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Determining The Correct Number Of Components To Extract From A Principal Components Analysis: A Monte Carlo Study Of The Accuracy Of The Scree Plot.

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This article pertains to the accuracy of the of the scree plot in determining the correct number of components to retain under different conditions of sample size, component loading and variable-to-component ratio. The study employs use of Monte Carlo simulations in which the population parameters were manipulated, and data were generated, and then the scree plot applied to the generated scores.

Key words: Monte Carlo, factor analysis, principal component analysis, scree plot

Introduction

In social science research, one of the decisions that quantitative researchers make is determining the number of components to extract from a given set of data. This is achieved through several factor analytic procedures. The scree plot is one of the most common methods used for determining the number of components to extract. It is available in most statistical software such as the Statistical Software for the Social Sciences (SPSS) and Statistical Analysis Software (SAS).

Factor analysis is a term used to refer to statistical procedures used in summarizing relationships among variables in a parsimonious but accurate manner. It is a generic term that includes several types of analyses, including (a) common factor analysis, (b) principal component analysis (PCA), and (c) confirmatory factor analysis (CFA). According to Merenda, (1997) common factor analysis may be used when a primary goal of the research is to investigate how well a new set of data fits a particular well-established model. On the other hand, Stevens (2002) noted that principal components analysis is usually used to identify the factor structure or model for a set of variables. In contrast; CFA is based on a strong theoretical foundation that allows the researcher to specify an exact model in advance. In this article, principal components analysis is of primary interest.

Principal component analysis

Principal component analysis develops a small set of uncorrelated components based on the scores on the variables. Tabachnick and Fidell (2001) pointed that components empirically summarize the correlations among the variables. PCA is the more appropriate method than CFA if there are no hypotheses about components prior to data collection, that is, it is used for exploratory work.

When one measures several variables, the correlation between each pair of variables can be arranged in a table of correlation coefficients between the variables. The diagonals in the matrix are all 1.0 because each variable theoretically has a perfect correlation with itself. The off-diagonal elements are the correlation coefficients between pairs of variables. The existence of clusters of large correlation coefficients between subsets of variables suggests that those variables are related and could be measuring the same underlying dimension or concept. These underlying dimensions are called components.

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A component is a linear combination of variables; it is an underlying dimension of a set of items. Suppose, for instance a researcher is interested in studying the characteristics of freshmen students. Next, a large sample of freshmen are measured on a number of characteristics like personality, motivation, intellectual ability, family socio-economic status, parents' characteristics, and physical characteristics. Each of these characteristics is measured by a set of variables, some of which are correlated with one another.

An analysis might reveal correlation patterns among the variables that are thought to show the underlying processes affecting the behavior of freshmen students. Several individual variables from the personality trait may combine with some variables from motivation and intellectual ability to yield an independence component. Variables from family socio-economic status might combine with other variables from parents' characteristics to give a family component. In essence what this means is that the many variables will eventually be collapsed into a smaller number of components.

Velicer et. al., (2000) noted that a central purpose of PCA is to determine if a set of p observed variables can be represented more parsimoniously by a set of m derived variables (components) such that $m < p$. In PCA the original variables are transformed into a new set of linear combinations (principal components). Gorsuch (1983) described the main aim of component analysis as to summarize the interrelationships among the variables in a concise but accurate manner. This is often achieved by including the maximum amount of information from the original variables in as few derived components as possible to keep the solution understandable.

Stevens (2002) noted that if we have a single group of participants measured on a set of variables, then PCA partitions the total variance by first finding the linear combination of variables that accounts for the maximum amount of variance. Then the procedure finds a second linear combination, uncorrelated with the first component, such that it accounts for the next largest amount of variance, after removing the variance attributable to the first component from the system. The third principal component is constructed to be uncorrelated with the first two, and accounts for the third largest amount of variance in the system. This process continues until all possible components are constructed. The final result is a set of components that are not correlated with each other in which each derived component accounts for unique variance in the dependent variable.

Uses of principal components analysis

Principal component analysis is important in a number of situations. When several tests are administered to the same examinees, one aspect of validation may involve determining whether there are one or more clusters of tests on which examinees display similar relative performances. In such a case, PCA functions as a validation procedure. It helps evaluate how many dimensions or components are being measured by a test.

