

A Bathtub Hazard Function Model Assuming a Mixture of Two Weibull Parametric Hazard Functions under a Bayesian Approach

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Abstract: This study introduces a parametric survival model based on a mixture of two Weibull hazard functions to accommodate bathtub-shaped hazard rates. The proposed formulation provides flexible modeling of lifetime data in applications such as engineering and medical studies. Inference for the model, in the presence of censored observations and covariates, is conducted within a Bayesian framework. Prior distributions for the model parameters are specified using empirical Bayesian methods based on a preliminary data analysis. Posterior summaries of interest are obtained via Markov Chain Monte Carlo (MCMC) simulation methods using the OpenBUGS software to generate samples from the joint posterior distribution of the model parameters. The methodology is illustrated through three real data applications, including two engineering datasets and one medical dataset from oncology.

Keywords: bathtub hazard; mixture hazard models; Weibull distribution; Bayes approach; MCMC simulation methods

1. Introduction

Survival analysis [1–4] has evolved significantly, encompassing both parametric and nonparametric methods. Challenges remain due to censoring, covariates, and the complexity of survival data. Advances in computation have enabled flexible parametric models and novel nonparametric approaches, yet highly parameterized models may face identifiability, convergence problems to get the estimators for the parameters of the model, and interpretability issues, while some nonparametric methods remain computationally intensive. This study proposes a practical, parsimonious parametric methodology with broad applicability.

Parametric methods dominate in engineering and industrial research, whereas nonparametric approaches are common in medical studies. Standard models rely on distributions such as exponential, Weibull, lognormal, and gamma, each defining a specific hazard function, often the primary quantity of interest.

Let T denote a positive random variable representing a unit's lifetime. A parametric survival model is characterized by its probability density function $f(t)$, the survival (or reliability) function $R(t) = P(T > t)$, and the hazard function $h(t)$, or instantaneous failure rate.

The hazard function $h(t)$ is a non-negative function defined as the limit of the probability of a component or unit to failure in the interval $[t, t + \Delta t]$ given that this component is surviving until time t , that is,

$$h(t) = \lim_{\Delta t \rightarrow 0} \frac{P(t \leq T < t + \Delta t \mid T \geq t)}{\Delta t} \quad (1)$$



The hazard function could have different behaviors in each application as constant hazard, increasing or decreasing hazard, unimodal hazard, presence of many peaks and bathtub hazard. For each case, we need to assume a probability distribution which characterizes the hazard function behavior required in applications.

The bathtub hazard function has three distinct regions in the plot of Figure 1: a burn-in region (decreasing hazard), an useful life region (approximately constant hazard) and a wear-out region (increasing hazard). In industrial and engineering applications these regions denote three phases that a newly manufactured product passes through its life span.

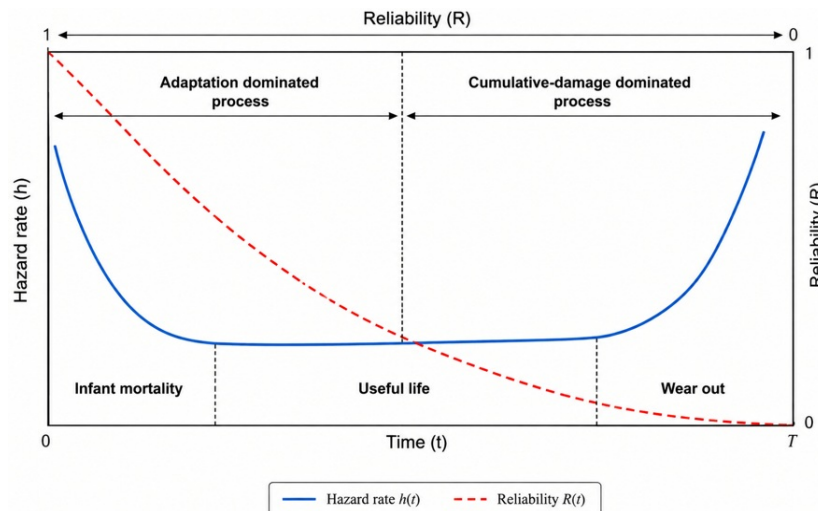


Figure 1. Schematic representation of the bathtub-shaped hazard function, showing the infant-mortality, useful-life, and wear-out phases. Adapted from [5].

In this situation, during the burn-in period, the hazard rate decreases, with failures typically caused by defects, poor workmanship, installation errors, or inadequate handling [6]. This stage is often termed the debugging, infant mortality, or break-in region.

Following the initial burn-in phase, the hazard rate remains approximately constant during the useful lifetime. In the subsequent wear-out phase, the hazard rate increases, with failures typically linked to aging, attritional wear, corrosion, inadequate maintenance, or finite design life [7].

Bathtub hazard or intensity functions are present in different areas of reliability applications, as in software reliability growth models, where many of which possess complex parametric forms. A framework for software reliability growth models possessing a bathtub-shaped fault detection rate and derived stable and efficient expectation conditional maximization algorithms to enable the fitting of these models [8]. To analyze failure data exhibiting non-monotonic behavior, a special subclass known as the exponential power-law process (EXPLP) has been proposed. Its baseline function is the intensity function of the power-law process, and several properties of this subclass have been investigated through one of its special cases, namely the EXPLP(1) intensity function [9]. Using two real datasets from the literature they provided evidence that the EXPLP(1) intensity function is well suited to analyze data which exhibits a bathtub behavior. Studies on generalizations and extensions of the Weibull model have been proposed in the literature to accommodate bathtub-shaped hazard rates. These studies include graphical representations of the mean residual life functions for several families of Weibull-related lifetime distributions, together with their corresponding hazard functions [10].

To model bathtub hazards the literature introduces many models as observed by [11], usually with three or more parameters. These parametric models in many cases could present non-identifiability problems and difficulties to obtain the maximum likelihood estimates of the parameters of the models using numerical methods depending on precise initial values to have convergence of the numerical algorithm. Among these bathtub hazard models we can cite the Perks distribution with four or five parameters [11], the modified Weibull distribution with three parameters [12], the exponentiated Weibull distribution with three parameters [13], the Haupt and Schabe distribution [14], the Additive Burr XII distribution [15], the new modified Weibull distribution [16], the truncated Weibull distribution [17], the FIRE distribution [18] and the competing risk model [19]. Also noteworthy is the development of a two-parameter bathtub (TPBT) model based on the generalized exponential (GE) distribution, which led to a new family of lifetime distributions, referred to as the four-parameter odd generalized exponential bathtub-shaped distribution [20].

Studies were carried out the underlying mechanism in a multiple-component scenario and provided a model based on hierarchy–structure systems to infer the appearance of the bathtub hazard rate via their structure induced failure modes. They point out that since most parametric models introduced in the literature to model bathtub hazard rate are usually not interpretable and the data could not bring explicit hazard shape information due to the limitation of its amount, then it is hard to select a proper model [21]. The additive Weibull model as a flexible parametric model capable of describing bathtub-shaped hazard functions introduced by [22] that was generalized by [23,24] proposing a log-normal modified Weibull distribution to model different shaped hazard rates including bathtub hazard rate. The corresponding hazard function is obtained by adding two Weibull-type hazard components, allowing the model to accommodate decreasing, increasing, and bathtub-shaped failure-rate patterns. Although the underlying additive Weibull hazard structure is not new, its Bayesian formulation and regression extension provide a useful framework for inference with censored survival data. In this study, we develop a Bayesian implementation of this model, incorporate covariate effects, and obtain posterior inference for the model parameters and relevant functions, including the survival and hazard functions and the location of the bathtub minimum.

Other studies have focused on the construction of parametric probability distributions characterized by bathtub-shaped hazard rates [11,25–28]. Alternative approaches have also been proposed, including the use of fuzzy methods to model bathtub hazard rates and estimate maintenance costs [29], as well as the application of the maximum entropy principle to derive bathtub-shaped hazard functions for entire systems when two or more observed moments fall within an appropriate region [21]. In addition, data analysis of the wear-out phase based on the q-exponential distribution has been investigated [30], while two phenomenological models of system-level degradation have been proposed to characterize bathtub-shaped hazard functions [5].

