sum_{i=0}^{\infty}\sum_{j=0}^{\omega} (-1)^{i+j} \binom{\omega}{j} \left(\frac{1}{(\theta(j+1))} + \mu\right) \frac{e^{-\theta(j+1)\mu}}{(\theta(j+1))} + 2\mu \frac{\ln\left(1 + \left(1 - e^{-\theta\mu}\right)^{\alpha}\right)}{\ln 2} - 2\mu \tag{20}$$

and the mean deviation about median $\overline{\mu}$ is

$$\begin{aligned} M_{\overline{\mu}}(x) &= \int_0^\infty |x - \overline{\mu}| f(x;\ \alpha,\ \theta) dx \\ &= 2\int_{\overline{\mu}}^\infty xF(x;\ \alpha,\ \theta) dx - \overline{\mu} \end{aligned} \tag{21}$$

Using (8) in (21), then after solving, we get the expression of the mean deviation about median $M_{\overline{\mu}}(x)$ is

$$M_{\overline{\mu}}(x) = \frac{1}{\ln(2)} \sum_{i=1}^{\infty}\sum_{j=0}^{i\alpha} \frac{(-1)^{i+j+1}}{i(j!)} \binom{i\alpha}{j} \left(\frac{1}{\theta j} + \overline{\mu}\right) \frac{e^{-\theta j \overline{\mu}}}{\theta j} - \overline{\mu} \tag{22}$$

3.5. Moment Generating Function

The MGF about origin of APNLTE distribution having PDF (7) is obtained as follows

$$\begin{aligned} M_X(t) &= E\left(e^{tX}\right) \\ &= \frac{\alpha\theta}{\ln(2)} \sum_{i=0}^{\infty}\sum_{j=0}^{\omega} (-1)^{i+j} \binom{\omega}{j} \frac{1}{(\theta(j+1) - t)} \end{aligned} \tag{23}$$

provided $t < \theta$.

3.6. Characteristics Function

Characteristics function of APNLTE distribution with PDF (7) is calculated as follows:

$$\begin{aligned} \phi_X(t) &= E\left(e^{itX}\right) \\ &= \int e^{itx} f(x;\ \alpha,\ \theta) dx \end{aligned} \tag{24}$$

Using (9) in (24) and we get

$$\phi_X(t) = \frac{\alpha\theta}{\ln(2)} \sum_{i=0}^{\infty}\sum_{j=0}^{\omega} (-1)^{i+j} \binom{\omega}{j} \frac{1}{(\theta(j+1) - it)} \tag{25}$$

4. Stochastic Ordering

Let X and Y be two independent random variables having APNLTE distribution with parameters (α_1, θ_1) and (α_2, θ_2) respectively, then X is said to be stochastically greater ($X \geq Y$) than Y if

- $F_X(x) \leq F_Y(x) \forall x$ in terms of distribution function

- $h_X(x) \le h_Y(x) \forall x$ in terms of hazard rate function and
- $m_X(x) \le m_Y(x) \forall x$ in terms of mean residual life.

Theorem: Let X and Y be any two independent random variables having APNLTE distribution with parameters (α_1, θ_1) and (α_2, θ_2) respectively. Then Y is stochastically greater than X (i.e., $X \le Y$) if $X \le Y$ than Y if $F_X(x) \le F_Y(x) \forall x$, provided $\theta_1 < \theta_2$ if $\alpha_1 = \alpha_2 = \alpha$.

Proof: It is given that $\theta_1 < \theta_2$, then

$$1 - e^{-\theta_1 x} < 1 - e^{-\theta_2 x}, \ \forall \ x$$

$$1 + \left(1 - e^{-\theta_1 x}\right)^{\alpha} < 1 + \left(1 - e^{-\theta_2 x}\right)^{\alpha}, \ \forall \ x$$

$$\frac{\ln\left(1 + \left(1 - e^{-\theta_1 x}\right)^{\alpha}\right)}{\ln 2} < \frac{\ln\left(1 + \left(1 - e^{-\theta_2 x}\right)^{\alpha}\right)}{\ln 2}, \ \forall \ x$$

$$F_X(x) < F_Y(x), \quad \forall \ x$$

This shows that X is stochastically greater than Y when $\theta_1 < \theta_2$.

The condition $F_X(x) < F_Y(x), \ \forall \ x$ is not satisfied for the shape parameter. Therefore the stochastic ordering only hold for scale parameters and if shape parameters are equal.

