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A Novel Correction for the Multivariate Ljung-Box Test

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Abstract: We propose a new analytical improvement to the multivariate Ljung-Box test that addresses the inherent pronounced deviations of the original test from nominal type I error rates under almost all scenarios. Prior attempts to mitigate this issue have been directed at modification of the test statistics or correction of the test distribution to achieve precise results in finite samples. In previous studies, focused on designing corrections to the univariate Ljung-Box, a method that specifically adjusts the test rejection region has been the most successful of attaining the best type I error rates. We adopt the same approach for the more complex, multidimensional time series scenarios. We use large sample simulation data (multivariate normal) for a range of sample sizes (n = 40 to 300), lags (m = 2 to $n-1$), and multivariate time series dimension (k = 2 to 10) to obtain an empirical estimation of the correct rejection regions for the particular combination of these variables. Furthermore, we use regression modeling with interactions and covariate power combinations to parametrically extend these precise rejection regions to all combinations of sample sizes, lags, and number of time series. Thus, we design and implement the correction based on regression models fit to empirically estimate rejection thresholds. Our results are validated to independent validation simulations and show that we attain almost perfect type I error rates with mean absolute deviation of 0.0011 across all scenarios compare to 0.0276 for the classical Ljung-Box test, which represents a 96% improvement. These findings will enhance the goodness-of-fit diagnostics for multivariate time series.

Keywords: multivariate time series; diagnostic test; type I error; portmanteau test

1. Introduction

The usual objective after selection of a time series model is forecasting future values of the variable of interest and corresponding confidence intervals for certain number of time steps ahead. Correct predications and confidence intervals are predicated on the appropriateness of the model that can be ascertained through goodness-of-fit statistics. The classical statistics were the Box-Pierce test [1] and the Ljung-Box test [2]. The latter was an improved weighted version of the former. They were defined as:

$$Q_{BP} = n\sum_{i=1}^{m} \hat{r}_i{}^2 \tag{1}$$

$$Q_{LB} = n(n+2)\sum_{i=1}^{m} \frac{\hat{r}_i^2}{n-i} = n\sum_{i=1}^{m} \frac{n+2}{n-i}\hat{r}_i^2 \tag{2}$$

where n, m, and $\hat{r}_i^2$ are the sample size, number of lags being tested and the sample autocorrelation function of the residuals at lag i, respectively.

Furthermore, the general linear process multivariate time series representation [3] is given by:

$$Z_t = \sum_{i=0}^{\infty} \psi_i a_{t-i} \tag{3}$$

where $\psi_0 = I_k$ is a $k \times k$ identity matrix, ψ_i and a_i, $i = 1,2,\ldots$ are constant $k \times k$ matrices and independent and identically distributed random vectors with mean zero and a positive-definite covariance matrix Σ_a respectively. The coefficient matrices ψ_i must satisfy $\sum_{i=1}^{\infty} \|\psi_i\| < \infty$ where $\|\psi_i\| = \sqrt{tr(\psi_i \psi_i^T)}$ is a Frobenius norm of the matrix ψ_i. Let Z_{it} be the *i*-th component of the multivariate time series Z_t,

$$Z_t = \begin{pmatrix} Z_{1t} \\ Z_{2t} \\ \vdots \\ Z_{kt} \end{pmatrix} \tag{4}$$

To measure the linear dynamic of a multivariate stationary time series $\{Z_t\}_{t=1}^{n}$, we define the lag l cross-covariance matrix (CCM) as:

$$\Gamma_l = Cov(Z_t, Z_{t-l}) = E[Z_t Z_{t-i}{}^T] = \begin{bmatrix} E(Z_{1t}Z_{1,t-l}) & E(Z_{1t}Z_{2,t-l}) & \cdots & E(Z_{1t}Z_{k,t-l}) \\ E(Z_{2t}Z_{1,t-l}) & E(Z_{2t}Z_{2,t-l}) & \cdots & E(Z_{2t}Z_{k,t-l}) \\ \vdots & \vdots & \ddots & \vdots \\ E(Z_{kt}Z_{1,t-l}) & E(Z_{kt}Z_{2,t-l}) & \cdots & E(Z_{kt}Z_{k,t-l}) \end{bmatrix} \tag{5}$$

For stationary time series the CCM is function of lag l only. Furthermore, given the sample $\{Z_t\}_{t=1}^{n}$, we can estimate all CCM matrices defined in (5). The sample variances are:

$$\hat{\Gamma}_0 = \frac{1}{n-1} \sum_{t=1}^{n} Z_t Z_t^T \, , \hat{\Gamma}_l = \frac{1}{n-1} \sum_{t=l+1}^{n} Z_t Z_{t-l}^T \, , l = 1,2,\ldots,n-1 \tag{6}$$

The original multivariate Ljung-Box test statistic is a tool used extensively in multivariate time series analysis to check for the presence of autocorrelation at multiple lag lengths in the residuals (error terms) of a model. Essentially, it tests whether there are significant correlations between lagged values of the multivariate time series or residuals, indicating that the model may not have fully captured the data's underlying structure. Therefore, traditional methods for assessing the goodness of fit essentially serve as comprehensive tests for detecting the absence of correlation within the estimated residuals. Consequently, numerous statistical methods have been developed specifically to evaluate whether there is no correlation present among the residuals of a time series model. The Portmanteau test of univariate time series was extended to the multivariate case by several authors [4–7]. Particularly, the multivariate Ljung-Box test statistic is defined as:

$$Q_k(m) = n^2 \sum_{l=1}^{m} \frac{1}{n-l} tr(\hat{\Gamma}_l^T \, \hat{\Gamma}_0^{-1} \hat{\Gamma}_l \hat{\Gamma}_0^{-1}) \tag{7}$$

where *tr*(*A*) is the trace of matrix *A*, *T* represents the transpose of a matrix, *m* is the lag, and *n* is the sample size.

Ref. [8] also defined the $Q_k(m)$ test statistic (7) using the residual autocorrelation matrix, $\hat{R}_l = \hat{L}^T \hat{\Gamma}_l \hat{L}$, where $\hat{\Gamma}_l = \frac{1}{n} \sum_{t=l+1}^{n} \hat{a}_t \, \hat{a}_{t-l}, \hat{\Gamma}_{-l} = \hat{\Gamma}_l, l \geq 0$ and $\hat{L}$ is the lower triangular Cholesky composition of $\hat{\Gamma}_0^{-1}$:

$$Q_m = n^2 \sum_{l=1}^{m} \hat{r}_l^T (\hat{R}_0^{-1} \otimes \hat{R}_0^{-1}) \hat{r}_l \tag{8}$$

where $\hat{r}_l$ is the row vector of length k^2 formed by stacking the rows of $\hat{R}_l$ and $\otimes$ is the Kronecker product of two matrices.

However, the tests described above showcase deviations from the expected type I error rates, which impact its efficacy. Several authors were attempting to adjust the test statistic itself. For instance, ref. [7] proposed an improved multivariate portmanteau test, where they extended the approach that ref. [9] used in univariate portmanteau test to multivariate scenarios by taking the log of the determinant of residual autocorrelation matrix:

$$\hat{D}_m = -n \, log \left| \hat{\Re}_m \right| \tag{9}$$

where

$$\hat{\Re}_m = \begin{pmatrix} \mathrm{I}_k & \hat{R}_1 & \cdots & \hat{R}_m \\ \hat{R}_1^T & \mathrm{I}_k & \cdots & \hat{R}_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{R}_m^T & \hat{R}_{m-1}^T & \cdots & \mathrm{I}_k \end{pmatrix} \tag{10}$$

and $\mathrm{I}_k = \hat{R}_0$. In their work, the authors have discovered that their test statistic outperform the Q_m test statistics and proven the improved multivariate portmanteau test was more powerful in most cases besides for the case of *n* = 50 and *m* = 30.

Although the improved multivariate Portmanteau test had provided better performances in most cases and Lin and McLeod also refined the Peña–Rodríguez portmanteau framework [10], it still did not improve the original multivariate Ljung-Box test in all scenarios. Later work has examined alternative portmanteau and residual-correlation diagnostics including an improved lack-of-fit measures for time series models [11], and foundational modeling and distributional results for multiple time series and sample autocorrelations [12,13]. Standard time series references also establish the broader modeling and diagnostic context for Box–Jenkins procedures and Ljung–Box-type statistics [14–17]. However, previous results have shown that correction of the rejection region outperforms analytical refinements [18,19]. Therefore, we extended the latter approach that was proven to be the most precise in previous studies in univariate cases to the multidimensional time series scenarios. This approach focuses on adjusting the rejection region, rather than altering the multivariate Ljung-Box test statistic itself to achieve the enhancement of the goodness-of-fit analytics for the multivariate time series studies.