This study introduces a simple and interpretable parametric model for bathtub shaped hazard functions. Based on a finite mixture of Weibull hazards within a Bayesian framework, the model extends the Weibull distribution to capture decreasing, constant, and increasing hazard rates. The approach allows straightforward inference while providing flexibility for modeling survival data.

Finite mixture models are becoming very popular in applications with regard to density estimation and clustering [5,21,30], given the great flexibility in fitting the obtained model to the data. If the number of components in the mixture is known, we have a parametric mixture model [31–35]. The use of a Bayesian approach with MCMC (Markov Chain Monte Carlo) simulation methods usually is considered in the literature to analyze data with mixture models [36–39].

The approach considered in this study differs from the usual standard mixture of parametric densities approach considered in different studies introduced in the literature. The prior distributions of the parameters of the model, especially for the incidence probability parameter in the mixture model, could be elicited using some prior information based on a preliminary data analysis using empirical Bayesian methods [40], especially for situations with small sample sizes, and the posterior summaries compared using other more non-informative priors. The posterior summaries of interest are obtained using standard MCMC (Markov Chain Monte Carlo) methods and the OpenBUGS software [41]. It is important to point out, that the basic construction of a bathtub hazard from two Weibull hazard components is not new. The main contribution of the present study is presented in terms of the Bayesian implementation, posterior inference, regression formulation, treatment interpretation, and applications to reliability and censored medical data.

From now, the manuscript is prepared as follows: Section 2 introduces the Weibull probability distribution; Section 3 introduces the bathtub hazard function defined as a mixture of two hazard functions; Section 4 introduces statistical inferences for the proposed model under a Bayesian approach; Section 5 shows two applications with engineering real data sets and an application with a medical real data set in presence of censored data and a covariate; finally, Section 6 presents some conclusions.

2. The Weibull Probability Distribution

The Weibull distribution [42] is defined by the probability density function,

$$f(t) = \gamma \lambda t^{\gamma-1} \exp(-\lambda t^\gamma), \quad t > 0, \lambda > 0, \gamma > 0 \quad (2)$$

where λ is a rate-type parameter and γ is the shape parameter. The hazard function is given by $h(t) = \gamma \lambda t^{\gamma-1}$ (increasing hazard if $\gamma > 1$, decreasing hazard if $\gamma < 1$ and constant hazard if $\gamma = 1$).

The mean and variance of the Weibull distribution (2) are given respectively by,

$$E(T) = \lambda^{-1/\gamma} \Gamma(1 + 1/\gamma) \quad (3)$$

and

$$\text{Var}(T) = \lambda^{-2/\gamma} \left(\Gamma \left[1 + \frac{2}{\gamma} \right] - \left(\Gamma \left[1 + \frac{1}{\gamma} \right] \right)^2 \right) \quad (4)$$

3. Bathtub Hazard Function Defined as a Mixture of Two Hazard Functions

A hazard function $h(t)$ with a bathtub behavior defined as a mixture of two hazard functions $h_1(t)$ and $h_2(t)$ proposed by [6], is expressed as,

$$h(t) = ch_1(t) + (1 - c)h_2(t) \quad (5)$$

where $h_1(t)$ should be decreasing (burn-in region) and $h_2(t)$ should be increasing (wear-out region); c is a relative weight controlling the contribution of the decreasing hazard component to the overall hazard, while $1 - c$ is the corresponding weight of the increasing component, $0 < c < 1$.

The accumulated hazard function $\Lambda(t)$ defined by $\Lambda(t) = \int_0^t h(u)du$ is given by $\Lambda(t) = c\Lambda_1(t) + (1 - c)\Lambda_2(t)$ and the reliability function is a fixed time t is given by

$$\begin{aligned} R(t) &= \exp[-\Lambda(t)] \\ &= \exp[-c\Lambda_1(t)]\exp[-(1 - c)\Lambda_2(t)] \\ &= [R_1(t)]^c[R_2(t)]^{1-c} \end{aligned} \quad (6)$$

where $R_1(t)$ and $R_2(t)$ are the reliability functions related to the hazard functions $h_1(t)$ and $h_2(t)$.

The probability density function $f(t)$ obtained from $h(t) = f(t)/R(t)$, is given by,

$$f(t) = h(t)R(t) = [ch_1(t) + (1 - c)h_2(t)][R_1(t)]^c[R_2(t)]^{1-c} \quad (7)$$

Remark: The proposed model combines the two component hazards directly: This is different from a conventional mixture of survival distributions, whose resulting hazard is $h(t) = [cf_1(t) + (1 - c)f_2(t)]/[cS_1(t) + (1 - c)S_2(t)]$.

3.1. Use of mixture of two Weibull hazard functions

We assume two Weibull hazard functions in (5) given by $h_1(t) = \lambda_1\gamma_1 t^{\gamma_1-1}$ with $\gamma_1 < 1$ (decreasing hazard) and $h_2(t) = \lambda_2\gamma_2 t^{\gamma_2-1}$ with $\gamma_2 > 1$ (increasing hazard), that is,

$$h(t) = c\lambda_1\gamma_1 t^{\gamma_1-1} + (1 - c)\lambda_2\gamma_2 t^{\gamma_2-1} \quad 0 < c < 1 \quad (8)$$

where the accumulated hazard functions are given, respectively, by $\Lambda_1(t) = \lambda_1 t^{\gamma_1}$, $\Lambda_2(t) = \gamma_2 t^{\gamma_2}$ and $\Lambda(t) = c\lambda_1 t^{\gamma_1} + (1 - c)\lambda_2 t^{\gamma_2}$. The corresponding reliability functions are given, respectively, by, $R_1(t) = \exp(-\Lambda_1(t)) = \exp(-\lambda_1 t^{\gamma_1})$, $R_2(t) = \exp(-\Lambda_2(t)) = \exp(-\lambda_2 t^{\gamma_2})$ and $R(t) = \exp(-\Lambda(t)) = \exp(-c\lambda_1 t^{\gamma_1} - (1 - c)\lambda_2 t^{\gamma_2})$.

Remark: for $0 < \gamma_1 < 1$ and $\gamma_2 > 1$, the derivative of the hazard has a unique positive zero t_0 , and that the hazard is decreasing for $t < t_0$ and increasing for $t > t_0$. Thus, the proposed hazard has a unique bathtub minimum (see Appendix C).

The probability density function for the random variable T is given by,

$$f(t) = -dR(t)/dt = (c\lambda_1\gamma_1 t^{\gamma_1-1} + (1 - c)\lambda_2\gamma_2 t^{\gamma_2-1})\exp(-c\lambda_1 t^{\gamma_1} - (1 - c)\lambda_2 t^{\gamma_2}) \quad (9)$$

The mean of the probability distribution with density (9) is given by,

$$E(T) = \int_0^\infty tf(t)dt \quad (10)$$

4. Stastical Inferences for the Model under a Bayesian Approach

Assuming the model originated from a mixture of two Weibull hazard function, the likelihood function for $c, \lambda_1, \gamma_1, \lambda_2$ and γ_2 is given by,

$$L(\theta) = \prod_{i=1}^n f(t_i) \quad (11)$$

where θ is the parameter vector, that is, $\theta = (c, \lambda_1, \gamma_1, \lambda_2, \gamma_2)$ and $f(t_i) = (c\lambda_1\gamma_1 t_i^{\gamma_1-1} + (1 - c)\lambda_2\gamma_2 t_i^{\gamma_2-1})\exp(-c\lambda_1 t_i^{\gamma_1} - (1 - c)\lambda_2 t_i^{\gamma_2})$, for $i = 1, \dots, n$.

In presence of the censored right data, the likelihood function is given by,

$$L(\theta) = \prod_{i=1}^n [f(t_i)]^{\delta_i} [R(t_i)]^{1-\delta_i} \quad (12)$$

where $h(t_i) = c\lambda_1\gamma_1 t_i^{\gamma_1-1} + (1-c)\lambda_2\gamma_2 t_i^{\gamma_2-1}$, $R(t_i) = \exp(-c\lambda_1 t_i^{\gamma_1} - (1-c)\lambda_2 t_i^{\gamma_2})$, $\delta_i = 1$ if we have a complete observation and $\delta_i = 0$ if we have a right censored observation.