5. Entropy

Renyi (1961) [15] introduced the measure of uncertainty in the distribution of a random variable X is termed as Renyi entropy, denoted by $J_R(\gamma)$ and is defined as

$$J_R(\gamma) = \frac{1}{(1-\gamma)} \ln\left(\int_0^{\infty} f^{\gamma}(x)dx\right) \text{ where, } \gamma > 0 \text{ and } \gamma \ne 1 \tag{26}$$

Using (7) in (26), we get

$$= \frac{1}{(1-\gamma)} \ln\left(\int_0^{\infty} \left(\frac{\theta \alpha e^{-\theta x}\left(1 - e^{-\theta x}\right)^{\alpha - 1}}{\ln 2\left(1 + \left(1 - e^{-\theta x}\right)^{\alpha}\right)}\right)^{\gamma} dx\right)$$

$$= \frac{1}{(1-\gamma)}\left(\frac{\alpha\theta}{\ln 2}\right)^{\gamma} \ln\left[\sum_{k=0}^{\infty}(-1)^{\gamma}\binom{\gamma}{k}\int_0^{\infty} e^{-\gamma\theta x}\left(1 - e^{-\theta x}\right)^{k\alpha + \gamma(\alpha - 1)}dx\right]$$

$$= \frac{1}{(1-\gamma)}\left(\frac{\alpha\theta}{\ln 2}\right)^{\gamma} \ln\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{\gamma + l}\binom{\gamma}{k}\binom{w}{l}\frac{1}{\theta\gamma l}\right]$$

where, $w = k\alpha + \gamma(\alpha - 1)$

$$J_R(\gamma) = \frac{1}{(1-\gamma)}\left(\frac{\alpha\theta}{\ln 2}\right)^{\gamma} \ln\left\{\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{\gamma + l}\binom{\gamma}{k}\binom{w}{l}\frac{1}{\gamma l}\right) - \ln\theta\right\} \tag{27}$$

6. Estimation of Unknown Parameters of the Proposed Lifetime Distribution

In this section, we compute maximum likelihood estimators of the parameters α and θ of APNLTE distribution having PDF (7) and their approximate confidence interval estimates for the complete sample $\underline{X} = (x_1, x_2, \ldots x_n)$ of size n from it.

The likelihood function $L(\alpha,\theta\,|\,\underline{X})$ is obtained as follows

$$L(\alpha,\theta\,|\,\underline{X})=\prod_{i=1}^{n} f(x_i;\alpha,\theta)$$
$$=\prod_{i=1}^{n}\frac{\theta\alpha e^{-\theta x_i}\left(1-e^{-\theta x_i}\right)^{\alpha-1}}{\ln 2\left(1+\left(1-e^{-\theta x_i}\right)^{\alpha}\right)} \tag{28}$$

taking logarithm on both sides of (28), we get

$$\ln L(\alpha,\theta\,|\,\underline{X})=n\ \ln\alpha+n\ \ln\theta-\theta\sum_{i=0}^{\infty}x_i+(\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-\theta x_i}\right)-n\ln(\ln 2)$$
$$-\sum_{i=1}^{n}\ln\left(1+(1-e^{-\theta x_i})^{\alpha}\right) \tag{29}$$

On differentiating (18) partially with respect to α and θ and equating them to zero, we get the log-likelihood equation as follows

$$\frac{\partial}{\partial\theta}\ln L(\alpha,\theta\,|\,\underline{X})=0$$
$$\Rightarrow\frac{n}{\theta}-\sum_{i=1}^{n}x_i+(\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{-\theta x_i}}{\left(1-e^{-\theta x_i}\right)}-\sum_{i=1}^{n}\frac{\alpha\left(1-e^{-\theta x_i}\right)^{\alpha-1}x_i e^{-\theta x_i}}{1+\left(1-e^{-\theta x_i}\right)^{\alpha}}=0 \tag{30}$$

and

$$\frac{\partial}{\partial\alpha}\ln L(\alpha,\theta\,|\,\underline{X})=0$$
$$\Rightarrow\frac{n}{\alpha}+(\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-\theta x_i}\right)-\sum_{i=1}^{n}\frac{\alpha\theta e^{-\theta x_i}\left(1-e^{-\theta x_i}\right)^{\alpha-1}}{1+\left(1-e^{-\theta x_i}\right)^{\alpha}}=0 \tag{31}$$

The above Equations (30) and (31) cannot be solved analytically for α and θ. Therefore, some numerical iteration techniques will impose for their numerical solutions; particularly here we use Newton-Raphson method.