2. Methods

The inspiration for this research project came from a study which designed and implemented optimal corrections to the rejection regions for the adjusted univariate Adjusted Box-Pierce test for all sample sizes and lags to achieve a better type I error rates. That study, ref. [10] proposed a new approach that inflates or deflates the rejection region test instead of redesigning the test statistic itself. According to the authors, at the time their study was conducted, the Adjusted Box-Pierce was the superior test in measuring the goodness-of-fit methods for time series at that time. In the same vein, we extended the same approach from the univariate to the multivariate time series case in this study.

2.1. Simulation Design

This section provides an overview of what we are planning through a large simulation study, which is an integral part of the project. There are three main input parameters, *k*, *n*, and *m* to define the simulation, where k ($k = 2,3,5,10$) denotes the multivariate time series dimension, n ($n = 40,50,\ldots,300$) corresponds to the sample size which totaled 27 elements, and m ($m = 2,3,\ldots,n-1$) represents the lag time variable or the delay representing a fixed amount of passing time. The total number of values that the variables k, n and m attain were 4, 27, and $4 \times \sum_{n=4}^{30}(10n-2)$ respectively.

2.2. The Simulation Design Adopted the Following Four Steps

1. For a given number of series *k* and a chosen sample size *n*, we simulated 10^6 independent multivariate white noise samples $s_{n_1}, s_{n_2}, \ldots, s_{n_{10^6}} \sim N(0_{k,1}, I_{k,k})$. Simulating the data this way allows us to mimic the real-world scenario of what residual will be obtained after fitting an appropriate multivariate time series model.
2. For given k and n, and from m running from *2* to $n-1$, we apply the multivariate Ljung-Box test to each unique combination of k, n, and m, and extract the corresponding *p*-values at an alpha level of 0.05. This results in producing 10^6 *p*-values.
3. For each combination of k, n and m we count the number of *p*-values less than 0.05 and divide such a number by the total number of *p*-values generated in the previous step.
4. Stack the results into a data frame of dimension $\sum_{n=4}^{30}(10n-2)$ by 4 (the number of possible combinations of k, n and m over the range of values that we defined above) where the rows denote the values of k, n, m, and the type I error rates calculated in step three.

2.3. Linear Model

As done in the univariate case, the primary concept of the study is not to construct a new statistic but to bring correction to the rejection region for the multivariate time series case. This idea was first proposed by [19] with the univariate Adjusted Box-Pierce test. For this, a linear model-based correction approach was conceived to either inflate or deflate the rejection region of the Multivariate Ljung-Box test in order to improve the Type I error rates for all dimensions ($k = 2,3,5,10$), sample sizes ($n = 40,50,\ldots,300$), and lags ($m = 2,3,\ldots,n-1$). Due to the dynamics of the simulated series, a data-driven model building was leading to the deployment of six different linear regressions models depending on the number of series involved in the study, their lengths, and the number of lags. The battery of models was trained on data as simulated and described. In the process of capturing the dynamics of the data, the *k* (2, 3, and 5) series were divided into subsets of different intervals [0, 50], [51, 70], [71, 90], [91, 120], [121, 200], and [201, 300]. To facilitate the training, testing and evaluation of the linear model in the case of the 10-dimensional time series, we split the data into intervals of [0, 50], [51, 70], [71, 90], [91, 110], [111, 200], and [201, 300]. Sample sizes selected are shown in Table 1. The variation in patterns of type I error rates (shown in Figures 1–4) across different sample sizes required the adoption of individual models to achieve the targeted level of accuracy. These linear models are intricate, incorporating various degrees of *n* and *m*, as well as their two-way interactions. The general model that we implemented in the study is shown in formula (11).

$$E(Z_t - 0.05) = \alpha_1 n^s + \alpha_2 m^p + \alpha_3(n^s m^p) + \alpha_4(n^{2s} m^{2p}) + \alpha_5 n^{2s} + \alpha_6(n^{3s} m^{2p}) + \alpha_7(n^{3s} m^{3p}) + \alpha_8 m^{4p} + \alpha_9 m^{5p} \tag{11}$$

Additionally, a comprehensive grid search was conducted within the general formula (11) to identify the optimal values for the power transformation parameters s and p. We performed a large-sample simulation study to empirically estimate the rejection regions, so the Law of Large Numbers guarantees consistency of the estimation. We performed a dense grid search to identify the powers and the coefficients of the bivariate polynomials that best fit the contours. This computational method has been successfully done before by [18,19]. The summary statistics of each best-fit model are presented in the tables within the results chapter of this research paper, along with the specific values for s and p.

Table 1. Summary of the subsets for different dimensions.

Dimensions of the Time Series (k)	Subsets of Sample Sizes (n)					
2	0–50	51–70	71–90	91–120	121–200	201–300
3	0–50	51–70	71–90	91–120	121–200	201–300
5	0–50	51–70	71–90	91–120	121–200	201–300
10	0–50	51–70	71–90	91–110	111–200	201–300

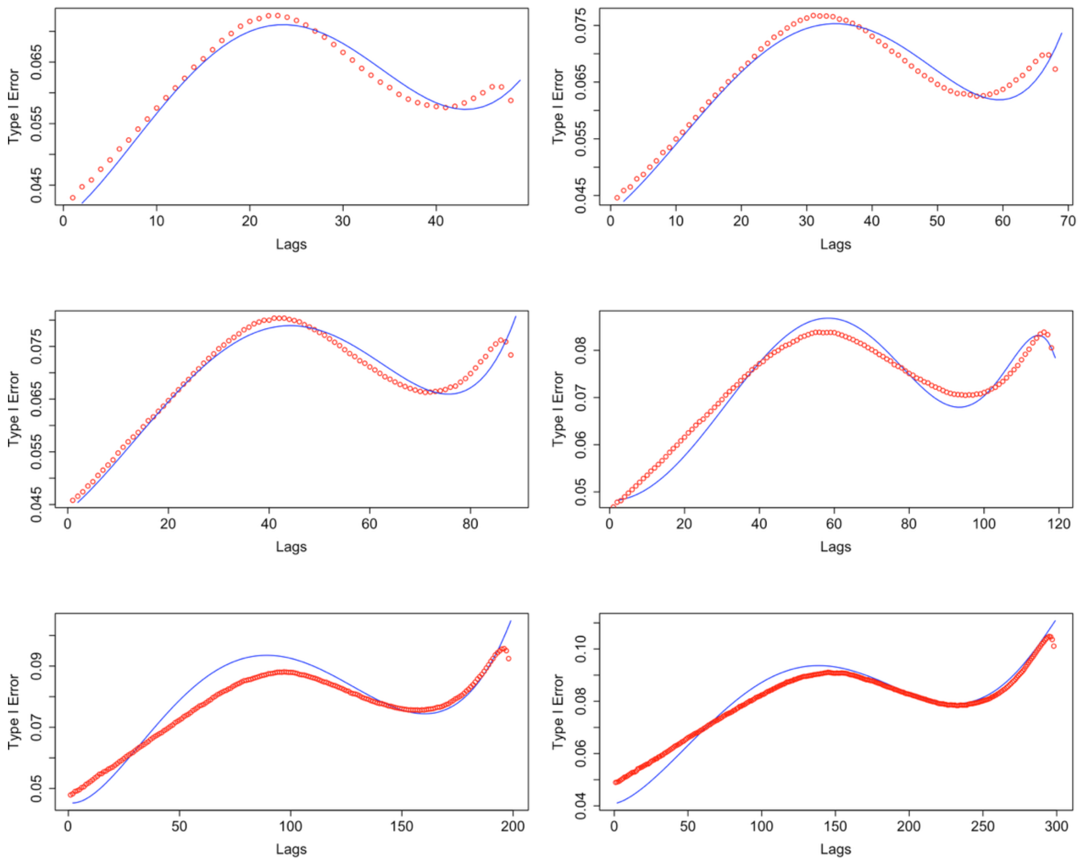

Figure 1. Patterns of type I error rates for $k = 2$.