In presence of a covariate x , we assume a logit model for the relative weight given by,

$$\text{logit}(c_i) = \log \left[\frac{c_i}{1-c_i} \right] = \beta_0 + \beta_1 x_i \quad (13)$$

for $i = 1, 2, \dots, n$. That is,

$$c_i = \frac{e^{\beta_0 + \beta_1 x_i}}{1 + e^{\beta_0 + \beta_1 x_i}} \quad (14)$$

Observe that the treatment affects the relative weight, rather than representing a conventional hazard ratio, that is, it is an odds ratio.

We also could generalize the logit model (13) for situations with k covariates. The regression specification (14) was motivated by the objective of investigating whether treatment changes the relative contribution of the decreasing and increasing hazard components while maintaining a common underlying bathtub hazard structure.

Other possibility is to assume that treatment is allowed to affect the rate-type parameters of both Weibull components. Specifically, we could consider $\log(\lambda_{1i}) = \alpha_1 + \eta_1 x_i$, $\log(\lambda_{2i}) = \alpha_2 + \eta_2 x_i$, together with $\text{logit}(c_i) = \beta_0 + \beta_1 x_i$. The shape parameters γ_1 and γ_2 are maintained common across groups to preserve the restrictions $0 < \gamma_1 < 1$ and $\gamma_2 > 1$ and to avoid unnecessary overparameterization given the sample size. We could compare the restricted and extended models using Bayesian model-selection/predictive criteria and to examine the posterior distributions of η_1 and η_2 .

We consider a Bayesian approach [43] to get inferences for the parameters of the bathtub hazard models. Assuming the bathtub model defined by the mixture of two Weibull hazard functions (10), we assume a beta prior distribution $\text{Beta}(a, b)$ with mean $a/[a + b]$ and variance $ab/[(a + b)^2(a + b + 1)]$ for the relative weight c assuming known hyperparameters a and b . In the elicitation of the beta prior distributions we could use the information provided by the observed outcomes using a preliminary data analysis, for example, the histogram of the lifetimes, especially for situations with small sample sizes. In this way we are assuming empirical Bayesian methods [44]. Other possibility is to assume a more non-informative prior for c , as for example a $\text{Beta}(1, 1)$ prior distribution for c , and to present a sensitivity analysis to check the effect of different priors on the posterior summaries of interest. We assume an uniform $U(0, 1)$ prior distribution for the shape parameter γ_1 (decreasing hazard), an uniform $U(1, v)$ with large value for the hyperparameter v for the shape parameter γ_2 (increasing hazard); alternatively, we also could consider more non-informative gamma $G(d, e)$ prior distributions for both shape parameters, where $G(d, e)$ denotes a gamma probability distribution with mean d/e and variance d/e^2 . We also assume approximately non-informative $G(d, e)$ prior distributions for the rate-type parameters λ_1 and λ_2 . We further assume prior independence among the parameters.

In the Bayesian framework, model parameters are treated as random variables, with prior distributions combined with the likelihood to form the posterior distribution, that is, $P(\theta|D) \propto L(D|\theta)P(\theta)$ where $P(\theta)$ denote the prior, $P(\theta|D)$ the posterior, and $L(D|\theta)$ the likelihood function. Posterior summaries, such as means, standard deviations, and credible intervals, are obtained using MCMC methods, including Gibbs sampling and the Metropolis-Hastings algorithm [45,46]. Sampling is performed with OpenBUGS [41], which requires only the specification of likelihood and priors.

5. Applications

5.1. Application 1

In this application we consider a data set introduced by [47] consisting of the times to failure (in weeks) of 50 devices (Table 1).

Table 1. Data set [47].

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18
18	18	18	18	21	32	36	40	45	46	47	50	55	60
63	63	67	67	67	67	72	75	79	82	82	83	84	84
84	85	85	85	85	85	86	86						

The histogram for the data is presented in Figure 2, from where it is observed that there is a number of devices with failures at the beginning of the follow-up period (burn-in region) with decreasing rates before a useful life region, and after this region, we have a wear-out region with increasing rates. Figure 2 (first panel) shows that the data could be fitted by a parametric statistical model constructed from a mixture of two parametric hazard functions. In this way, we assume a mixture of two Weibull hazard functions introduced in Section 3.1 under a Bayesian approach.

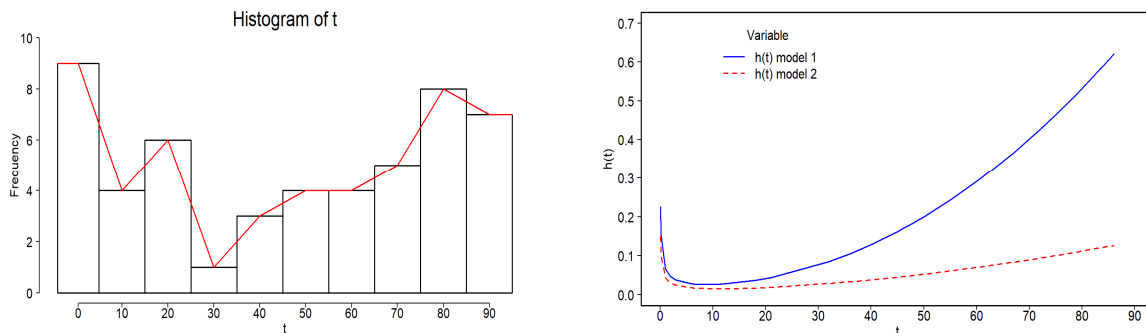


Figure 2. Histogram of the data and the fitted hazard function (mixture of two Weibull hazard functions).

For a Bayesian analysis, we first assume a Beta(1, 1) prior distribution, that is, a uniform U(0, 1) prior, for the relative weight parameter c in (7), where the mean of the beta distribution is given by $1/[1 + 1] = 0.5$. Let us denote this model as “model 1”. In this case, the choice of the hyperparameters of the beta distribution was made without assuming any prior information based on a preliminary data analysis. A second assumed prior distribution is given by a Beta(5,5) prior distribution. Let us denote this model as “model 2”. The resulting posterior distributions are compared to assess the robustness of the inference to prior specification.

We assume uniform prior distributions for the shape parameters γ_1 given, respectively, by U(0, 1) and U(1, 5) (observe that we should have the restrictions $0 < \gamma_1 < 1$ and $\gamma_2 > 1$ to have bathtub shaped hazard function) and γ_2 and non-informative $G(0.001, 0.001)$ and $G(0.00001, 0.00001)$ prior distributions for the transformed rate-type parameters $\theta_1 = 1/\lambda_1$ and $\theta_2 = 1/\lambda_2$. Observe that a gamma distribution $G(0.00001, 0.00001)$ has mean equals to one and variance equals to 100,000. The transformations θ_1 and θ_2 were used to get better performance for the MCMC simulation method. We further assume independence among all parameters.

Using the OpenBUGS software considering 3 parallel chains, a burn-in sample of size 11,000 and simulating new 1000 Gibbs samples (every 100th sample in 100,000 simulated Gibbs samples) to get the Monte Carlo Bayesian estimates, Table 2 shows the posterior summaries of interest (posterior means, posterior standard deviations and 95% credible interval for each parameter of the model and the minimum t_0 of the bathtub hazard function). The convergence of the simulation algorithm was verified from traceplots of the simulated values for each parameter (see Figure 3, assuming a Beta(1,1) prior distribution for c). In Table 2, it is observed that using a more non-informative Beta(1,1) prior for the relative weight parameter c (“model 1”), the 95% credible for each parameter, in general, are larger than assuming “model 2”. Figure 2 (second panel), also shows the fitted hazard function assuming the model obtained from a mixture of two Weibull hazard functions. From the second panel of Figure 2, it is observed a better fit (better bathtub shape) for the model with hazard function assuming a Beta(1,1) prior (“model 1”) for the relative weight parameter c . The obtained DIC is 454.5 for “model 2” and the obtained DIC for “model 1” is 448.5 indicating better fitted model for “model 1” (smaller DIC indicates better fitted model). The estimated R index introduced by [48] was close to one in all cases.

Table 2. Posterior summaries (application 1).