Another situation is in exploratory regression analysis when a researcher gathers a moderate to a large number of predictors to predict some dependent variable. If the number of predictors is large relative to the number of participants, PCA may be used to reduce the number of predictors. If so, then the sample size to variable ratio increases considerably and the possibility of the regression equation holding up under cross-validation is much better (Stevens 2002). Here, PCA is used as a variable reduction scheme because the number of simple correlations among the variables can be very large. It also helps in determining if there is a small number of underlying components, which might account for the main sources of variation in such a complex set of correlations. If there are 30 variables or items, 30 different components are probably not being measured. It therefore makes sense to use some variable reduction scheme that will indicate how the variables or items cluster or "hang" together.

The use of PCA on the predictors is also a way of attacking the multicollinearity problem (Stevens, 2002). Multicollinearity occurs when predictors are highly correlated with each other. This is a problem in multiple regression because the predictors account for the same variance in the dependent variable. This redundancy makes the regression model less accurate in as far as the number of predictors required to explain the variance in the dependent variable in a parsimonious way is concerned. This is so because several predictors will have common variance in the dependent variable. The use of PCA creates new components, which are uncorrelated; the order in which they enter the regression equation makes no difference in terms of how much variance in the dependent variable they will account for.

Principal component analysis is also useful in the development of a new instrument. A researcher gathers a set of items, say 50 items designed to measure some construct like attitude toward education, sociability or anxiety. In this situation PCA is used to cluster highly correlated items into components. This helps determine empirically how many components account for most of the variance on an instrument. The original variables in this case are the items on the instrument.

Stevens (2002) pointed out several limitations (e.g., reliability consideration and robustness) of the *k* group MANOVA (Multivariate Analysis of Variance) when a large number of criterion variables are used. He suggests that when there are a large number of potential criterion variables, it is advisable to perform a PCA on them in an attempt to work with a smaller set of new criterion variables.

The scree plot

The scree plot is one of the procedures used in determining the number of factors to retain in factor analysis, and was proposed by Cattell (1966). With this procedure eigenvalues are plotted against their ordinal numbers and one examines to find where a break or a leveling of the slope of the plotted line occurs. Tabachnick and Fidell (2001) referred to the break point as the point where a line drawn through the points changes direction. The number of factors is indicated by the number of eigenvalues above the point of the break. The eigenvalues below the break indicate error variance. An eigenvalue is the amount of variance that a particular variable or component contributes to the total variance. This corresponds to the equivalent number of variables that the component represents. Kachigan, (1991) provided the following explanation: a component associated with an eigenvalue of 3.69 indicates that the component accounts for as much variance in the data collection as would 3.69 variables on average. The concept of an eigenvalue is important in determining the number of components retained in principal component analysis.

The scree plot is an available option in most statistical packages. A major weakness of this procedure is that it relies on visual interpretation of the graph. Because of this, the scree plot has been accused of being subjective. Some authors have attempted to develop a set of rules to help counter the subjectivity of the scree plot. Zoski and Jurs (1990) presented rules for the interpretation of the scree plot. Some of their rules are: (a) the minimum number of break points for drawing the scree plot should be three, (b) when more than one break point exists in the curve, the first one should be used, and (c) the slope of the curve should not approach vertical. Instead, it should have an angle of 40 degrees or less from the horizontal.

Previous studies found mixed results on the accuracy of the scree plot. Zwick and Velicer (1986) noted that “the scree plot had moderate overall reliability when the mean of two trained raters was used” (p.440). Cattell and Jaspers (1967) discovered that the scree plot displayed very good reliability. On the other hand, Crawford and Koopman (1979) reported very poor reliability of the scree plot.

Monte Carlo study

Hutchinson and Bandalos, (1997) pointed that Monte Carlo studies are commonly used to study the behavior of statistical tests and psychometric procedures in situations where the underlying assumptions of a test are violated. They use computer-assisted simulations to provide evidence for problems that cannot be solved mathematically. Robey and Barcikowski (1992) stated that in Monte Carlo simulations, the values of a statistic are observed in many samples drawn from a defined population.

Monte Carlo studies are often used to investigate the effects of assumption violations on statistical tests. Statistical tests are typically developed mathematically using algorithms based on the properties of known mathematical distributions such as the normal distribution. Hutchinson and Bandalos, (1997) further noted that these distributions are chosen because their properties are understood and because in many cases they provide good models for variables of interest to applied researchers. Using Monte Carlo simulations in this study has the advantage that the population parameters are known and can be manipulated; that is, the internal validity of the design is strong although this will compromise the external validity of the results.