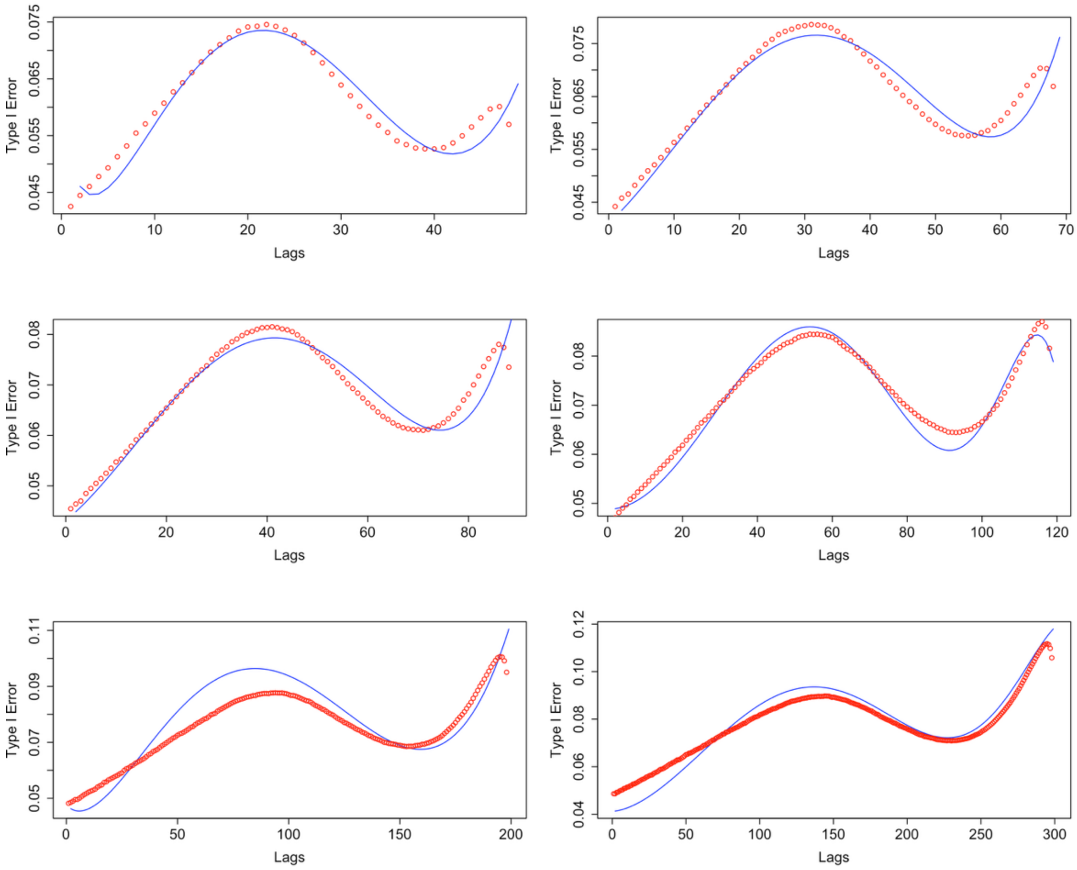

Figure 2. Patterns of type I error rates for *k* = 3.

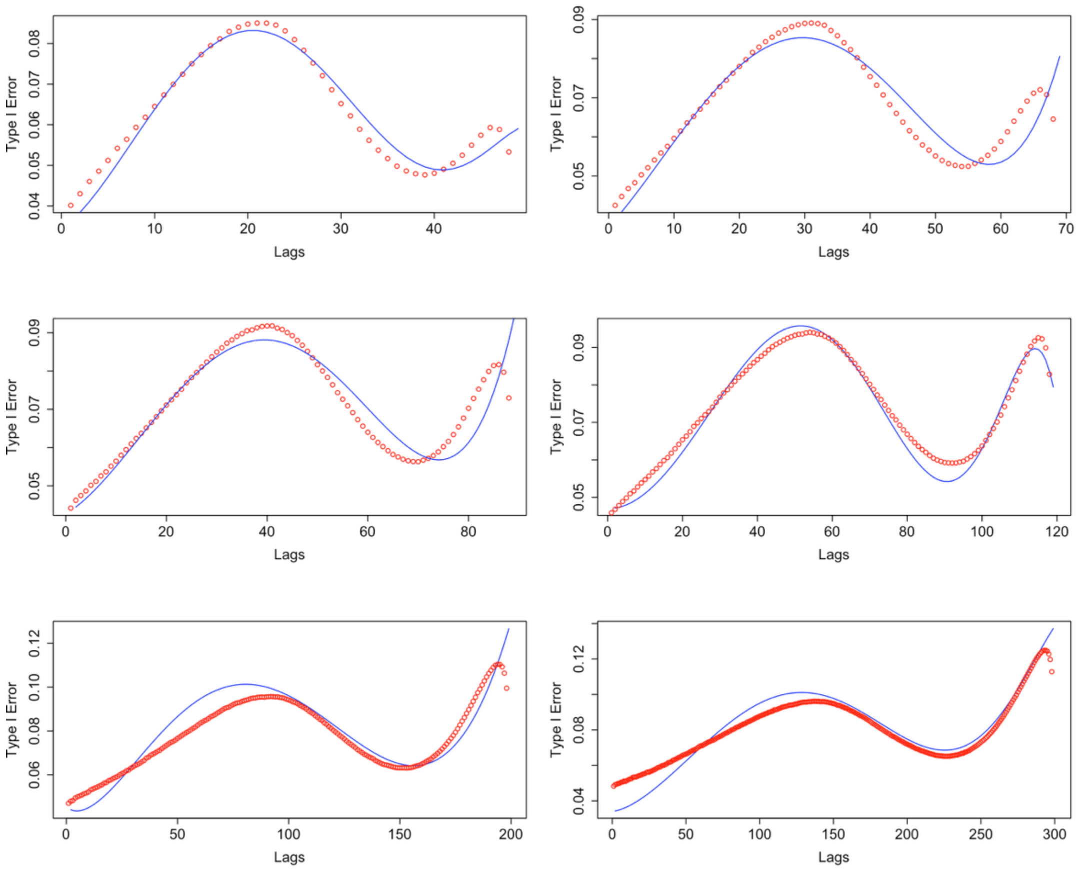

Figure 3. Patterns of type I error rates for *k* = 5.

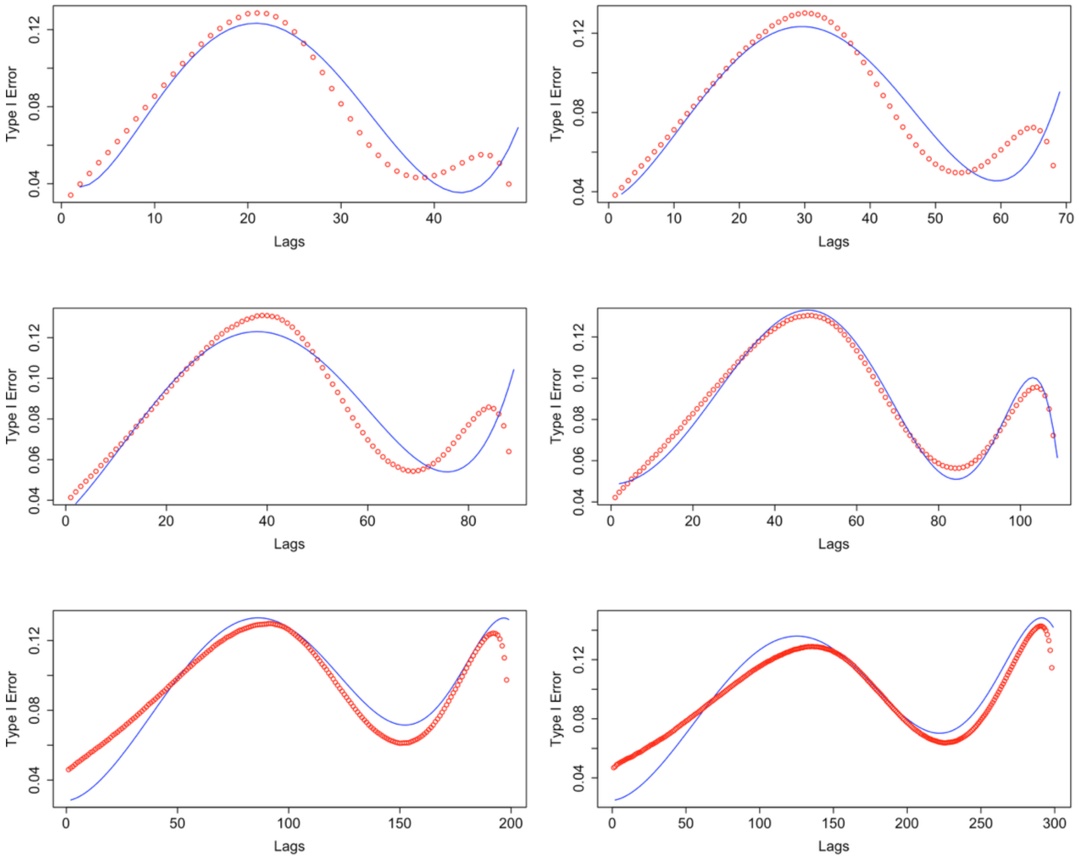

Figure 4. Patterns of type I error rates for k = 10.