(a) Beta(1,1) prior for c				
	mean	sd	Lower 95%	Upper 95%
c	0.8079	0.2377	0.1485	0.9993
γ_1	0.4724	0.1096	0.2841	0.7304
γ_2	3.1480	0.4640	2.3580	4.1800
λ_1	0.1751	0.8806	0.0348	0.5777
λ_2	0.000071	0.00028	0.000003	0.00042
θ_1	11.4200	6.4710	1.7500	28.8000
θ_2	86,870.0	94,750.0	2399.0	342,600.0
t_0	17.7700	3.4230	11.2500	24.5900

Table 2. Cont.

(b) Beta(5,5) prior for c				
	mean	sd	Lower 95%	Upper 95%
c	0.5528	0.1572	0.2440	0.8363
γ_1	0.4575	0.1051	0.2677	0.6832
γ_2	2.7550	0.2703	2.1510	3.2100
λ_1	0.1688	0.09994	0.05161	0.4289
λ_2	0.00004	0.000085	0.000003	0.00023
θ_1	7.8870	4.5590	2.3330	19.5200
θ_2	93,330.0	89,480.0	4385.0	32,8300.0
t_0	15.3800	2.8890	9.6930	21.1300

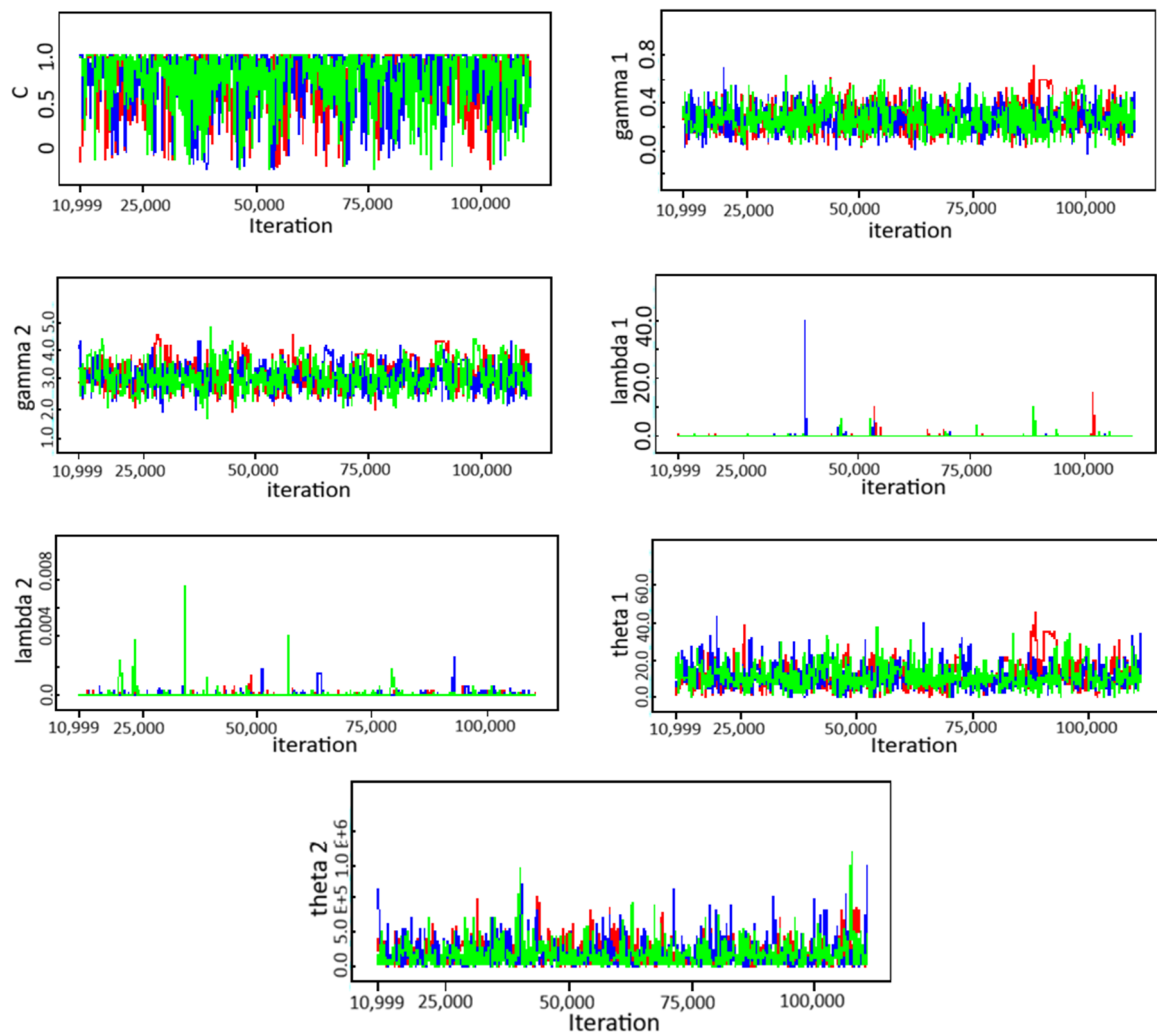


Figure 3. Trace plots of the simulated Gibbs samples (mixture of two Weibull hazard).

As illustrative purposes, we also assume a parametric lifetime model with four parameters introduced in the literature which includes bathtub hazard shapes. A special parametric model for this situation was introduced by [20] assuming a two parameter bathtub (TPBT) model and the generalized exponential (GE) distribution to propose a new family of lifetime distributions, named the odd generalized exponential four-parameter bathtub shaped distribution (OGE-TPBT). Maximum likelihood and Bayesian procedures were used by [20] to estimate the model parameters. They pointed out that the OGE-TPBT model fits the data better than both the GE and TPBT distributions.

The cumulative distribution function of a random variable T with a OGE-TPBT distribution [20] with a vector of parameters $\zeta = (\lambda, \alpha, \beta, \theta)$ is given by,

$$F(t) = \left[1 - \exp \left(-\lambda \left[\exp \left(\alpha \left(e^{t^\beta} - 1 \right) \right) - 1 \right] \right) \right]^\theta \quad (15)$$

where $\lambda, \alpha, \beta, \theta > 0$ and $t > 0$.

The corresponding probability density function is given by,

$$f(t) = \lambda \alpha \beta \theta t^{\beta-1} \exp(\alpha(e^{t^\beta} - 1)) \exp\{-\lambda[\exp(\alpha(e^{t^\beta} - 1)) - 1]\} [1 - \exp\{-\lambda[\exp(\alpha(e^{t^\beta} - 1)) - 1]\}]^{\theta-1} \quad (16)$$

The hazard function $h(t) = f(t)/S(t)$ is given by,

$$h(t) = \frac{\lambda \alpha \beta \theta t^{\beta-1} \exp(\alpha(e^{t^\beta} - 1)) \exp\{-\lambda[\exp(\alpha(e^{t^\beta} - 1)) - 1]\} [1 - \exp\{-\lambda[\exp(\alpha(e^{t^\beta} - 1)) - 1]\}]^{\theta-1}}{1 - [1 - \exp\{-\lambda[\exp(\alpha(e^{t^\beta} - 1)) - 1]\}]^\theta} \quad (17)$$

The MLE (maximum likelihood estimators) for the four parameters $(\lambda, \alpha, \beta, \theta)$ of the OGE-TPBT distribution with density (16) obtained by [20] are given respectively by 0.15959, 0.00037, 0.48769, 0.26697). The Bayes estimates assuming gamma prior distributions (fixed values for the hyperparameters) for the four parameters $(\lambda, \alpha, \beta, \theta)$ of the OGE-TPBT distribution obtained by [20] are given respectively by 0.07635, 0.00032, 0.49710, 0.22130). Considering the classical and Bayesian estimates, we get the estimates for the hazard function (17) of the OGE-TPBT distribution. Figure 4 shows the graphs of the fitted hazard functions considering the mixture Weibull hazard model proposed in this study and the OGE-TPBT model fitted using a classical (SM-MLE) approach and a Bayesian approach (SM-Bayes). From these graphs, it is observed that the mixture Weibull hazard model is more in agreement with the histogram of the data of Table 1. Furthermore, the OGE-TPBT model presents several parameters that have no direct interpretation for researchers in the field of engineering or medicine, which is a practical problem in applications.