According to Brooks et al. (1999), Monte Carlo simulations perform functions empirically through the analysis of random samples from populations whose characteristics are known to the researcher. That is, Monte Carlo methods use computer assisted simulations to provide evidence for problems that cannot be solved mathematically, such as when the sampling distribution is unknown or hypothesis is not true.

Mooney, (1997) pointed that the principle behind Monte Carlo simulation is that the behavior of a statistic in a random sample can be assessed by the empirical process of actually drawing many random samples and observing this behavior. The idea is to create a pseudo-population through mathematical procedures for generating sets of numbers that resemble samples of data drawn from the population.

Mooney (1997) further noted that other difficult aspects of the Monte Carlo design are writing the computer code to simulate the desired data conditions and interpreting the estimated sampling plan, data collection, and data analysis. An important point to note is that a Monte Carlo design takes the same format as a standard research design. This was noted by Brooks et al., (1999) when they wrote “It should be noted that Monte Carlo design is not very different from more standard research design, which typically includes identification of the population, description of the sampling plan, data collection and data analysis” (p. 3).

Methodology

Sample size (n)

Sample size is the number of participants in a study. In this study, sample size is the number of cases generated in the Monte Carlo simulation. Previous Monte Carlo studies by (Velicer et al. 2000, Velicer and Fava, 1998, Guadanoli & Velicer, 1988) found sample size as one of the factors that influences the accuracy of procedures in PCA. This variable had three levels (75, 150 and 225). These values were chosen to cover both the lower and the higher ends of the range of values found in many applied research situations.

Component loading (a_{ij})

Field (2000) defined a component loading as the Pearson correlation between a component and a variable. Gorsuch, 1983 defined it as a measure of the degree of generalizability found between each variable and each component. A component loading reflects a quantitative relationship and the further the component loading is from zero, the more one can generalize from that component to the variable. Velicer and Fava, (1998), Velicer et al., (2000) found the magnitude of the component loading to be one of the factors having the greatest effect on accuracy within PCA. This condition had two levels (.50 and .80). These values were chosen to represent a moderate coefficient (.50) and a very strong coefficient (.80).

Variable-to-component ratio (p:m)

This is the number of variables per component. The number of variables per component will be measured counting the number of variables correlated with each component in the population conditions. The number of variables per component has repeatedly been found to influence the accuracy of the results, with more variables per component producing more stable results. Two levels for this condition were used (8:1 and 4:1). Because the number of variables in this study was fixed at 24, these two ratios yielded three and six variables per factor respectively.

Number of variables

This study set the number of variables a constant at 24, meaning that for the variable-to-component ratio of 4:1, there were six variables loading onto one component, and for variable-to-component ratio of 8:1, eight variables loaded onto a component (see Appendixes A to D).

Generation of population correlation matrices

A pseudo-population is an artificial population from which samples used in Monte Carlo studies are derived. In this study, the underlying population correlation matrices were generated for each possible a_{ij} and p:m combination, yielding a total of four matrices (see Appendixes E to H).

The population correlation matrices were generated in the following manner using RANCORR programme by Hong (1999):

1. The factor pattern matrix was specified based on the combination of values for p:m and a_{ij} (see Appendixes A to D).
2. After specifying the factor pattern matrix and the program is executed, a population correlation matrix was produced for each combination of conditions.
3. The program was executed four times to yield four different population correlation matrices, one correlation matrix for each combination of conditions (see Appendixes E to H).

After the population correlation matrices were generated, the Multivariate Normal Data Generator (MNDG) program (Brooks, 2002) was used to generate samples from the population correlation matrices. This program generated multivariate normally distributed data. A total of 12 cells were created based on the combination of n, p: m and a_{ij}. For each cell, 30 replications were done to give a total of 360 samples, essentially meaning that 360 scree plots were generated. Each of the samples had a pre-determined factor structure since the parameters were set by the researcher. The scree plots were then examined to see if they extracted the exact number of components as set by the researcher.

Interpretation of the scree plots

The scree plots were given to two raters with some experience in interpreting scree plots. These raters were graduate students in Educational Research and Evaluation and had taken a number of courses in Educational Statistics and Measurement.

First, the raters were asked to look at the plots independently to determine the number of components extracted. Second, they were asked to interpret the scree plots together. The raters had no prior knowledge of how many components were built into the data. The accuracy of the scree plot was measured by how many times it extracted the exact number of components.