For Figures 1–4, the blue lines represent the general model described in formula (11), where the red lines are the data obtained from the simulation. The red lines represent the empirically derived contours of the critical regions at α level 0.05 for all scenarios. Their seemingly chaotic fluctuations are results of the deviations of the test statistics from its asymptotic chi-square distribution in small samples. The blue lines represent the best multivariate regression fits to this empirical data. The specific and optimal s and p values for each individual model will be presented in the next chapter of this research paper.

In essence, this computational study aims to parametrically define rejections regions for all cases through dense grid search of data points which are extended to all data points in the study range so the results are obtained through extensive simulation and are chosen based on error minimization of validation data.

3. Results

While conducting our study, it was evident that there are apparent differences in the patterns of the type I error rates. Accordingly, individual models were built according to subsets of the sample sizes shown in Table 1. For each model, the optimal s and p values in the general models were obtained by doing a grid search, where the values ranged from 0.1 to 2 with an increment of 0.1 (400 unique combinations). The best values are presented in Table 2 and Appendix Table A1, A8 and A15 along with the validation test set sample sizes for each value of k, and the summary statistics for different sample size of k = 2 are presented in Tables 3–8. (*** is highly significant, ** is very significant, * is significant, and no * is not significant).

Table 2. Model summary for k = 2.

Sample Sizes (n)	s	p
45	1.8	1.3
65	2.0	1.1
85	2.0	1.1
100	2.0	2.0
160	0.3	0.9
280	1.5	1.5

Table 3. Summary statistics for selected variables, $n \leq 50$ and $k = 2$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	-3.71×10^{-5}	2.77×10^{-6}	−13.37	$<2 \times 10^{-16}$ ***
m^p	2.51×10^{-3}	1.71×10^{-4}	14.672	$<2 \times 10^{-16}$ ***
$n^s m^p$	-1.37×10^{-6}	1.85×10^{-7}	−7.403	1.42×10^{-10} ***
$n^{2s} m^{2p}$	-8.71×10^{-11}	3.71×10^{-12}	−23.46	$<2 \times 10^{-16}$ ***
n^{2s}	2.46×10^{-8}	2.70×10^{-9}	9.134	6.61×10^{-14} ***
$n^{3s} m^{2p}$	7.45×10^{-14}	3.70×10^{-15}	20.116	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-8.68×10^{-17}	7.17×10^{-18}	−12.103	$<2 \times 10^{-16}$ ***
m^{4p}	1.07×10^{-9}	6.59×10^{-11}	16.256	$<2 \times 10^{-16}$ ***
m^{5p}	-2.21×10^{-12}	2.50×10^{-13}	−8.865	2.18×10^{-13} ***

Table 4. Summary statistics for selected variables, $50 < n \leq 70$ and $k = 2$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	-8.99×10^{-6}	9.09×10^{-7}	−9.888	$<2 \times 10^{-16}$ ***
m^p	4.35×10^{-3}	3.27×10^{-4}	13.317	$<2 \times 10^{-16}$ ***
$n^s m^p$	-7.12×10^{-7}	8.05×10^{-8}	−8.843	1.14×10^{-14} ***
$n^{2s} m^{2p}$	-6.84×10^{-12}	4.22×10^{-13}	−16.213	$<2 \times 10^{-16}$ ***
n^{2s}	1.50×10^{-9}	2.05×10^{-10}	7.343	3.02×10^{-11} ***
$n^{3s} m^{2p}$	1.49×10^{-15}	9.72×10^{-17}	15.364	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-3.37×10^{-18}	2.67×10^{-19}	−12.629	$<2 \times 10^{-16}$ ***
m^{4p}	1.28×10^{-9}	3.13×10^{-10}	4.09	7.95×10^{-5} ***
m^{5p}	9.21×10^{-12}	1.98×10^{-12}	4.64	9.14×10^{-6} ***

Table 5. Summary statistics for selected variables, $70 < n \leq 90$ and $k = 2$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	-4.63×10^{-6}	5.52×10^{-7}	−8.392	2.62×10^{-14} ***
m^p	3.38×10^{-3}	2.57×10^{-4}	13.163	$<2 \times 10^{-16}$ ***
$n^s m^p$	-3.28×10^{-7}	3.70×10^{-8}	−8.862	1.61×10^{-15} ***
$n^{2s} m^{2p}$	-1.29×10^{-12}	7.76×10^{-14}	−16.655	$<2 \times 10^{-16}$ ***
n^{2s}	4.77×10^{-10}	7.44×10^{-11}	6.419	1.55×10^{-9} ***
$n^{3s} m^{2p}$	1.70×10^{-16}	1.09×10^{-17}	15.596	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-3.12×10^{-19}	2.26×10^{-20}	−13.791	$<2 \times 10^{-16}$ ***
m^{4p}	3.52×10^{-10}	9.16×10^{-11}	3.837	0.00018 ***
m^{5p}	2.69×10^{-12}	4.25×10^{-13}	6.327	2.49×10^{-9} ***

Table 6. Summary statistics for selected variables, $90 < n \leq 120$ and $k = 2$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	2.82×10^{-6}	1.71×10^{-7}	16.505	$<2.00 \times 10^{-16}$ ***
m^p	-1.33×10^{-5}	1.09×10^{-6}	−12.253	$<2.00 \times 10^{-16}$ ***
$n^s m^p$	2.71×10^{-9}	1.10×10^{-10}	24.544	$<2.00 \times 10^{-16}$ ***
$n^{2s} m^{2p}$	-3.98×10^{-17}	1.63×10^{-18}	−24.479	$<2.00 \times 10^{-16}$ ***
n^{2s}	-2.05×10^{-10}	1.35×10^{-11}	−15.204	$<2.00 \times 10^{-16}$ ***
$n^{3s} m^{2p}$	1.13×10^{-21}	1.28×10^{-22}	8.863	$<2.00 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	3.89×10^{-26}	4.95×10^{-27}	7.866	1.50×10^{-13} ***
m^{4p}	2.76×10^{-17}	1.13×10^{-18}	24.559	$<2.00 \times 10^{-16}$ ***
m^{5p}	-1.40×10^{-21}	6.32×10^{-23}	−22.157	$<2.00 \times 10^{-16}$ ***

Table 7. Summary statistics for selected variables, $120 < n \leq 200$ and $k = 2$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	4.57×10^{-3}	1.31×10^{-3}	3.492	0.000495 ***
m^p	7.11×10^{-3}	4.15×10^{-4}	17.117	$<2 \times 10^{-16}$ ***
$n^s m^p$	-1.45×10^{-3}	9.54×10^{-5}	−15.155	$<2 \times 10^{-16}$ ***
$n^{2s} m^{2p}$	-9.22×10^{-6}	3.51×10^{-7}	−26.259	$<2 \times 10^{-16}$ ***
n^{2s}	-1.16×10^{-3}	2.85×10^{-4}	−4.058	5.25×10^{-5} ***
$n^{3s} m^{2p}$	2.38×10^{-6}	8.84×10^{-8}	26.971	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-9.59×10^{-9}	3.03×10^{-10}	−31.645	$<2 \times 10^{-16}$ ***
m^{4p}	5.94×10^{-9}	2.67×10^{-10}	22.285	$<2 \times 10^{-16}$ ***
m^{5p}	-2.70×10^{-12}	9.80×10^{-13}	−2.756	0.005928 **

Table 8. Summary statistics for selected variables, $n > 200$ and $k = 2$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	1.00×10^{-5}	2.00×10^{-7}	50.307	$<2 \times 10^{-16}$ ***
m^p	-2.22×10^{-5}	9.53×10^{-7}	−23.321	$<2 \times 10^{-16}$ ***
$n^s m^p$	1.82×10^{-8}	3.28×10^{-10}	55.568	$<2 \times 10^{-16}$ ***
$n^{2s} m^{2p}$	-2.60×10^{-15}	3.58×10^{-17}	−72.599	$<2 \times 10^{-16}$ ***
n^{2s}	-2.27×10^{-9}	4.67×10^{-11}	−48.582	$<2 \times 10^{-16}$ ***
$n^{3s} m^{2p}$	3.09×10^{-19}	7.29×10^{-21}	42.368	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-3.09×10^{-24}	9.32×10^{-25}	−3.314	0.000931
m^{4p}	1.34×10^{-15}	2.77×10^{-17}	48.205	$<2 \times 10^{-16}$ ***
m^{5p}	-1.31×10^{-19}	4.68×10^{-21}	−27.957	$<2 \times 10^{-16}$ ***

Each model summary across varying sample sizes and dimensions indicates exceptionally low *p*-values for the covariates. Symbols such as *, **, *** indicate the level of statistical significance each variable contributes to the respective models. Summary statistics for $k = 3$, 5, and 10 are presented in the Appendix A.