It is important to point out that the literature also introduces many other models including bathtub hazard functions even with only two parameters, as the Chen probability distribution [49], and the Pham probability distribution [50,51]. The Chen distribution has cumulative distribution function, hazard function and probability density function given respectively by, $F(t) = 1 - \exp[\lambda(1 - \exp(t^\beta))]$, $h(t) = \lambda \beta t^{\beta-1} \exp(t^\beta)$ and $f(t) = \lambda \beta t^{\beta-1} \exp(t^\beta) \exp[\lambda(1 - \exp(t^\beta))]$, $t > 0, \lambda > 0, \beta > 0$; if $\beta < 1$ we have a bathtub hazard. The Pham distribution has cumulative distribution function, hazard function and probability density function given respectively by, $F(t) = 1 - \exp[(1 - \sigma^{\alpha(t)})]$, $h(t) = \alpha[\log(\sigma)]\sigma^{\alpha(t)}t^{\alpha-1}$ and $f(t) = h(t)S(t) = \alpha[\log(\sigma)]\sigma^{\alpha(t)}t^{\alpha-1}\exp[1 - \sigma^{\alpha(t)}]$, $\alpha(t) = t^\alpha, t > 0, \alpha > 0$ and $\sigma > 1$. Also as comparative purposes, we assumed the Chen and the Pham distributions in the data analysis under a Bayesian approach. Assuming independent prior distributions $\lambda \sim \text{Gamma}(0.1, 0.1)$ and $\beta \sim \text{Gamma}(0.1, 0.1)$ for the parameters of the Chen distribution, the posterior means(standard error) for λ and β (use of the OpenBUGS software with a burn-in sample of size 11,000) are given respectively by 0.978(0.1169) and 0.1211(0.01129). Similar results were obtained considering the priors $\lambda \sim \text{Uniform}(0,5)$ and $\beta \sim \text{Uniform}(0,1)$. Assuming independent prior distributions $\sigma \sim \text{Gamma}(0.1, 0.1)$ and $\alpha \sim \text{Uniform}(1,5)$ for parameters of the Pham distribution, the posterior means(standard error) for α and σ (use of the OpenBUGS software with a burn-in-sample of size 11,000) are given respectively by 0.7785(0.1035) and 1.039(0.0178). Figure 4 also shows the graphs of the fitted hazard functions considering the Chen and Pham distributions.

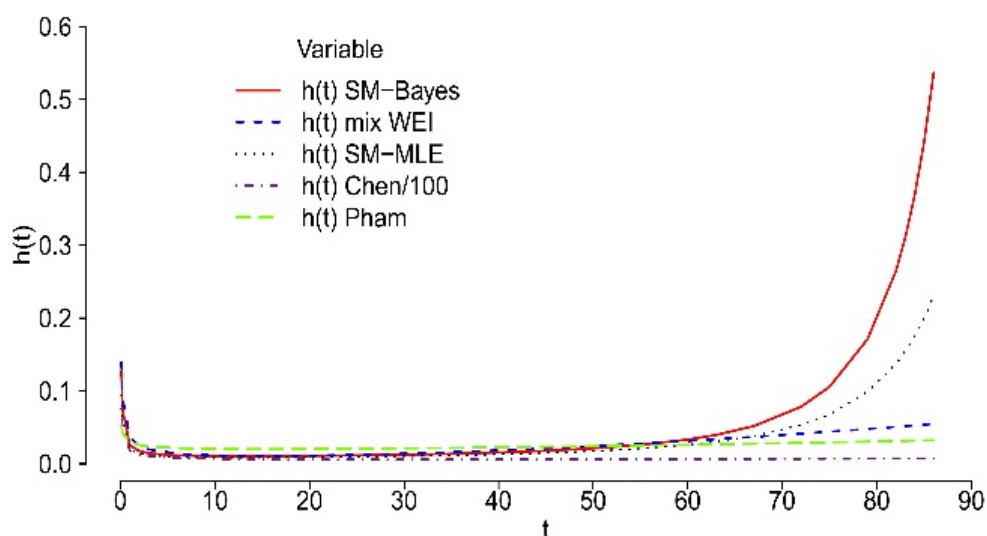


Figure 4. The fitted hazard functions.

5.2. Application 2

In this application, we consider the lifetime data of 52 transformers in UK National Grid [52]. These transformers are used for 275 kV and 400 kV electric power net work. They were installed between 1952 and 2008. Up to 2009, 52 transformers failed. The data set is presented in Table 3.

Table 3. Data set [52].

1	1	2	3	4	4	6	6	7	8	8	9	10	10	10
13	14	14	15	15	16	19	19	20	21	22	22	22	23	24
24	25	26	26	27	27	27	28	28	29	29	30	31	34	34
35	35	37	37	39	39	48								

The histogram of the data is presented in Figure 5 showing that the data could be fitted by a parametric model obtained from a mixture of two parametric Weibull hazard functions.

For a Bayesian analysis, we assume a Beta(1, 1) prior distribution for the relative weight parameter c . We also assume an uniform U(0, 1) prior distributions for the shape parameter γ_1 (decreasing hazard), an uniform U(1, 5) prior for the parameter γ_2 (increasing hazard), a $G(0.001, 0.001)$ prior distribution for the transformed rate-type parameter $\theta_1 = 1/\lambda_1$ and a non-informative $G(0.00001, 0.00001)$ prior distribution for the transformed rate-type parameter $\theta_2 = 1/\lambda_2$. We further assume independence among all parameters.

Using the OpenBUGS software considering 3 parallel chains, a burn-in sample of size 211,000 and simulating new 1000 Gibbs samples (every 100th sample in 100,000 additional simulated Gibbs samples) to get the Monte Carlo Bayesian estimates of interest, Table 4 shows the posterior summaries of interest (posterior means, posterior standard deviations and 95% credible interval for each parameter of the model and the minimum t_0 of the bathtub hazard function). The convergence of the simulation algorithm was verified from traceplots of the simulated values for each parameter. Figure 5 also shows the fitted hazard function assuming the mixture of two parametric Weibull hazard functions model.

Table 4. Posterior summaries (Application 2).

	Mean	sd	Lower 95%	Upper 95%
c	0.4211	0.2942	0.01460	0.9585
γ_1	0.6236	0.2562	0.09353	0.9878
γ_2	1.7050	0.5463	0.07633	2.5560
λ_1	0.1616	0.9239	0.01088	0.6759
λ_2	0.01256	0.02555	0.000073	0.0785
θ_1	25.6800	23.8500	1.4780	91.8900
θ_2	1704.00	8420.00	12.7400	13,920.00
t_0	0.9357	1.0030	0.00309	3.0840

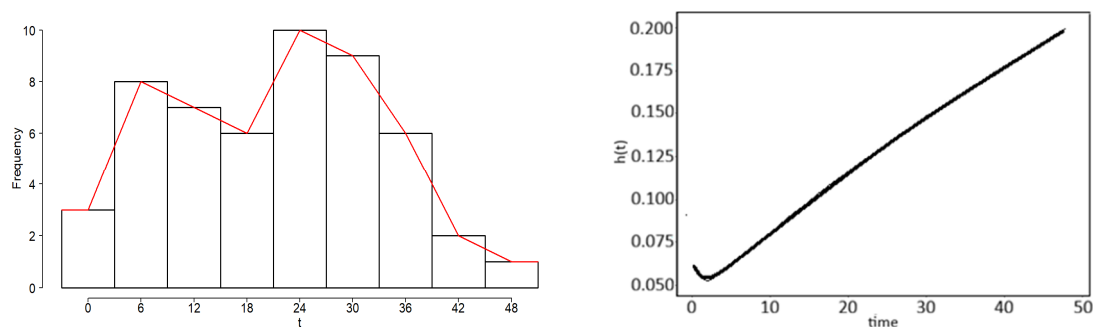


Figure 5. Histogram of the data and the fitted hazard (mixture of two Weibull hazards).