Results

The first research question of the study is: How accurate is the scree plot in determining the correct number of components? This question was answered in two parts. First, this question was answered by considering the degree of agreement between the two raters. Table 1 is of the measure of agreement between the two raters when component loading was .80. To interpret Table 1, the value of 1 indicates a correct decision and a value of 0 indicates a wrong decision by the raters as they interpreted the scree plots. A correct decision means that the scree plot extracted the correct number of components (either three components for 8:1 ratio or six components for 4:1). Thus, from Table 1, the two raters agreed correctly 108 of the times while they agreed wrongly 52 times.

Table 1. A cross tabulation of the measure of agreement when component loading was .80 between rater 1 and rater 2.

		Rater 2: 0	Rater 2: 1	Total
Rater 1	0	52	11	63
	1	9	108	117
Total		61	119	180

An examination of Figures 1 and 2 show that when component loading was .80, it was relatively clear where the cut-off point was for determining the number of components to extract. Figure 1 clearly shows that six the

Figure 1. The scree plot for variable-to-component ratio of 4:1, component loading of .80

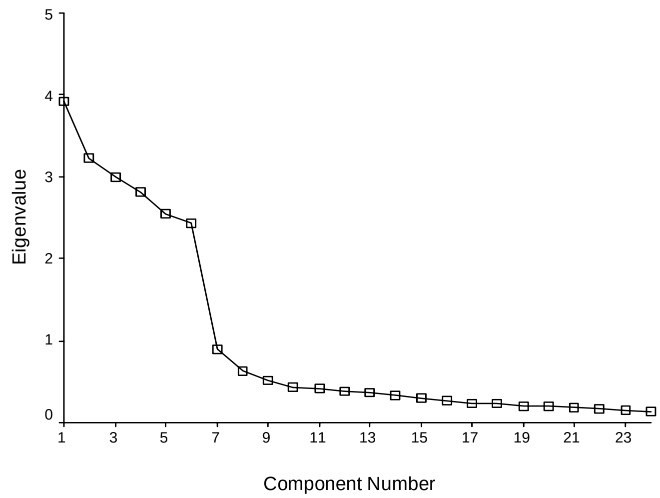
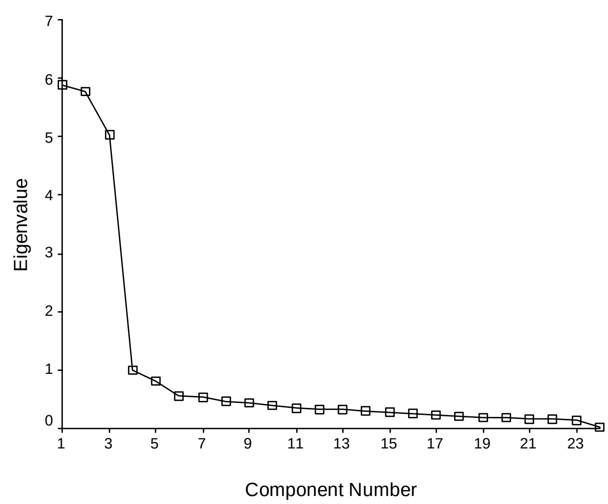


Figure 2. The scree plot for variable-to-component ratio of 8:1, component loading of .80



components were extracted and in Figure 2, three components were extracted. These two plots show why it was easy for the raters to have more agreement for component loading of .80. This was not the case when component loading was .50 as the raters had few cases of agreement and more cases of disagreement.

In Table 2, when component loading was .50, the two raters agreed correctly only 28 times and agreed wrongly 97 times. Compared to component loading of .80, the scree plot was not as accurate when component loading was .50. This finding is consistent with that of Zwick and Velicer (1986) who noted in their study that, "The raters in this study showed greater agreement at higher than at lower component loading levels." (p. 440). Figures 1 and 2 show typical scree plots that were obtained for component loading of .50. In Figure 3, the number of components extracted was supposed to be six, but it is not clear from the plot were the cut-off point is for six components. One can see why there were a lot of disagreements between the two raters when component loading was low. In Figure 4, the plot was supposed to extract three components, but it is not quite clear even to an experienced rater, how many components to be extracted with this plot. These cases show how it is difficult to use the scree plots especially in exploratory studies when the researcher does not know the number of components that exist.