7. Approximated Confidence Intervals

As MLE of α and θ can't be obtained analytically, so that exact confidence intervals cannot be obtained. Thus, we derived approximate confidence intervals for α and θ by using Fisher's information matrix, which is given by

$$I(\alpha,\theta)=\begin{bmatrix}\frac{\partial^2}{\partial\theta^2}\ln L(\alpha,\theta\,|\,\underline{X}) & \frac{\partial^2}{\partial\theta\partial\alpha}\ln L(\alpha,\theta\,|\,\underline{X})\\ \frac{\partial^2}{\partial\alpha\partial\theta}\ln L(\alpha,\theta\,|\,\underline{X}) & \frac{\partial^2}{\partial\alpha^2}\ln L(\alpha,\theta\,|\,\underline{X})\end{bmatrix}$$

and the inverse matrix of estimated Fisher information matrix is obtained as follows

$$\left[I\left(\hat{\alpha},\hat{\theta}\right)\right]^{-1}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$$

where, $a_{11}=\mathrm{var}(\hat{\theta}),\ a_{22}=\mathrm{var}(\hat{\alpha})\,and\ a_{12}=a_{21}=\mathrm{cov}(\hat{\theta},\hat{\alpha})$. Thus two sided $100(1-\alpha)\%$ confidence intervals for θ and α are $\hat{\theta}\pm Z_{\alpha/2}\sqrt{a_{11}}$ $\hat{\alpha}\pm Z_{\alpha/2}\sqrt{a_{22}}$ respectively. Where $Z_{\alpha/2}$ denote the upper $\frac{\alpha}{2}\%$ points of standard normal distribution.

8. Application in Real Scenario

In order to check wellness in the sense of having fitting, applicability and superiority of the proposed model APNLTE distribution in real scenario. We have considered two different real data sets in medical field. The data set I is remission times of 128 bladder cancer patients (see, Lee and Wang (2003) [16]). The data of remission times of 128 bladder cancer patients is considered by several authors such as Khan et al. (2014) [17], Kumar et al. (2015 and 2018) [7,9,18], Khan et al. (2014) [17] introduced transmuted Inverse Weibull distribution (TIWD) and showed that TIWD out performs some well-known distributions such as TIWD, transmuted Inverse Rayleigh distribution (TIRD), transmuted Inverse Exponential distribution (TIED) and transmuted Weibull distribution (TWD) for the data of remissions times. The criterions used were AIC, BIC, K-S test statistic and mean squared error. Kumar et al. (2015) [7,9] developed $\mathrm{DUS_E}(\theta)$-distribution and $\mathrm{SS_E}(\theta)$-distibution respectively and showed that they outperform some well-known distributions such as TIWD, TIRD, TIED and TWD for the same bladder cancer patients data. Kumar et al. (2018) [18] proposed $\mathrm{MG_L}(\theta)$-distribution and showed that it fits bladder cancer patient's data. Next, data set II is survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, originally observed and reported by Bjerkedal (1960) [19]. Recently, Ibrahim, M., & Yousof, H. M. (2020) [20] and Elbiely, M. M., & Yousof, H. M. (2018) [21] used this data to check the suitability and applicability of the proposed Poisson Burr X generalized Lomax (PBXGL) and Weibull Generalized Lx (WGLx) lifetime distributions on the basis of AIC and BIC respectively.

Here we consider L-R test criterion for testing $H_0 : \alpha = 1$ vs. $H_1 : \alpha \neq 1$ to the consider data sets, in order to check whether insertion of shape parameter α to the baseline distribution improves its applicability to the real scenario or not. The L-R test statistic for testing H_0 against H_1 for data set I & data set II are defined as follows

$$\zeta_1 = -2(l_1 - l_0) \tag{32}$$

and

$$\zeta_2 = -2(l_1^{'} - l_0^{'}) \tag{33}$$

where, l_0 & l_1 and $l_0^{'}$ and $l_1^{'}$ are the estimated log-likelihood under H_0 against H_1 for data set I and II respectively. For large *n*, ζ is asymptotically distributed as χ^2_r-distribution. Where, *r* is the number of unknown parameters under H_0. If

$$\zeta_{cal} > \chi^2_{(r)}(\gamma) \tag{34}$$

then, we reject H_0 at $100\gamma\%$ level of significance, where $\chi^2_{(r)}(\gamma)$ denotes the upper $100\gamma\%$ quantile of $\chi^2_{(r)}$ distribution.