5.3. Application 3

In this application, we consider a medical oncology data set consisting of the lifetimes in months of 201 patients introduced by [53] and reanalyzed by [54] related to a retro-spective study in patients with gastric adenocarcinoma who underwent curative resection with D2 lymphadenectomy in the Barretos Cancer Hospital (Barretos, São Paulo state, Brazil) between January 2002 and December 2007. Table 5 shows the dataset, which includes 76 patients that received adjuvant CRT (adjuvant chemoradiotherapy) and 125 that received resection

alone, where 53.2% of all the data set are censored, 57.9% censored observations for the patients treated with CRT and 50.4% censored observations for the patients treated with resection alone. We assume non-informative right censoring, meaning that, conditional on the observed covariates and model parameters, the censoring process is assumed to be independent of the event-time process. Under this assumption, the contribution of a censored observation to the likelihood is given by the corresponding survival probability.

The histograms of the complete data (only the non censored data) are presented in Figure 6 (all data, CRT data and resection alone data) showing that the data could be fitted by a parametric model obtained from a mixture of two parametric Weibull hazard functions introduced in Section 3.1 (there are many deaths at the very beginning of the follow-up time).

Table 5. Data from 76 patient received adjuvant CRT and 125 receiving resection alone.

CRT:	5.76	7.89	8.85	8.95	9.05	9.47	10.72	11.97	12.50
12.83	13.09	13.49	13.78	13.82	14.70	14.77	16.38	16.51	17.07
17.07	17.14	17.14+	17.70	18.39	19.21	19.38+	20.49+	20.76+	21.02
22.86	23.39+	23.82+	24.21	24.21+	24.31	24.28+	24.97+	25.23	25.33+
25.56+	25.59+	25.76+	25.79	25.79+	26.05	27.89+	28.22+	28.59	28.65+
29.08+	29.31+	30.26	30.69+	30.95+	31.64+	31.84+	32.40+	32.47+	32.53+
33.88+	34.74+	34.90+	35.72+	35.89+	36.00+	36.00+	36.00+	36.00+	36.00+
36.00+	36.00+	36.00+	36.00+	36.00+	36.00+	36.00+	36.00+	36.00+	36.00+
Resection:	0.10	0.20	0.23	0.26	0.30	0.33	0.49	0.53	0.56
0.63	0.66	0.66	1.18	1.45+	1.61	1.78	2.63+	2.73	2.80
2.89	2.96	3.32	3.49+	4.01	4.54	4.67	4.67	4.93	6.15
6.55+	6.91	7.17	7.70	7.93+	8.32	8.36	8.39	8.78	8.91
9.28	9.70	10.03	10.2	10.53	10.76	11.41	11.88	12.50	13.13+
13.95	14.01	14.05	14.34	14.38+	15.43	15.76+	16.09	16.18	16.94
16.94+	17.14	17.24	17.43	18.62	19.14	19.44	19.84+	19.93	20.49+
21.38+	21.48	21.84+	21.88+	22.14	22.99+	23.39+	23.52+	23.55+	23.85+
24.01+	24.57+	24.80+	25.26+	25.30	25.33+	25.36+	26.15+	26.32	23.32+
26.78+	27.37+	28.98+	29.28+	29.31+	29.97+	30.16+	30.49+	30.63+	31.38+
31.68+	32.50+	32.80+	33.09+	33.36+	33.65+	33.91+	34.08+	34.21+	34.41+
35.00+	35.03+	35.20+	35.30+	35.59+	36.00+	36.00+	36.00+	36.00+	36.00+
36.00+	36.00+	36.00+	36.00+	36.00+	36.00+	36.00+	36.00+	36.00+	36.00+

(+ indicates censored data). Lifetimes: months since surgery.

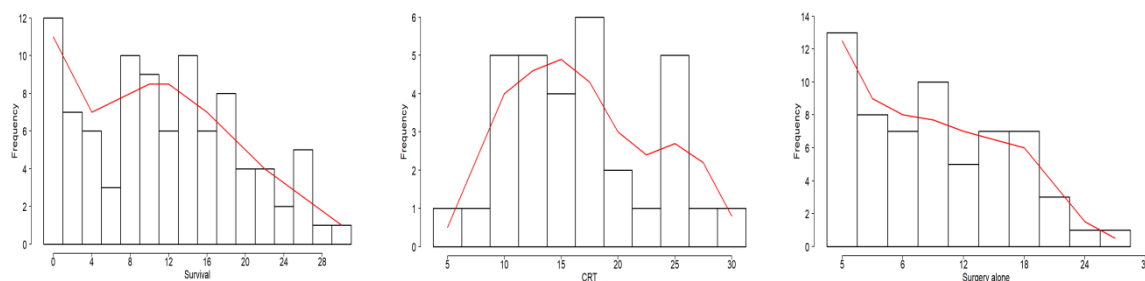


Figure 6. Histograms of complete data (all data, CRT data and resection alone data).

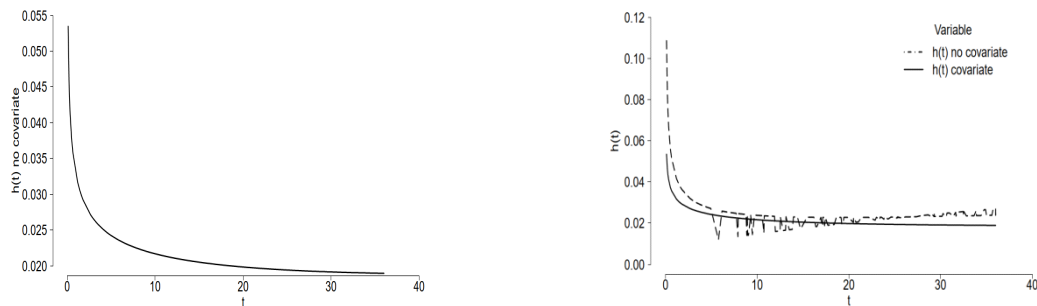
5.3.1. A Bayesian Analysis not Considering the Presence of the Covariate Treatment

For a first Bayesian analysis of the data of 201 patients not considering the presence of the covariate treatment (CRT or resection alone), we assume a Beta(1, 1) prior distribution for the relative weight parameter c , an uniform $U(0, 1)$ prior distribution for the shape parameter γ_1 (decreasing hazard), an uniform $U(1, 3)$ prior distribution for the parameter γ_2 (increasing hazard), a $G(0.001, 0.001)$ prior distribution for the transformed rate-type parameter $\theta_1 = 1/\lambda_1$ and a non-informative $G(0.00001, 0.00001)$ prior distribution for the transformed rate-type parameter $\theta_2 = 1/\lambda_2$. We further assume independence among all parameters.

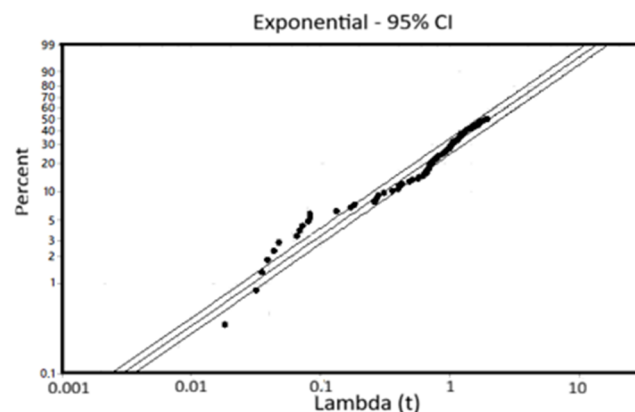
Using the OpenBUGS software considering a burn-in sample of size 11,000 and simulating new 1000 Gibbs samples (every 100th sample in 100,000 additional simulated Gibbs samples) to get the Monte Carlo Bayesian estimates of interest, Table 6 shows the posterior summaries of interest. The convergence of the simulation algorithm was verified from traceplots of the simulated values for each parameter. Figure 7 shows the fitted hazard function assuming the mixture of two parametric Weibull hazard functions model. The Monte Carlo estimate of DIC [55] used as a Bayesian discrimination criterion (see Appendix A) is given by 897.4.

Table 6. Posterior summaries (application 3 not considering the covariate treatment).