Table 2. A cross tabulation of the measure of agreement when component loading was .50 between rater 1 and rater 2.

		Rater 2		
		0	1	Total
Rater 1	0	97	50	147
	1	5	28	33
Total		102	78	180

Reports of rater reliability on the scree plot have ranged from very good (Cattell & Jaspers, 1967) to quite poor (Crawford & Koopman, 1979). This wide range and the fact that data encountered in real life situations rarely have perfect structure with high component loading makes it difficult to recommend this procedure as a stand-alone procedure for practical uses in determining the number of components. Generally, most real data have low to moderate component loading, which makes the scree plot an unreliable procedure of choice (Zwick & Velicer, 1986).

The second part of question one was to consider the percentages of time that the scree plots were accurate in determining the exact number of components, and those percentages were computed for each cell (see table 5). In Table 5, results of the two raters are presented according to variable-to-component ratio, component loading and sample size. The table shows mixed results of the interpretation of the scree plot by the two raters. However, the scree plot appeared to do well when component loading was high (.80) with a small number of variables (three). When variable-to-component ratio was 8:1 and component loading was .80, the scree plot was very accurate. The lowest performance of the scree plot in this cell was 87% for a sample size of 75. On the other hand, when variable-to-component ratio was 4:1, component loading was .80, and sample size was 225, the scree plot was only accurate 3% of the time with rater 1. With rater 2 under the same conditions, the scree plot was correct 13% of the time.

The second question was: Does the accuracy of the scree plot change when two experienced raters interpret the scree plots together? For this question, percentages were computed of how many times the two raters were correct when they interpreted the scree plots together. The results are presented in table 5 in the row Consensus row. These results show that even if two raters work together, the accuracy of the scree plot does not necessarily improve when component loading was .50. When variable-to-component ratio was 8:1 and component loading was .50, rater 2 was actually better than when the two raters worked together. This is again an example of the mixed results obtained by the scree plot which makes it unreliable. On the other hand, the accuracy of the scree plot improved when component loading was .80, and variable-to-component ratio was 4:1. When component loading was .80, and variable-to-component ratio was 8:1, having two rates work together did not change anything since the scree plot was very accurate when the two raters work independently.

The bottom line is in this study, the scree plot produced mixed results and this is mainly due to its subjectivity. Although it was 100% accurate under certain conditions, it was also terrible under other conditions. It however emerged from this study that the accuracy of the scree plot improves when the component loading is high, and the number of variables per component is few.

Figure 3. The scree plot for variable-to-component ratio of 4:1, component loading of .50

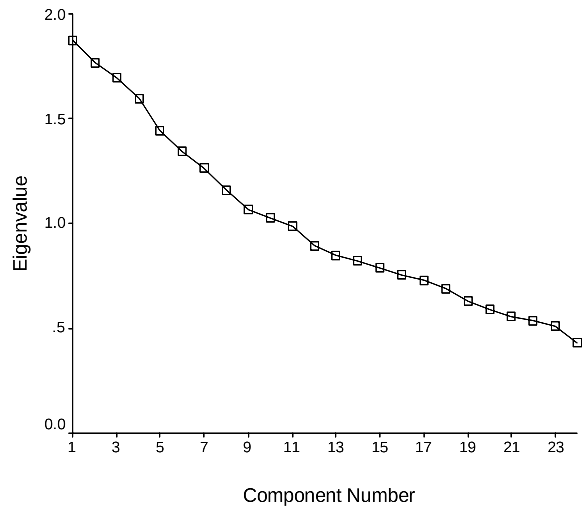


Figure 4. The scree plot for variable-to-component ratio of 8:1, component loading of .50

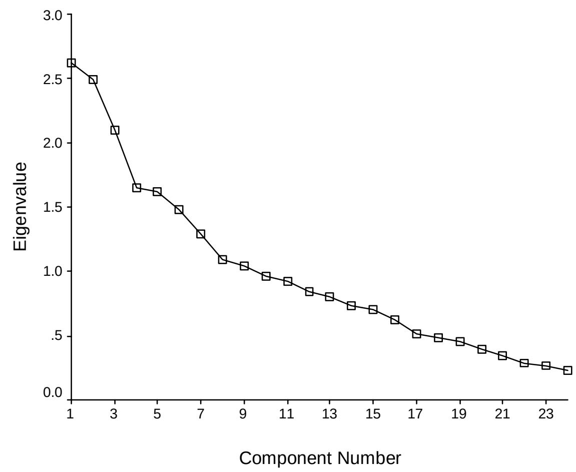


Table 5. Performance of scree plot (as a percentage) under different conditions of variable-to-component ratio, component loading and sample size.