Now, for the considered data sets I and II, we get $\zeta_{1_{cal}}$ = 7.046 and $\zeta_{2_{cal}}$ = 41.5322 which are greater than $\chi^2_{(1)}(0.05)$ = 3.841. Therefore, the null hypothesis is rejected at 5% level of significance, and we may conclude that our proposed distribution fits better as compared to the baseline distribution for the considered data sets at 5% level of significance.

We also checked the superiority of the proposed model over some well-known and useful models having decreasing, increasing, non- monotonic and constant hazard rates, which are as follows;

1. Exponential Distribution with probability density distribution given by

$$f(x;\theta) = \theta e^{-\theta x}; \quad x > 0, \theta > 0$$

and it has constant hazard rate.

2. Exponentiated Exponential Distribution having probability density function given by

$$f(x;\theta,\alpha) = \alpha\theta e^{-\theta x}\left(1 - e^{-\theta x}\right)^{\alpha-1}; \; x > 0, \; \theta > 0, \; \alpha > 0$$

and it has an decreasing or increasing hazard function if $\alpha < 1$ or $\alpha > 1$ respectively and for $\alpha = 1$, the hazard function is constant.

3. Transmuted Exponential Distribution having probability density function is given by

$$f(x;\theta)=\left(1+\lambda-2\lambda\left(1-e^{-\theta x}\right)\right)\theta e^{-\theta x};\quad x>0,\ \theta>0$$

and it has increasing and decreasing hazard rate depending on the values of parameters.

4. DUS Exponential $DUS_E(\theta)$ -Distribution proposed by Kumar, D. (2015) [7] and having PDF is given by

$$f(x;\theta)=\frac{\theta}{e-1}\,e^{-\theta x}\,e^{1-e^{\theta x}};\ x>0,\ \theta>0$$

it has increasing hazard rate for all values of parameter θ.

5. Logarithmic transformed Exponential (LTE) Distribution proposed by Maurya, et al. (2016) [11] having PDF given by

$$f(x;\theta)=\frac{\theta e^{-\theta x}}{\ln 2\left(1+e^{-\theta x}\right)};\ x>0,\ \theta>0$$

it has increasing hazard rate for all values of parameter θ.

6. New Logarithmic transformed Exponential (NLTE) Distribution proposed by Maurya, et al. (2018) [14] having probability density function given by

$$f(x;\theta)=\frac{\theta e^{-\theta x}}{\ln 2\left(2-e^{-\theta x}\right)};\ x>0,\ \theta>0$$

it has decreasing hazard rate for all values of parameter θ.

The criterions used are Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC) and Kolmogorov-Smirnov (KS) test statistics D_n which are defined as follows

$$AIC=-2\ln\ L\left(\hat{\alpha},\hat{\theta}\mid \underline{X}\right)+2k$$

$$BIC=-2\ln\ L\left(\hat{\alpha},\hat{\theta}\mid \underline{X}\right)+k\ln\ n$$

and K-S distance $D_n=\sup|F(x)-F_n(x)|$

Where, $F_n(x)$ and k are the empirical CDF and the number of unknown parameters in the considered model. The values of AIC, BIC and D_n for the proposed model and 6 other models are discussed above for the data set I and data set II are shown in Tables 2 and 3 respectively. From Tables 2 and 3, it is clear that our proposed distribution is best fitted as compared to the other 6 models in the sense of having smallest AIC, BIC and D_n values.

The TTT plot of the considered real data sets are shown in Figure 4 and the plots of estimated CDF and PDF of APNLTE distribution for the considered data are shown in Figures 5 and 6 respectively, we have also shown PP-plots of the proposed model and considered competitive models in Figure 7 and 8 respectively.