In summary, we attain type I error rates with mean absolute deviation of 0.0011 across all scenarios compare to 0.0276 for the classical Ljung-Box test, which represents a 96% improvement. These findings will enhance the goodness-of-fit diagnostics for multivariate time series.

Data Example

An application of the correction to the multivariate Ljung-Box test was done using 10 Fortune 500 stock data obtained from Yahoo finance, including Apple, Tesla, Walmart, etc. For a sample size of 50, numerous discrepancies were discovered at different lags. Here, we provide an example of two such discrepancies in different directions using the classical multivariate Ljung-Box test on these 10 sets of stock data. Thus, the model presented in Table A15 was implemented for this dataset.

An instance of discrepancy type I (classical Ljung-Box fails to reject and the proposed method rejects the null) is found at lag 35 (red dot in Figure 5), where the *p*-values were obtained by the standard multivariate Ljung-Box test. By the proposed correction to the original test, the *p*-value of 0.0614 is compared to critical values of 0.05 and 0.0646. By using another timeframe from the same stocks, an instance of a discrepancy type II (classical Ljung-Box rejects and the proposed method fails to reject the null) is also found at lag 39 (blue dot in Figure 6) by the standard multivariate Ljung-Box test. With our proposed correction to the original test, the *p*-value of 0.0472 is compared to critical values of 0.05 and 0.0439. Both instances are shown in Figures 5 and 6 respectively.

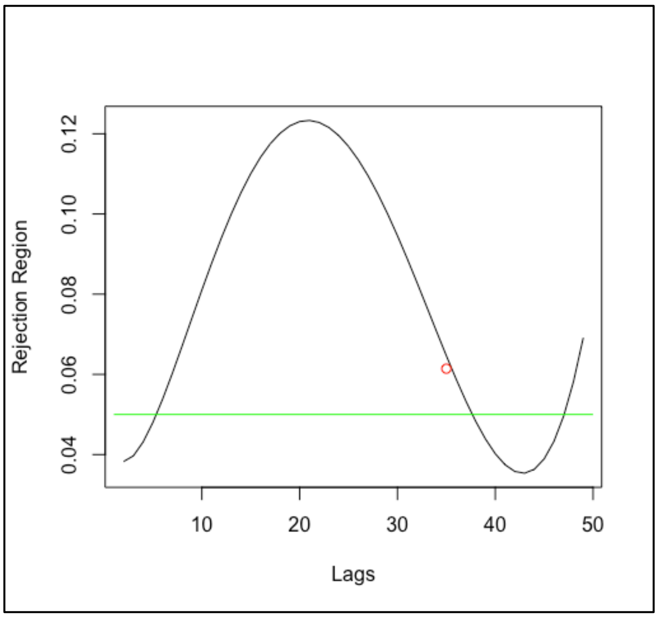

Figure 5. Discrepancy type I between corrected and naïve rejection region for sample size 50.

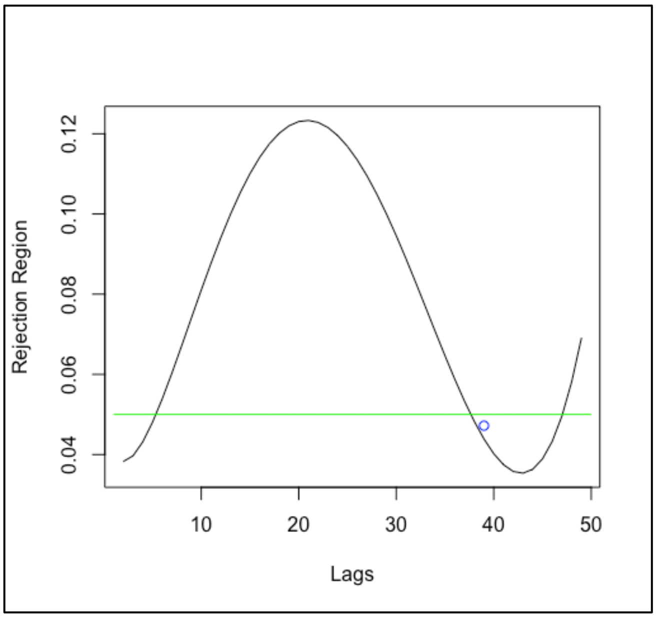

Figure 6. Discrepancy type II between corrected and naïve rejection region for sample size 50.

4. Discussion

In the study, a new approach was introduced to improve the original multivariate Ljung-Box test by adopting the work of [19] in univariate time series analysis. The method combines large-scale simulations with scenario-specific regression modeling, incorporating complex interaction terms to achieve an excellent fit, while maintaining nominal type I error rates across all dimensions, sample sizes, and lags used in the test statistic. This improvement effectively addresses almost all scenarios in multivariate time series analysis. The regression models that were constructed in the study depend on the number of time series k, sample sizes n, and the lags m. The exponents s and p of different variables in the models are treated as hyperparameters to guide the learning process, with an extensive grid search performed to find their optimal values. The simulation study showed that this test outperforms all existing competing goodness-of-fit methods for sample sizes up to 300, which represents common data analysis setting and are most-affected by deviations of the test statistics from asymptotic distribution. Further studies for large sample size could elucidate the behavior of the test statistics in such scenarios.

The key benefit of the proposed correction to the original multivariate Ljung-Box test is its significant improvement in type I error rates across all dimensions, sample sizes, and lag values. This correction technique has direct implications for more precise decision-making by investors and financial institutions and can also be easily extended to larger sample sizes. We have verified the applicability of the methods to cases with VARMA models with several parameters. The results were very similar to presented one and were thus omitted from the manuscript. The standard multivariate time series theory is based on multivariate normal distribution of the errors, which is what we adopted in our study as well. Stock market and other financial data have shown deviations from normality and therefore additional exploration on the behavior of the multivariate Ljung-Box test statistics in such cases are needed. Furthermore, bootstrapping-based multivariate time series inferences are definitely a viable alternative. However, the computational cost of such methods is by several orders of magnitude larger than that of the proposed method.

Author Contributions

M.H. conducted the simulation study, interpreted the results, drafted, and revised the manuscript. S.D. participated in the design of the models and reviewed the final draft. C.R. designed and conceived the study, reviewed, and revised the manuscript. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

The model developed in this study is written into a R function and it's publicly available on GitHub at https://github.com/CADS-CR-lab/Multivariate-Time-Series-GOF.git.

Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for this paper.

Appendix A

Table A1. Model summary for $k = 3$.

Sample Sizes (n)	s	p
45	1.3	0.7
65	2.0	1.1
85	2.0	1.1
100	0.4	2.0
160	0.3	0.9
280	2.0	1.7

Table A2. Summary statistics for selected variables, $n \leq 50$ and $k = 3$.

Variables	Estimate	Std. Error	t-Value	p-Value
n^s	-9.19×10^{-4}	1.22×10^{-4}	−7.549	7.47×10^{-11} ***
m^p	6.39×10^{-2}	7.78×10^{-3}	8.218	3.86×10^{-12} ***
$n^s m^p$	-5.04×10^{-4}	5.86×10^{-5}	−8.586	7.53×10^{-13} ***
$n^{2s} m^{2p}$	-1.63×10^{-7}	5.08×10^{-8}	−3.204	0.00197 **
n^{2s}	6.19×10^{-6}	8.39×10^{-7}	7.378	1.59×10^{-10} ***
$n^{3s} m^{2p}$	2.13×10^{-9}	3.54×10^{-10}	6.008	5.85×10^{-8} ***
$n^{3s} m^{3p}$	-6.15×10^{-11}	6.14×10^{-12}	−10.018	1.34×10^{-15} ***
m^{4p}	-1.23×10^{-5}	1.66×10^{-6}	−7.423	1.31×10^{-10} ***
m^{5p}	9.14×10^{-7}	6.99×10^{-8}	13.077	$<2 \times 10^{-16}$ ***

Table A3. Summary statistics for selected variables, $50 < n \leq 70$ and $k = 3$.