V-C-R	4: 1						8: 1					
Comp.loading	**.50**			**.80**			**.50**			**.80**		
Sample size	75	150	225	75	150	225	75	150	225	75	150	225
Rater 1	73%	20%	10%	10%	16%	3%	33%	13%	27%	87%	100%	100%
Rater 2	67%	10%	16%	26%	23%	13%	63%	57%	77%	100%	100%	100%
Consensus	23%	20%	10%	75%	100%	100%	47%	23%	47%	100%	97%	100%

Conclusion

Generally, the findings of this study are in agreement with previous studies that found mixed results on the scree plot. The subjectivity in the interpretation of the procedure makes it such an unreliable procedure to use as a stand-alone procedure. The scree plot would probably be useful in confirmatory factor analysis to provide a quick check of the factor structure of the data. In that case the researcher already knows the structure of the data as opposed to using it in exploratory studies where the structure of the data is unknown. If used in exploratory factor analysis, the scree plot can be misleading even for experienced researcher because of its subjectivity.

Based on the findings of this study, it is recommended that the scree plot not be used as a stand-alone procedure in determining the number of components to retain. Researchers should use it with other procedures like parallel analysis or Velicer's Minimum Average Partial (MAP) and parallel analysis. In situations where the scree plot is the only procedure available, users should be very cautious in using it and they can do so in confirmatory studies but not exploratory studies.

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Appendix:

Appendix A : Population Pattern Matrix $p:m$ = 8:1 (p = 24, m = 3 a_{ij} = .80).

	Components (*m*)		
p	1	2	3
1	**.80**	.00	.00
2	**.80**	.00	.00
3	**.80**	.00	.00
4	**.80**	.00	.00
5	**.80**	.00	.00
6	**.80**	.00	.00
7	**.80**	.00	.00
8	**.80**	.00	.00
9	.00	**.80**	.00
10	.00	**.80**	.00
11	.00	**.80**	.00
12	.00	**.80**	.00
13	.00	**.80**	.00
14	.00	**.80**	.00
15	.00	**.80**	.00
16	.00	**.80**	.00
17	.00	.00	**.80**
18	.00	.00	**.80**
19	.00	.00	**.80**
20	.00	.00	**.80**
21	.00	.00	**.80**
22	.00	.00	**.80**
23	.00	.00	**.80**
24	.00	.00	**.80**

Appendix B: Population Pattern Matrix p:m = 8:1 (p = 24, m = 3 aij = .50).

	Components (*m*)		
p	1	2	3
1	**.50**	.00	.00
2	**.50**	.00	.00
3	**.50**	.00	.00
4	**.50**	.00	.00
5	**.50**	.00	.00
6	**.50**	.00	.00
7	**.50**	.00	.00
8	**.50**	.00	.00
9	.00	**.50**	.00
10	.00	**.50**	.00
11	.00	**.50**	.00
12	.00	**.50**	.00
13	.00	**.50**	.00
14	.00	**.50**	.00
15	.00	**.50**	.00
16	.00	**.50**	.00
17	.00	.00	**.50**
18	.00	.00	**.50**
19	.00	.00	**.50**
20	.00	.00	**.50**
21	.00	.00	**.50**
22	.00	.00	**.50**
23	.00	.00	**.50**
24	.00	.00	**.50**

Appendix C: Population Pattern Matrix p:m = 4:1 (p = 24, m = 6, aij = .80).

	Components (*m*)					
p	1	2	3	4	5	6
1	**.80**	.00	.00	.00	.00	.00
2	**.80**	.00	.00	.00	.00	.00
3	**.80**	.00	.00	.00	.00	.00
4	**.80**	.00	.00	.00	.00	.00
5	.00	**.80**	.00	.00	.00	.00
6	.00	**.80**	.00	.00	.00	.00
7	.00	**.80**	.00	.00	.00	.00
8	.00	**.80**	.00	.00	.00	.00
9	.00	.00	**.80**	.00	.00	.00
10	.00	.00	**.80**	.00	.00	.00
11	.00	.00	**.80**	.00	.00	.00
12	.00	.00	**.80**	.00	.00	.00
13	.00	.00	.00	**.80**	.00	.00
14	.00	.00	.00	**.80**	.00	.00
15	.00	.00	.00	**.80**	.00	.00
16	.00	.00	.00	**.80**	.00	.00
17	.00	.00	.00	.00	**.80**	.00
18	.00	.00	.00	.00	**.80**	.00
19	.00	.00	.00	.00	**.80**	.00
20	.00	.00	.00	.00	**.80**	.00
21	.00	.00	.00	.00	.00	**.80**
22	.00	.00	.00	.00	.00	**.80**
23	.00	.00	.00	.00	.00	**.80**
24	.00	.00	.00	.00	.00	**.80**