	Mean	sd	Lower 95%	Upper 95%
c	0.5724	0.3058	0.0292	0.9970
γ_1	0.7966	0.0809	0.6389	0.9497
γ_2	1.5700	0.4103	1.0260	2.5180
λ_1	0.1959	0.6002	0.0314	1.3610
λ_2	0.0077	0.0328	< 0.0001	0.0706
θ_1	14.71	8.6470	0.7348	31.90
θ_2	19,070.0	33,070.0	14.15	127,900.0

**Figure 7.** The fitted hazard function considering all data where the left panel refers to data not considering the presence of the covariate treatment and the right panel refers to data in presence of the covariate treatment (mixture of two Weibull hazards).

Since the accumulated hazard function $\Lambda(t) = -\log[S(t)]$ has an exponential distribution with parameter 1, we can verify the fit of the proposed model considering plots of $\Lambda(t) = -\log[\hat{S}(t)]$ versus t [56]. Since this plot (Figure 8) has an approximately linear relation, we can conclude that the model is satisfactorily fitted by the data.

**Figure 8.** Graphical comparison involving the estimated cumulative hazard.

5.3.2. A Bayesian Analysis Considering the Presence of the Covariate Treatment

For a second Bayesian analysis of the data considering the presence of the covariate treatment (CRT or resection alone), we assume that treatment affects the rate-type parameters of both Weibull components, that is, $\log(\lambda_{1i}) = \alpha_1 + \eta_1 x_i$, $\log(\lambda_{2i}) = \alpha_2 + \eta_2 x_i$, together with the logistic regression model (13) for the relative weight parameter c , with $x = 1$ (CRT) and $x = 0$ (resection alone). Let us denote this model as “model 1”. We assume a uniform $U(0, 1)$ prior distribution for the shape parameter γ_1 , an uniform $U(1, 3)$ prior distribution for the parameter γ_2 , a normal $N(0, 10)$ prior distribution for the regression parameter β_0 , a normal $N(0, 1000)$ prior distribution for the regression parameter β_1 , a normal $N(0, 10)$ prior distribution for the regression parameter α_j and a normal $N(0, 1000)$ prior distribution for the regression parameter η_j , $j = 1, 2$. We further assume independence among all parameters.

Using the OpenBUGS software considering a burn-in sample of size 51,000 and simulating new 1200 Gibbs samples (every 50th sample in 60,000 additional simulated Gibbs samples) to get the Monte Carlo Bayesian estimates of interest, Table 7 shows the posterior summaries of interest. The convergence of the simulation algorithm was verified from traceplots of the simulated values for each parameter.

Table 7. Posterior summaries (application 3 with “model 1”).

	Mean	sd	Lower 95%	Upper95%
α_1	-2.438	0.4282	-3.109	-1.346
α_2	-4.677	2.406	-9.213	-0.154
β_0	4.032	2.594	-0.8017	8.974
β_1	-27.52	20.57	-74.68	1.623
η_1	-16.21	27.22	-70.9	34.22
η_2	-0.7297	2.645	-5.654	4.495
γ_1	0.643	0.08721	0.4696	0.8165
γ_2	1.547	0.2378	1.133	2.082

Assuming a second model, denoted as “model 2”, with only a logistic regression model (13) for the relative weight parameter c , we consider an uniform $U(0, 1)$ prior distribution for the shape parameter γ_1 , an uniform $U(1, 3)$ prior distribution for the parameter γ_2 , a normal $N(4,1)$ prior distribution for the regression parameter β_0 , a normal $N(-27,1)$ prior (data-informed priors, that is, using empirical Bayesian methods, [40]) for the regression parameter β_1 , a uniform $U(0, 1)$ prior distribution for the rate-type parameters λ_1 and a uniform $U(0, 0.05)$ prior distribution for the rate-type parameters λ_2 . Observe that we are using some prior information of the obtained results assuming “model 1”. We further assume independence among all parameters

Using the OpenBUGS software considering a burn-in sample of size 11,000 and simulating new 1000 Gibbs samples (every 100th sample in 100,000 additional simulated Gibbs samples) to get the Monte Carlo Bayesian estimates of interest, Table 8 shows the posterior summaries of interest. The convergence of the simulation algorithm was verified from traceplots of the simulated values for each parameter. The Monte Carlo estimate of DIC [55] for “model 2” is given by 877.3, that is, “model 2” is better fitted by the data since the estimated DIC is smaller than the estimated DIC for “model 1” (DIC estimated by 918.2). Observe that in “model 1” the 95% credible intervals for the regression parameters η_1 and η_2 include the zero value showing not significance effect of the covariate in this formulation.

Table 8. Posterior summaries (application 3 With “model 2”).

	Mean	sd	Lower 95%	Upper 95%
β_0	4.1730	1.0300	2.114	6.0280
β_1	-26.910	1.0120	-28.87	-24.960
γ_1	0.6601	0.0741	0.5228	0.8134
γ_2	1.4730	0.2183	1.0820	1.9130
λ_1	0.0826	0.0213	0.0467	0.1315
λ_2	0.0048	0.0036	0.0008	0.0142

6. Conclusions

This study considered a practical and simple modeling methodology for reliability data where the hazard function has a bathtub shape, a common situation in industrial, engineering or medical applications. The proposed model obtained from the mixture of two Weibull hazard functions (a decreasing hazard and an increasing hazard) presents great computational facility to obtain the fitted hazard curve under a Bayesian approach and using standard existing MCMC (Markov Chain Monte Carlo) simulation methods to get the posterior summaries of interest. Using usual Bayesian empirical methods to elicit the prior distribution for the parameter representing the relative weight parameter in each component of the mixture, especially for small sample sizes, the obtained results can be of great practical interest. It is important to point out that the literature presents many parametric lifetime distributions usually with three or more parameters which could model bathtub shape hazards. In general these models with many parameters could present problems in applications since many parameters do not have practical interpretations for industrial, engineering, or medical studies. The proposed model consisting of a mixture of two Weibull hazard functions have simple practical interpretations for all parameters of the model and simple regression formulations in presence of one or more covariates. Other practical point of interest explored in this study: the use of existing free Bayesian software like the OpenBUGS software gives great simplification to get the inferences of interest assuming the proposed mixture Weibull hazards model. The three applications considered in the study to illustrate the proposed methodology show the ease and practicability of the methodology that can be of great practical interest, especially with engineering or medical lifetime data.

A possible limitation of the proposed mixture of two Weibull hazard functions model is the choice of prior information on the incidence probability c in (5), especially with small samples. In this situation there is some

limitation to use histograms. It is important to point out that Monte Carlo Bayes estimate of the relative weight parameter c is very sensitivity on the choice of the prior. As an alternative in this situation, we could get prior information on the mixture based on empirical estimates of the reliability function with no censoring data or the use of the non-parametric [57] estimate in presence of censored data, but these plots could be very non-informative with small sample sizes.

Observe that while practical, empirical Bayesian approaches estimate prior hyperparameters from the data, ignoring their uncertainty and effectively using the data twice, which departs from fully Bayesian principles and may impact posterior summaries.

Other important problem with the proposed model: in a two-component mixture model, the likelihood is invariant to an interchange of the component labels. Consequently, without an identifiability constraint, the MCMC sampler may switch the labels of the two components during sampling, resulting in multimodal marginal posterior distributions and potentially misleading posterior summaries. In the present model, however, the two Weibull components have distinct roles in constructing the bathtub-shaped hazard. Specifically, one component is intended to represent the decreasing (early-failure) component, whereas the second represents the increasing (late-failure) component. We have therefore imposed an explicit identifiability constraint on the Weibull shape parameters. This constraint assigns a unique scientific interpretation to the two components and prevents arbitrary permutation of their labels during MCMC sampling. Thus, component 1 consistently represents the early/decreasing failure mechanism and component 2 consistently represents the late/increasing failure mechanism.

As a future work, we also could consider regression models in presence of a vector of covariates for the shape parameters γ_1 and γ_2 in (8) especially considering applications in medical and engineering studies. We also could consider other lifetime distributions with decreasing hazard at the beginning of the follow-up period. Other point that could be explored in the future: use of mixture models with more than two components leading to hazard functions with decreasing, increasing, decreasing shapes as it is common in medical oncology data with cure fraction. A model with a mixture of three hazard functions could be assumed in application 3 especially for the resection alone data group (third histogram in Figure 6) but our interest in this study was to model data in presence of hazards with bathtub hazards.