Variables	Estimate	Std. Error	t-Value	p-Value
n^s	-1.09×10^{-5}	1.36×10^{-6}	−8.059	7.34×10^{-13} ***
m^p	5.46×10^{-3}	4.88×10^{-4}	11.189	$<2 \times 10^{-16}$ ***
$n^s m^p$	-9.01×10^{-7}	1.20×10^{-7}	−7.494	1.39×10^{-11} ***
$n^{2s} m^{2p}$	-9.38×10^{-12}	6.30×10^{-13}	−14.88	$<2 \times 10^{-16}$ ***
n^{2s}	1.86×10^{-9}	3.06×10^{-10}	6.098	1.42×10^{-8} ***
$n^{3s} m^{2p}$	2.02×10^{-15}	1.45×10^{-16}	13.92	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-4.58×10^{-18}	3.99×10^{-19}	−11.478	$<2 \times 10^{-16}$ ***
m^{4p}	2.51×10^{-9}	4.67×10^{-10}	5.379	3.89×10^{-7} ***
m^{5p}	8.08×10^{-12}	2.96×10^{-12}	2.726	0.0074 **

Table A4. Summary statistics for selected variables, $70 < n \leq 90$ and $k = 3$.

Variables	Estimate	Std. Error	t-Value	p-Value
n^s	-6.41×10^{-6}	8.67×10^{-7}	−7.393	8.06×10^{-12} ***
m^p	4.57×10^{-3}	4.04×10^{-4}	11.328	$<2 \times 10^{-16}$ ***
$n^s m^p$	-4.73×10^{-7}	5.81×10^{-8}	−8.134	1.18×10^{-13} ***
$n^{2s} m^{2p}$	-1.79×10^{-12}	1.22×10^{-13}	−14.701	$<2 \times 10^{-16}$ ***
n^{2s}	6.89×10^{-10}	1.17×10^{-10}	5.891	2.26×10^{-8} ***
$n^{3s} m^{2p}$	2.38×10^{-16}	1.72×10^{-17}	13.875	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-4.49×10^{-19}	3.56×10^{-20}	−12.617	$<2 \times 10^{-16}$ ***
m^{4p}	6.63×10^{-10}	1.44×10^{-10}	4.604	8.51×10^{-6} ***
m^{5p}	3.06×10^{-12}	6.68×10^{-13}	4.58	9.43×10^{-6} ***

Table A5. Summary statistics for selected variables, $90 < n \leq 120$ and $k = 3$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	1.34×10^{-2}	1.81×10^{-3}	7.414	1.14×10^{-12} ***
m^p	-1.02×10^{-4}	1.06×10^{-5}	−9.612	$<2 \times 10^{-16}$ ***
$n^s m^p$	1.93×10^{-5}	1.68×10^{-6}	11.479	$<2 \times 10^{-16}$ ***
$n^{2s} m^{2p}$	1.32×10^{-10}	4.24×10^{-11}	3.117	0.001994 **
n^{2s}	-2.00×10^{-3}	2.77×10^{-4}	−7.231	3.65×10^{-12} ***
$n^{3s} m^{2p}$	-4.14×10^{-11}	7.42×10^{-12}	−5.577	5.26×10^{-8} ***
$n^{3s} m^{3p}$	1.20×10^{-15}	2.09×10^{-16}	5.741	2.21×10^{-8} ***
m^{4p}	1.63×10^{-17}	4.77×10^{-18}	3.422	0.000703 ***
m^{5p}	-1.28×10^{-21}	1.34×10^{-22}	−9.561	$<2 \times 10^{-16}$ ***

Table A6. Summary statistics for selected variables, $120 < n \leq 200$ and $k = 3$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	-7.18×10^{-4}	2.04×10^{-3}	−0.352	0.725
m^p	1.33×10^{-2}	6.73×10^{-4}	19.81	$<2 \times 10^{-16}$ ***
$n^s m^p$	-2.92×10^{-3}	1.55×10^{-4}	−18.831	$<2 \times 10^{-16}$ ***
$n^{2s} m^{2p}$	-1.70×10^{-5}	5.90×10^{-7}	−28.847	$<2 \times 10^{-16}$ ***
n^{2s}	5.32×10^{-5}	4.47×10^{-4}	0.119	0.905
$n^{3s} m^{2p}$	4.48×10^{-6}	1.49×10^{-7}	30.02	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-1.86×10^{-8}	5.11×10^{-10}	−36.318	$<2 \times 10^{-16}$ ***
m^{4p}	1.25×10^{-8}	4.54×10^{-10}	27.577	$<2 \times 10^{-16}$ ***
m^{5p}	-1.34×10^{-11}	1.72×10^{-12}	−7.821	1.22×10^{-14} ***

Table A7. Summary statistics for selected variables, $n > 200$ and $k = 3$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	6.81×10^{-7}	1.31×10^{-8}	51.965	$<2 \times 10^{-16}$ ***
m^p	-1.04×10^{-5}	3.29×10^{-7}	−31.602	$<2 \times 10^{-16}$ ***
$n^s m^p$	4.17×10^{-10}	7.51×10^{-12}	55.453	$<2 \times 10^{-16}$ ***
$n^{2s} m^{2p}$	-1.19×10^{-18}	1.63×10^{-20}	−72.703	$<2 \times 10^{-16}$ ***
n^{2s}	-8.64×10^{-12}	1.83×10^{-13}	−47.165	$<2 \times 10^{-16}$ ***
$n^{3s} m^{2p}$	7.96×10^{-24}	1.74×10^{-25}	45.865	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	1.59×10^{-30}	7.71×10^{-30}	0.207	0.836
m^{4p}	2.19×10^{-17}	4.08×10^{-19}	53.727	$<2 \times 10^{-16}$ ***
m^{5p}	-7.84×10^{-22}	2.28×10^{-23}	−34.387	$<2 \times 10^{-16}$ ***

Table A8. Model summary for $k = 5$.

Sample Sizes (*n*)	*s*	*p*
45	1.5	1.4
65	2.0	1.1
85	2.0	1.0
100	0.4	2.0
175	0.3	0.9
280	2.0	1.6

Table A9. Summary statistics for selected variables, $n \leq 50$ and $k = 5$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	-1.38×10^{-4}	2.19×10^{-5}	−6.309	1.64×10^{-8} ***
m^p	3.11×10^{-3}	3.20×10^{-4}	9.705	5.30×10^{-15} ***
$n^s m^p$	-4.98×10^{-6}	1.09×10^{-6}	−4.552	1.96×10^{-5} ***
$n^{2s} m^{2p}$	-8.97×10^{-10}	4.42×10^{-11}	−20.293	$<2 \times 10^{-16}$ ***
n^{2s}	2.70×10^{-7}	6.85×10^{-8}	3.94	0.000178 ***
$n^{3s} m^{2p}$	2.40×10^{-12}	1.46×10^{-13}	16.461	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-2.09×10^{-15}	2.09×10^{-16}	−10.008	1.40×10^{-15} ***
m^{4p}	7.26×10^{-10}	3.76×10^{-11}	19.321	$<2 \times 10^{-16}$ ***
m^{5p}	-1.36×10^{-12}	8.73×10^{-14}	−15.578	$<2 \times 10^{-16}$ ***

Table A10. Summary statistics for selected variables, $50 < n \leq 70$ and $k = 5$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	-1.59×10^{-5}	2.48×10^{-6}	−6.415	3.11×10^{-9} ***
m^p	8.31×10^{-3}	8.92×10^{-4}	9.317	8.85×10^{-16} ***
$n^s m^p$	-1.36×10^{-6}	2.20×10^{-7}	−6.172	9.99×10^{-9} ***
$n^{2s} m^{2p}$	-1.54×10^{-11}	1.15×10^{-12}	−13.328	$<2 \times 10^{-16}$ ***
n^{2s}	2.72×10^{-9}	5.59×10^{-10}	4.858	3.72×10^{-6} ***
$n^{3s} m^{2p}$	3.24×10^{-15}	2.66×10^{-16}	12.209	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-6.98×10^{-18}	7.30×10^{-19}	−9.567	2.28×10^{-16} ***
m^{4p}	5.18×10^{-9}	8.54×10^{-10}	6.06	1.71×10^{-8} ***
m^{5p}	3.84×10^{-12}	5.42×10^{-12}	0.708	0.48