The dispersion matrix V_1 & V_2 of the MLEs of the parameters θ and α under the APNLTE distribution for the considered data sets I and II are obtained as follows

$$V_1=\begin{bmatrix}0.00016 & 0.00140\\ 0.00140 & 0.02420\end{bmatrix}$$

and

$$V_2=\begin{bmatrix}0.0140961 & 0.0636660\\ 0.0636660 & 0.4338214\end{bmatrix}$$

Consequently, the asymptotic confidence intervals for the parameters θ and α of the proposed distribution at 95% level of confidence are (0.08459, 0.13486) & (1.05817, 1.66798) and (0.7189401, 1.18435) & (2.245266, 4.827178) for data sets I and II respectively.

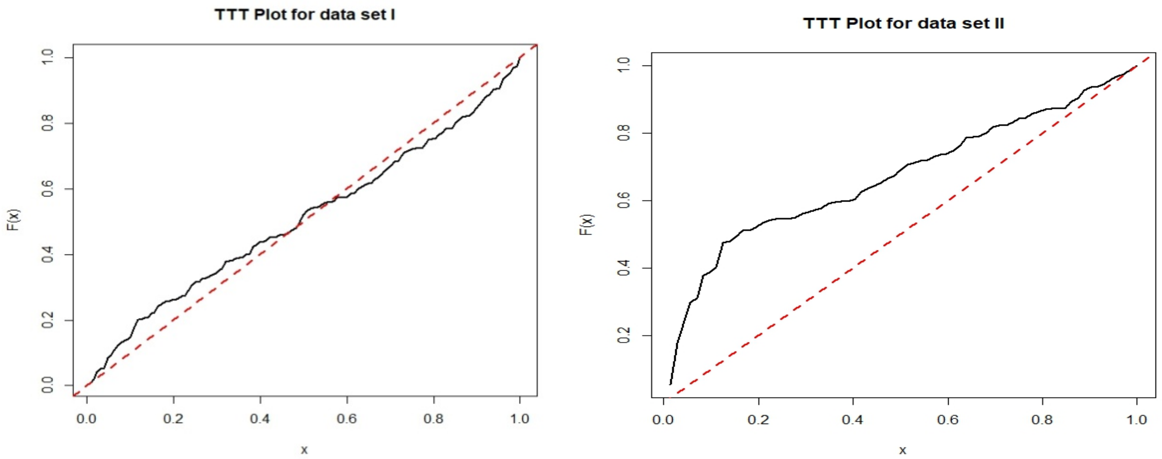

Figure 4. TTT plot for considered real data set I & II respectively.

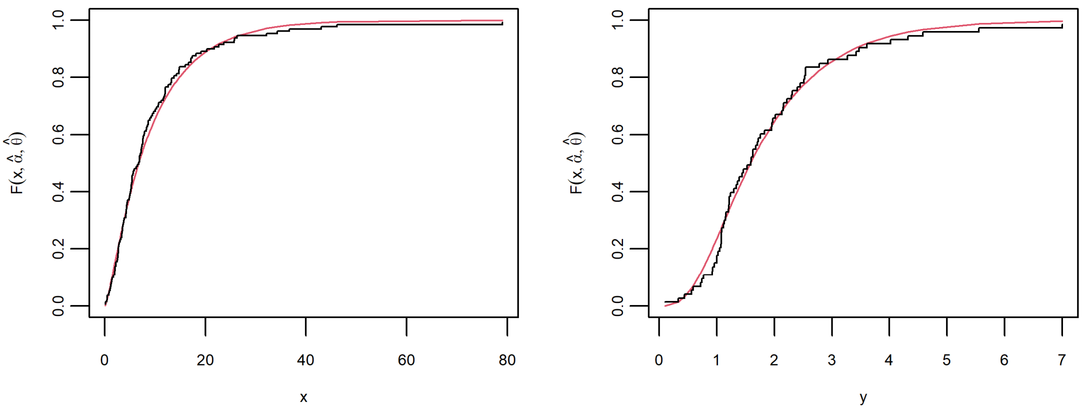

Figure 5. Fitted CDF of APNLTE distribution for real data set I and II respectively.