Author Contributions

J.A.A.: contributed to the text's development and statistical analysis. E.B.: contributed to the creation of figures and the article's structure. All authors have read and agreed to the published version of the manuscript.

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Informed Consent Statement

Not applicable.

Data Availability Statement

The datasets generated and/or analyzed during the current study are available from the corresponding author upon reasonable request. To facilitate scientific verification and reproducibility, qualified researchers may request access to the underlying data and supporting documentation. Data will be provided subject to applicable ethical, privacy, legal, and institutional requirements. The authors will ensure that the data remain accessible for at least 10 years after publication.

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Conflicts of Interest

The authors declare no conflict of interest.

Use of AI and AI-Assisted Technologies

No AI tools were utilized for the this paper.

Appendix A. (DIC—Deviance Information Criterion)

Deviance Information Criterion (DIC) is a model-selection criterion used in Bayesian analysis to compare competing Bayesian models. It is particularly useful when models are fitted using Markov Chain Monte Carlo (MCMC) methods.

Basic idea: Suppose we have several Bayesian models that could explain our data. We want a model that fits the observed data well, and is not unnecessarily complicated. DIC balances these two goals, much like AIC and BIC in frequentist statistics.

The general form is:

$$DIC = \bar{D} + 2p_D \quad (A1)$$

where $\bar{D}$ is posterior mean deviance and p_D is the effective number of parameters, measuring model complexity.

Deviance: the deviance is defined as

$$D(\theta) = -2 \ln p(y | \theta) + C \quad (A2)$$

where y is the observed data, θ denotes the model parameters, and $p(y | \theta)$ denotes the likelihood function. C is a constant that does not affect comparisons between models. A smaller deviance means the model provides a better fit to the observed data. In Bayesian analysis, we obtain many values of θ from the posterior distribution using MCMC. Therefore, we can calculate the deviance for each MCMC sample and obtain its posterior mean: $\bar{D} = E_{\theta|y}[D(\theta)]$.

Effective number of parameters p_D : A Bayesian model may not have a straightforward number of parameters, particularly when there are hierarchical or random effects. DIC therefore uses an effective number of parameters:

$$p_D = \bar{D} - D(\hat{\theta}) \quad (A3)$$

where $\hat{\theta} = E(\theta | y)$ is the posterior mean of the parameters. Thus,

$$DIC = D(\hat{\theta}) + 2p_D \quad (A4)$$

Both expressions are commonly used.

Interpretation: DIC can be understood as: DIC = lack of fit + complexity penalty. A model with many parameters can fit the data extremely well, but it may overfit. The p_D term penalizes this complexity. When comparing Bayesian models fitted to the same data, generally lower DIC is the preferred model. The DIC criterion is widely used in Bayesian inference applied especially using Monte Carlo methods on Markov Chains. DIC is considered as a relative model-comparison. Smaller DIC values indicate better relative fit under the criterion being compared, but do not establish that the selected model adequately describes the data.

Appendix B. OpenBUGS Code

Appendix B.1. Application 1

```
model {
  for (i in 1:N) {
    zeros[i] <- 0
    phi[i] <- -log(L[i])
    zeros[i] ~ dpois(phi[i])
    a1[i] <- c*lambda1*gamma1*pow(t[i],gamma1-1)
    a2[i] <- (1-c)*lambda2*gamma2*pow(t[i],gamma2-1)
    b1[i] <- c*lambda1*pow(t[i],gamma1)
    b2[i] <- (1-c)*lambda2*pow(t[i],gamma2)
    h[i] <- a1[i]+a2[i]
```

```

L[i] <- (a1[i]+a2[i])*exp(-b1[i]-b2[i])
}
c~ dbeta(5,5)
gamma1 ~ dunif(0,1)
gamma2 ~ dunif(1,5)
lambda1 <- 1/theta1
lambda2 <- 1/theta2
theta1 ~ dgamma(0.001,0.001)
theta2 ~ dgamma(0.00001,0.00001)
}
list(t=c(0.1,0.2,1.0,1.0,1.0,1.0,1.0,2.0,3.0,6.0,7.0,11.0,12.0,18.0,18.0,18.0,18.0,21.0,32.0,36.0,40.0,45.0,46.
0,47.0,50.0,55.0,60.0,63.0,63.0,67.0,67.0,67.0,67.0,72.0,75.0,79.0,82.0,82.0,83.0,84.0,84.0,84.0,85.0,85.0,85.0,8
5.0,85.0,86.0,86.0), N=50)

list(c=0.6, gamma1=0.8, gamma2=1.5, theta1=5, theta2=20000)

```

Appendix B.2. Application 3 (Censored Data and Covariate)

```

model {
  for (i in 1:N) {
    zeros[i] <- 0
    phi[i] <- -log(L[i])
    zeros[i] ~ dpois(phi[i])
    a1[i] <- c[i]*lambda1*gamma1*pow(t[i],gamma1-1)
    a2[i] <- (1-c[i])*lambda2*gamma2*pow(t[i],gamma2-1)
    b1[i] <- c[i]*lambda1*pow(t[i],gamma1)
    b2[i] <- (1-c[i])*lambda2*pow(t[i],gamma2)
    h[i] <- a1[i]+a2[i]
    logit(c[i]) <- beta*treatment[i]
    log(L[i]) <- delta[i]*log(h[i]) - b1[i]-b2[i]
  }
  gamma1 ~ dunif(0,1)
  gamma2 ~ dunif(1,3)
  beta ~ dnorm(0,0.001)
  lambda1 ~ dgamma(0.001,0.001)
  lambda2 ~ dgamma(0.001,0.001)
}
list(t=c(0.1000,0.2000,0.2300,0.2600,0.3000,0.3300,0.4900,0.5300,0.5600
,0.6300,0.6600,0.6600,1.1800,1.4500,1.6100,1.7800,2.6300,2.7300
,2.8000,2.8900,2.9600,3.3200,3.4900,4.0100,4.5400,4.6700,4.6700
,4.9300,5.7600,6.1500,6.5500,6.9100,7.1700,7.7000,7.8900,7.9300
,8.3200,8.3600,8.3900,8.7800,8.8500,8.9100,8.9500,9.0500,9.2800
,9.4700,9.7000,10.0300,10.2000,10.5300,10.7200,10.7600,11.4100,11.8800
,11.9700,12.5000,12.5000,12.8300,13.0900,13.1300,13.4900,13.7800,13.8200
,13.9500,14.0100,14.0500,14.3400,14.3800,14.7000,14.7700,15.4300,15.7600
,16.0900,16.1800,16.3800,16.5100,16.9400,16.9400,17.0700,17.1400,17.1400

```

[illegible]

```
list(gamma1=0.8,gamma2=1.5, beta=0, lambda1=0,lambda2=0)
```

Appendix C

The minimum of the bathtub hazard function can be written as follows: Assume two Weibull hazard functions, $h_1(t) = \lambda_1 \gamma_1 t^{\gamma_1 - 1}$, $0 < \gamma_1 < 1$, representing a decreasing hazard, and $h_2(t) = \lambda_2 \gamma_2 t^{\gamma_2 - 1}$, $\gamma_2 > 1$, representing an increasing hazard. The resulting bathtub-shaped hazard function is $h(t) = c \lambda_1 \gamma_1 t^{\gamma_1 - 1} + (1 - c) \lambda_2 \gamma_2 t^{\gamma_2 - 1}$, $0 < c < 1$. The minimum of $(h(t))$ can be obtained by differentiating with respect to (t) : $h'(t) = c \lambda_1 \gamma_1 (\gamma_1 - 1) t^{\gamma_1 - 2} + (1 - c) \lambda_2 \gamma_2 (\gamma_2 - 1) t^{\gamma_2 - 2}$. Setting $h'(t) = 0$, the unique time at which the hazard reaches its minimum is

$$t_0 = \left[\frac{c\lambda_1\gamma_1(1-\gamma_1)}{(1-c)\lambda_2\gamma_2(\gamma_2-1)} \right]^{1/(\gamma_2-\gamma_1)}$$

The stationary point above corresponds to the unique global minimum of the hazard function.

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