Table A11. Summary statistics for selected variables, $70 < n \leq 90$ and $k = 5$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	-1.05×10^{-5}	1.80×10^{-6}	−5.848	2.80×10^{-8} ***
m^p	1.12×10^{-2}	1.24×10^{-3}	9.053	5.12×10^{-16} ***
$n^s m^p$	-1.29×10^{-6}	1.79×10^{-7}	−7.208	2.24×10^{-11} ***
$n^{2s} m^{2p}$	-5.86×10^{-12}	5.70×10^{-13}	−10.277	$<2 \times 10^{-16}$ ***
n^{2s}	1.19×10^{-9}	2.43×10^{-10}	4.892	2.45×10^{-6} ***
$n^{3s} m^{2p}$	8.35×10^{-16}	7.97×10^{-17}	10.478	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-2.75×10^{-18}	2.51×10^{-19}	−10.951	$<2 \times 10^{-16}$ ***
m^{4p}	1.41×10^{-9}	1.62×10^{-9}	0.866	0.388
m^{5p}	8.05×10^{-11}	1.18×10^{-11}	6.809	1.97×10^{-10} ***

Table A12. Summary statistics for selected variables, $90 < n \leq 120$ and $k = 5$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	2.26×10^{-2}	2.79×10^{-3}	8.103	1.20×10^{-14} ***
m^p	-1.79×10^{-4}	1.63×10^{-5}	−11	$<2 \times 10^{-16}$ ***
$n^s m^p$	3.26×10^{-5}	2.59×10^{-6}	12.601	$<2 \times 10^{-16}$ ***
$n^{2s} m^{2p}$	3.33×10^{-10}	6.54×10^{-11}	5.088	6.23×10^{-7} ***
n^{2s}	-3.39×10^{-3}	4.27×10^{-4}	−7.955	3.26×10^{-14} ***
$n^{3s} m^{2p}$	-8.39×10^{-11}	1.14×10^{-11}	−7.343	1.80×10^{-12} ***
$n^{3s} m^{3p}$	2.33×10^{-15}	3.22×10^{-16}	7.234	3.59×10^{-12} ***
m^{4p}	1.26×10^{-17}	7.35×10^{-18}	1.716	0.0871
m^{5p}	-1.66×10^{-21}	2.07×10^{-22}	−8.026	2.01×10^{-14} ***

Table A13. Summary statistics for selected variables, $120 < n \leq 200$ and $k = 5$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	2.93×10^{-3}	2.78×10^{-3}	1.055	0.291
m^p	1.60×10^{-2}	8.80×10^{-4}	18.22	$<2 \times 10^{-16}$ ***
$n^s m^p$	-3.47×10^{-3}	2.02×10^{-4}	−17.174	$<2 \times 10^{-16}$ ***
$n^{2s} m^{2p}$	-2.14×10^{-5}	7.44×10^{-7}	−28.727	$<2 \times 10^{-16}$ ***
n^{2s}	-7.86×10^{-4}	6.04×10^{-4}	−1.303	0.193
$n^{3s} m^{2p}$	5.54×10^{-6}	1.87×10^{-7}	29.559	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-2.28×10^{-8}	6.42×10^{-10}	−35.552	$<2 \times 10^{-16}$ ***
m^{4p}	1.62×10^{-8}	5.65×10^{-10}	28.755	$<2 \times 10^{-16}$ ***
m^{5p}	-1.99×10^{-11}	2.08×10^{-12}	−9.567	$<2 \times 10^{-16}$ ***

Table A14. Summary statistics for selected variables, $n > 200$ and $k = 5$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	8.62×10^{-7}	1.90×10^{-8}	45.304	$<2 \times 10^{-16}$ ***
m^p	-2.80×10^{-5}	8.19×10^{-7}	−34.21	$<2 \times 10^{-16}$ ***
$n^s m^p$	1.00×10^{-9}	1.89×10^{-11}	53.08	$<2 \times 10^{-16}$ ***
$n^{2s} m^{2p}$	-4.90×10^{-18}	6.92×10^{-20}	−70.838	$<2 \times 10^{-16}$ ***
n^{2s}	-1.15×10^{-11}	2.67×10^{-13}	−43.244	$<2 \times 10^{-16}$ ***
$n^{3s} m^{2p}$	3.30×10^{-23}	7.18×10^{-25}	46.017	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	-4.62×10^{-29}	5.76×10^{-29}	−0.803	0.422
m^{4p}	2.81×10^{-16}	5.41×10^{-18}	52.068	$<2 \times 10^{-16}$ ***
m^{5p}	-1.72×10^{-20}	5.39×10^{-22}	−31.999	$<2 \times 10^{-16}$ ***

Table A15. Model summary for k = 10.

Sample Sizes (n)	s	p
45	1.4	0.8
65	2.0	0.9
85	2.0	1.0
100	2.0	2.0
185	2.0	1.7
280	2.0	1.7

Table A16. Summary statistics for selected variables, $n \leq 50$ and k = 10.

Variables	Estimate	Std. Error	t-Value	p-Value
n^s	-1.72×10^{-3}	2.56×10^{-4}	−6.729	2.70×10^{-9} ***
m^p	1.36×10^{-1}	1.76×10^{-2}	7.759	2.95×10^{-11} ***
$n^s m^p$	-6.27×10^{-4}	9.00×10^{-5}	−6.967	9.58×10^{-10} ***
$n^{2s} m^{2p}$	-2.53×10^{-7}	3.96×10^{-8}	−6.39	1.16×10^{-8} ***
n^{2s}	7.19×10^{-6}	1.19×10^{-6}	6.03	5.33×10^{-8} ***
$n^{3s} m^{2p}$	1.39×10^{-9}	1.88×10^{-10}	7.426	1.28×10^{-10} ***
$n^{3s} m^{3p}$	-1.88×10^{-11}	2.28×10^{-12}	−8.242	3.48×10^{-12} ***
m^{4p}	-2.07×10^{-6}	1.34×10^{-6}	−1.539	0.128
m^{5p}	2.54×10^{-7}	3.84×10^{-8}	6.604	4.62×10^{-9} ***

Table A17. Summary statistics for selected variables, $50 < n \leq 70$ and k = 10.

Variables	Estimate	Std. Error	t-Value	p-Value
n^s	-4.52×10^{-5}	7.66×10^{-6}	−5.903	3.57×10^{-8} ***
m^p	4.37×10^{-2}	5.64×10^{-3}	7.759	3.54×10^{-12} ***
$n^s m^p$	-8.76×10^{-6}	1.39×10^{-6}	−6.289	5.71×10^{-9} ***
$n^{2s} m^{2p}$	-1.16×10^{-10}	1.59×10^{-11}	−7.279	4.19×10^{-11} ***
n^{2s}	8.62×10^{-9}	1.73×10^{-9}	4.989	2.13×10^{-6} ***
$n^{3s} m^{2p}$	2.86×10^{-14}	3.59×10^{-15}	7.969	1.18×10^{-12} ***
$n^{3s} m^{3p}$	-1.81×10^{-16}	2.14×10^{-17}	−8.447	9.44×10^{-14} ***
m^{4p}	-9.48×10^{-8}	6.18×10^{-8}	−1.534	0.128
m^{5p}	6.19×10^{-9}	9.25×10^{-10}	6.692	8.00×10^{-10} ***

Table A18. Summary statistics for selected variables, $70 < n \leq 90$ and k = 10.

Variables	Estimate	Std. Error	t-Value	p-Value
n^s	-2.03×10^{-5}	4.04×10^{-6}	−5.017	1.41×10^{-6} ***
m^p	2.13×10^{-2}	2.78×10^{-3}	7.658	1.82×10^{-12} ***
$n^s m^p$	-2.33×10^{-6}	4.00×10^{-7}	−5.807	3.43×10^{-8} ***
$n^{2s} m^{2p}$	-1.19×10^{-11}	1.28×10^{-12}	−9.348	$<2 \times 10^{-16}$ ***
n^{2s}	2.24×10^{-9}	5.44×10^{-10}	4.122	6.06×10^{-5} ***
$n^{3s} m^{2p}$	1.62×10^{-15}	1.79×10^{-16}	9.082	4.32×10^{-16} ***
$n^{3s} m^{3p}$	-4.75×10^{-18}	5.62×10^{-19}	−8.451	1.85×10^{-14} ***
m^{4p}	6.13×10^{-9}	3.63×10^{-9}	1.688	0.0934
m^{5p}	1.12×10^{-10}	2.65×10^{-11}	4.22	4.12×10^{-5} ***

Table A19. Summary statistics for selected variables, $90 < n \leq 110$ and k = 10.