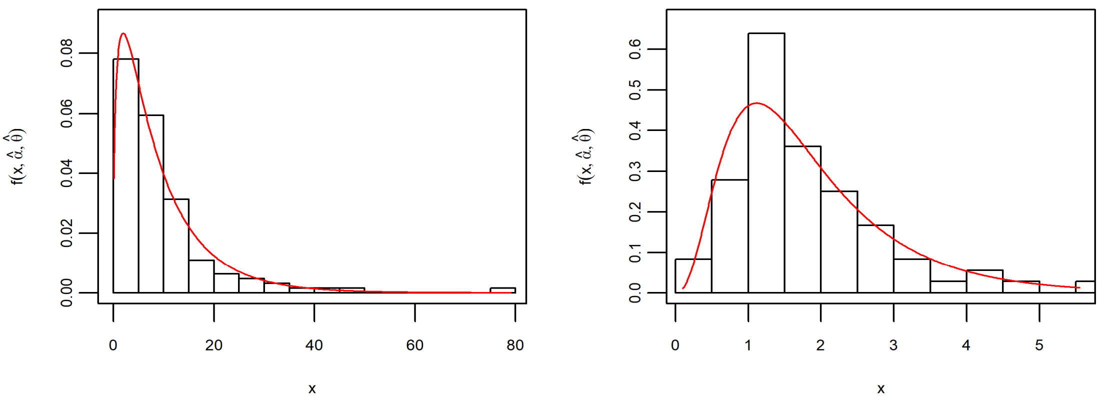

Figure 6. Fitted PDF of APNLTE distribution for real data set I and II respectively.

Table 2. MLE of the parameters, log-likelihood, AIC, BIC, KS distance, of the fitted models for the data set I.

Models	$\hat{\theta}$	$\hat{\alpha}$	ln $L(\hat{\alpha},\hat{\theta})$	AIC	BIC	KS
$Exp(\theta)$	0.1059	-	415.4052	832.8104	835.8624	0.0754
$Generalized\ Exp(\theta,\alpha)$	0.1204	1.2227	415.0903	832.1806	837.8847	0.8669
$DUS\ \ Exp(\theta)$	0.1342	-	416.0220	834.0440	836.8960	0.0813
$LT\ \ Exp(\theta)$	0.1059	-	415.8921	833.7842	836.6363	0.0686
$APNLT\ Exp(\theta,\alpha)$	0.1097	1.3631	413.0790	830.1589	835.8329	0.0582
$NLT\ \ Exp(\theta)$	0.0872	-	416.6022	835.2043	838.0564	0.0998
Transmuted $Exp(\theta,\alpha)$	0.0601	-	414.6408	833.2815	838.9856	0.0854

Table 3. MLE of the parameters, log-likelihood, AIC, BIC, KS distance, of the fitted models for data set II.

Models	$\hat{\theta}$	$\hat{\alpha}$	$\ln L(\hat{\alpha},\hat{\theta})$	AIC	BIC	KS
$Exp(\theta)$	0.5054	-	116.3150	234.6300	236.9066	0.2667
Generalized $Exp(\theta,\alpha)$	1.0370	3.3037	99.7196	203.4392	207.9925	0.6857
$DUS\ Exp(\theta)$	0.7080	-	110.8362	223.6724	225.9490	0.2323
$LT\ Exp(\theta)$	0.6656	-	113.1395	228.2789	230.5556	0.2519
$APNLT\ Exp(\theta,\alpha)$	0.9516	3.5362	99.5906	203.1812	207.7345	0.0750
$NLT\ Exp(\theta)$	0.4297	-	120.3567	242.7133	244.9900	0.2844
Transmuted $Exp(\theta,\alpha)$	0.8403	-	101.3567	207.7351	212.2884	0.1322