Variables	Estimate	Std. Error	t-Value	p-Value
n^s	6.56×10^{-6}	1.08×10^{-6}	6.077	6.21×10^{-9} ***
m^p	-2.82×10^{-5}	1.30×10^{-5}	−2.174	0.0309 *
$n^s m^p$	8.96×10^{-9}	1.22×10^{-9}	7.351	5.15×10^{-12} ***
$n^{2s} m^{2p}$	-2.89×10^{-16}	2.25×10^{-17}	−12.839	$<2 \times 10^{-16}$ ***
n^{2s}	-5.52×10^{-10}	9.61×10^{-11}	−5.746	3.43×10^{-8} ***
$n^{3s} m^{2p}$	1.20×10^{-20}	2.19×10^{-21}	5.469	1.36×10^{-7} ***
$n^{3s} m^{3p}$	3.05×10^{-25}	6.31×10^{-26}	4.827	2.77×10^{-6} ***
m^{4p}	2.12×10^{-16}	9.17×10^{-18}	23.158	$<2 \times 10^{-16}$ ***
m^{5p}	-1.31×10^{-20}	4.35×10^{-22}	−30.107	$<2 \times 10^{-16}$ ***

Table A20. Summary statistics for selected variables, $110 < n \leq 200$ and $k = 10$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	3.95×10^{-6}	1.18×10^{-7}	33.57	$<2 \times 10^{-16}$ ***
m^p	-5.70×10^{-5}	2.26×10^{-6}	−25.29	$<2 \times 10^{-16}$ ***
$n^s m^p$	4.37×10^{-9}	1.33×10^{-10}	32.84	$<2 \times 10^{-16}$ ***
$n^{2s} m^{2p}$	-5.82×10^{-17}	1.59×10^{-18}	−36.548	$<2 \times 10^{-16}$ ***
n^{2s}	-1.13×10^{-10}	3.79×10^{-12}	−29.718	$<2 \times 10^{-16}$ ***
$n^{3s} m^{2p}$	8.91×10^{-22}	3.47×10^{-23}	25.687	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	2.76×10^{-27}	2.95×10^{-27}	0.936	0.349
m^{4p}	8.60×10^{-16}	3.05×10^{-17}	28.161	$<2 \times 10^{-16}$ ***
m^{5p}	-6.55×10^{-20}	3.28×10^{-21}	−19.955	$<2 \times 10^{-16}$ ***

Table A21. Summary statistics for selected variables, $n > 200$ and $k = 10$.

Variables	Estimate	Std. Error	*t*-Value	*p*-Value
n^s	1.59×10^{-6}	3.46×10^{-8}	45.959	$<2 \times 10^{-16}$ ***
m^p	-3.61×10^{-5}	8.69×10^{-7}	−41.532	$<2 \times 10^{-16}$ ***
$n^s m^p$	1.14×10^{-9}	1.98×10^{-11}	57.66	$<2 \times 10^{-16}$ ***
$n^{2s} m^{2p}$	-2.93×10^{-18}	4.32×10^{-20}	−67.956	$<2 \times 10^{-16}$ ***
n^{2s}	-2.08×10^{-11}	4.84×10^{-13}	−42.929	$<2 \times 10^{-16}$ ***
$n^{3s} m^{2p}$	1.73×10^{-23}	4.58×10^{-25}	37.74	$<2 \times 10^{-16}$ ***
$n^{3s} m^{3p}$	1.78×10^{-28}	2.04×10^{-29}	8.763	$<2 \times 10^{-16}$ ***
m^{4p}	6.16×10^{-17}	1.08×10^{-18}	57.1	$<2 \times 10^{-16}$ ***
m^{5p}	-2.54×10^{-21}	6.02×10^{-23}	−42.184	$<2 \times 10^{-16}$ ***

References

1. Pierce, D.A. Distribution of Residual Autocorrelations in Regression Model with Autoregressive-Moving Average Errors. *J. Roy. Stat. Soc. B Methodol.* **1971**, *33*, 140–146.
2. Ljung, G.M.; Box, G.E.P. On a Measure of Lack of Fit in Time-Series Models. *Biometrika* **1978**, *65*, 297–303. https://doi.org/10.1093/biomet/65.2.297.
3. Tsay, R.S. *Multivariate Time Series Analysis: With R and Financial Applications*; Wiley Series in Probability and Statistics; John Wiley & Sons: Hoboken, NJ, USA, 2013.
4. Hosking, J.R.M. Lagrange-Multiplier Tests of Multivariate Time-Series Models. *J. Roy. Stat. Soc. B Methodol.* **1981**, *43*, 219–230.
5. Li, W.K.; Mcleod, A.I. Distribution of the Residual Autocorrelations in Multivariate ARMA Time-Series Models. *J. Roy. Stat. Soc. B Methodol.* **1981**, *43*, 231–239.
6. Li, W.K. *Diagnostic Checks in Time Series*; Chapman & Hall/CRC: Boca Raton, FL, USA, 2004.
7. Mahdi, E.; Ian McLeod, A. Improved multivariate portmanteau test. *J. Time Ser. Anal.* **2012**, *33*, 211–222. https://doi.org/10.1111/j.1467-9892.2011.00752.x.
8. Hosking, J.R.M. The Multivariate Portmanteau Statistic. *J. Am. Stat. Assoc.* **1980**, *75*, 602–608. https://doi.org/10.1080/01621459.1980.10477520.
9. Peña, D.; Rodríguez, J. The log of the determinant of the autocorrelation matrix for testing goodness of fit in time series. *J. Stat. Plan. Infer.* **2006**, *136*, 2706–2718. https://doi.org/10.1016/j.jspi.2004.10.026.
10. Lin, J.W.; McLeod, A.I. Improved Pena-Rodriguez portmanteau test. *Comput. Stat. Data An.* **2006**, *51*, 1731–1738. https://doi.org/10.1016/j.csda.2006.06.010.
11. Fisher, T.J.; Robbins, M.W. An Improved Measure for Lack of Fit in Time Series Models. *Stat. Sinica* **2018**, *28*, 1285–1305. https://doi.org/10.5705/ss.202016.0286.
12. Tiao, G.C.; Box, G.E.P. Modeling Multiple Time-Series with Applications. *J. Am. Stat. Assoc.* **1981**, *76*, 802–816. https://doi.org/10.1080/01621459.1981.10477728.
13. Kan, R.; Wang, X. On the distribution of the sample autocorrelation coefficients. *J. Econom.* **2010**, *154*, 101–121. https://doi.org/10.1016/j.jeconom.2009.06.010.
14. Box, G.E.; Jenkins, G.M.; Reinsel, G.C.; et al. *Time Series Analysis: Forecasting and Control*; Wiley Series in Probability and Statistics; John Wiley & Sons: Hoboken, NJ, USA, 2015.
15. Hassani, H.; Yeganegi, M.R. Sum of squared ACF and the Ljung-Box statistics. *Phys. A Stat. Mech. Appl.* **2019**, *520*, 81–86. https://doi.org/10.1016/j.physa.2018.12.028.
16. Hassani, H.; Yeganegi, M.R. Selecting optimal lag order in Ljung-Box test. *Phys. A Stat. Mech. Appl.* **2020**, *541*, 123700. https://doi.org/10.1016/j.physa.2019.123700.

17. Gallagher, C.M.; Fisher, T.J. On Weighted Portmanteau Tests for Time-Series Goodness-of-Fit. *J. Time Ser. Anal.* **2015**, *36*, 67–83. https://doi.org/10.1111/jtsa.12093.
18. Ehwerhemuepha, L.; Sok, H.; Rakovski, C. A more powerful unconditional exact test of homogeneity for 2 × c contingency table analysis. *J. Appl. Stat.* **2019**, *46*, 2572–2582. https://doi.org/10.1080/02664763.2019.1601689.
19. Danioko, S.; Zheng, J.; Anderson, K.; et al. A Novel Correction for the Adjusted Box-Pierce Test. *Front. Appl. Math. Stat.* **2022**, *8*, 873746. https://doi.org/10.3389/fams.2022.873746.