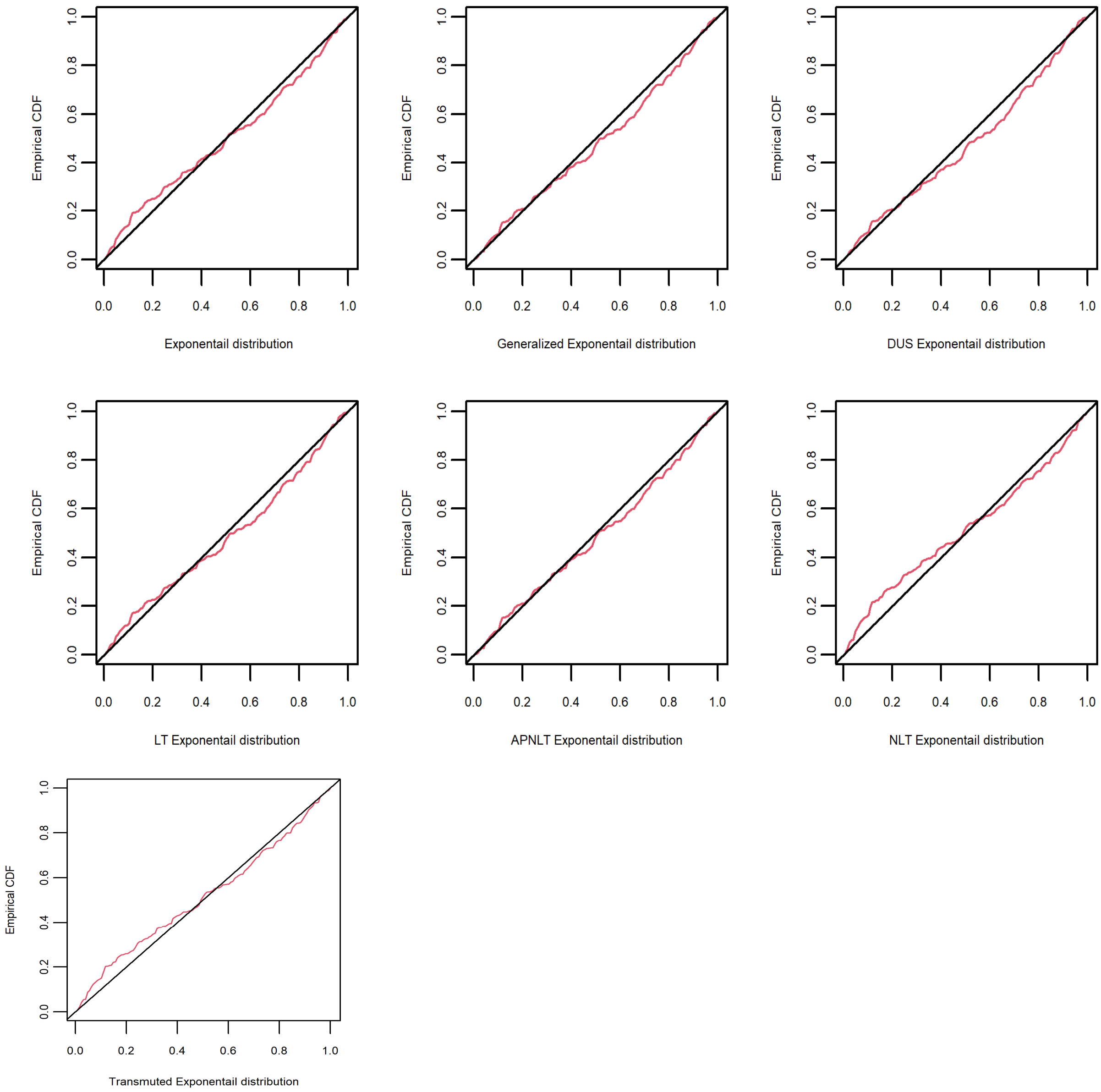

Figure 7. P-P plots of the considered distribution for the real data set I.

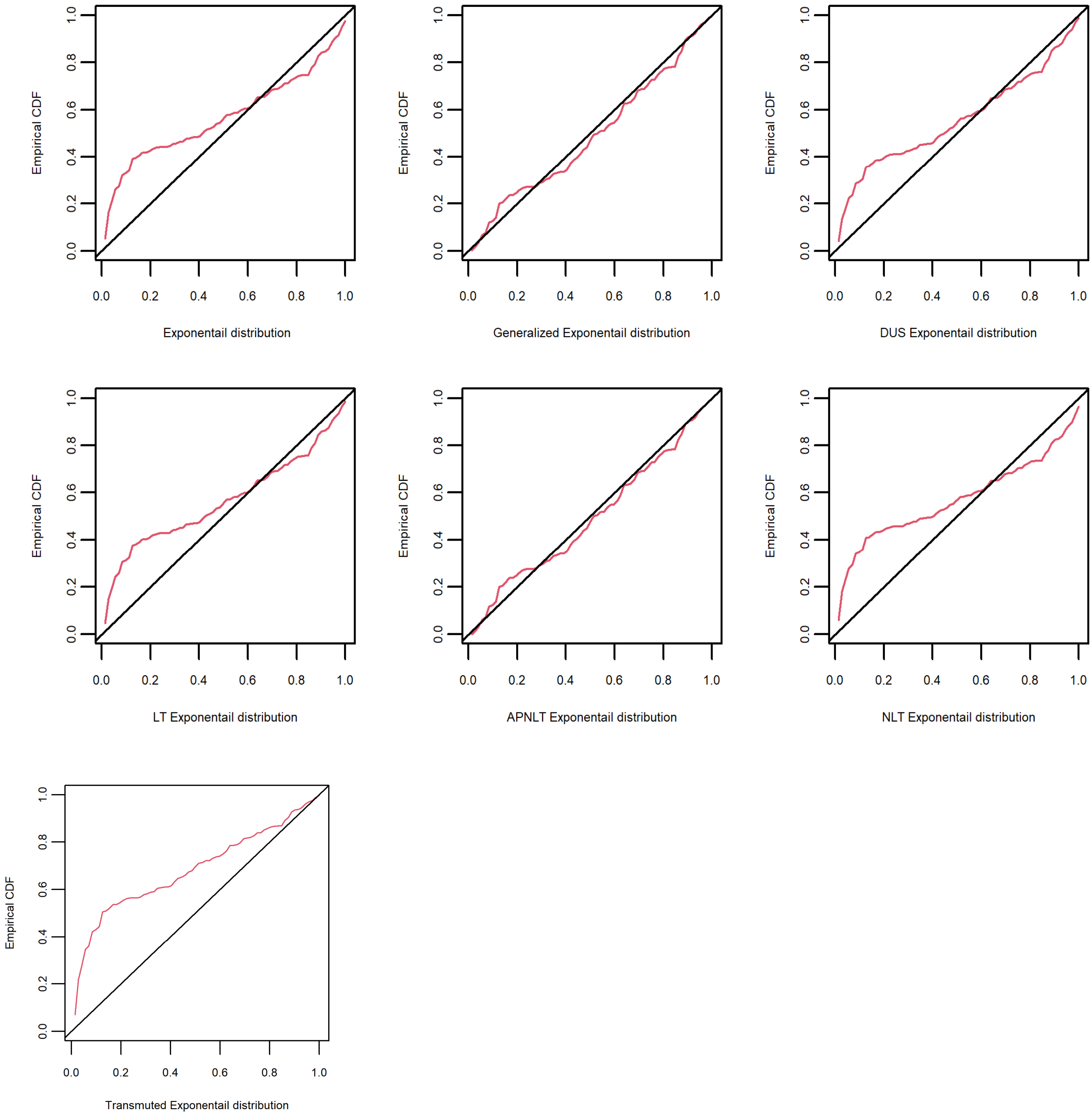

Figure 8. P-P plots of the considered distribution for the real data set II.

9. Conclusions

In the present article, a new generalization technique is developed, in order to get flexible lifetime distributions in the sense of having different shapes of hazard rate function. It is capable to generalize any available lifetime distribution called baseline distribution. This new technique is termed as APNLT. To demonstrate its real problem application, we have considered baseline distribution as Exp(θ)-distribution and the resulting distribution is termed as APNLTE($\alpha;\theta$) distribution. It has mmonotone as well as non-monotone shapes of hazard rate function, for certain choice of parameters θ and α. It has leptokurtic & platykurtic and positively skewed. Maximum likelihood estimation technique is used to estimate the parameters θ and α. The two different real data sets, a data of remission times of 128 bladder cancer patients and another data of survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, originally observed and reported by Bjerkedal (1960) [19] are considered and we obtained that our proposed distribution is an appropriate model as compared to $Exp(\theta)$, $E.\,Exp(\theta,\alpha)$, $DUS\ Exp(\theta)$, $LT\ Exp(\theta)$, $NLT\ Exp(\theta)$ and transmuted $Exp(\theta,\alpha)$ distributions under log-likelihood, AIC, BIC and KS- distance criterions of fitting and consequently we recommend our proposed lifetime distribution for its further use in medical and other field.

Author Contributions

Pa.K.: Conceptualization, Methodology, Investigation; Pr.K.: Conceptualization, Methodology, Visualization, Writing-Original draft preparation; S.P.: Conceptualization, Methodology, Visualization, Investigation, Software; D.K.: Supervision, Validation, Writing—Reviewing and Editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare no conflicts of interest.

Use of AI and AI-Assisted Technologies

The authors declare that they did not use AI tools for data analysis, image generation, or text drafting in this study.

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