Appendix D: Population Pattern Matrix p:m = 4:1 (p = 24, m = 6, aij = .50).

	Components (*m*)					
p	1	2	3	4	5	6
1	**.50**	.00	.00	.00	.00	.00
2	**.50**	.00	.00	.00	.00	.00
3	**.50**	.00	.00	.00	.00	.00
4	**.50**	.00	.00	.00	.00	.00
5	.00	**.50**	.00	.00	.00	.00
6	.00	**.50**	.00	.00	.00	.00
7	.00	**.50**	.00	.00	.00	.00
8	.00	**.50**	.00	.00	.00	.00
9	.00	.00	**.50**	.00	.00	.00
10	.00	.00	**.50**	.00	.00	.00
11	.00	.00	**.50**	.00	.00	.00
12	.00	.00	**.50**	.00	.00	.00
13	.00	.00	.00	**.50**	.00	.00
14	.00	.00	.00	**.50**	.00	.00
15	.00	.00	.00	**.50**	.00	.00
16	.00	.00	.00	**.50**	.00	.00
17	.00	.00	.00	.00	**.50**	.00
18	.00	.00	.00	.00	**.50**	.00
19	.00	.00	.00	.00	**.50**	.00
20	.00	.00	.00	.00	**.50**	.00
21	.00	.00	.00	.00	.00	**.50**
22	.00	.00	.00	.00	.00	**.50**
23	.00	.00	.00	.00	.00	**.50**
24	.00	.00	.00	.00	.00	**.50**

Appendix E. Population correlation matrix $p{:}m$ = 8: 1 (p = 24, m = 3, a_{ij} = .80).

V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11	V12	V13	V14	V15	V16	V17	V18	V19	V20	V21	V22	V23	V24
1.00																							
.620	1.00																						
.569	.674	1.00																					
.613	.697	.698	1.00																				
.562	.679	.750	.732	1.00																			
.626	.655	.708	.711	.740	1.00																		
.706	.606	.565	.579	.542	.585	1.00																	
.686	.602	.579	.568	.543	.573	.716	1.00																
-.04	-.01	.035	-.02	.018	-.02	.000	.019	1.00															
-.01	-.02	.030	.003	.041	.064	-.02	-.04	.637	1.00														
-.02	.006	.049	.046	.066	.058	-.05	-.02	.644	.632	1.00													
-.07	.012	.085	.041	.108	.054	-.07	-.07	.682	.669	.681	1.00												
-.05	.024	.058	.039	.068	.031	-.07	-.04	.653	.623	.700	.693	1.00											
-.05	.020	.102	.107	.157	.148	-.09	-.11	.630	.733	.701	.738	.674	1.00										
-.04	.002	.032	.005	.024	.011	-.05	-.01	.648	.636	.681	.654	.692	.638	1.00									
.000	-.02	-.02	-.04	-.04	-.06	.034	.046	.679	.582	.638	.642	.640	.564	.625	1.00								
.033	.018	-.03	.004	-.04	-.01	.000	.021	-.04	-.04	.027	-.05	.017	-.05	.030	-.01	1.00							
-.03	.011	.001	-.02	-.01	-.06	.011	.013	.045	-.04	-.03	.020	.000	-.05	-.02	.053	.608	1.00						
.002	.033	-.04	-.02	-.06	-.09	.015	.030	.000	-.09	-.02	-.04	.017	-.13	.012	.054	.683	.695	1.00					
.06	-.01	-.04	.001	-.04	.028	.033	.021	-.05	.015	.004	-.05	-.03	.002	-.02	-.03	.675	.588	.610	1.00				
-.04	.038	.034	.046	.060	.017	-.05	-.07	-.01	.006	-.01	.051	.024	.051	-.01	-.02	.608	.663	.648	.608	1.00			
-.03	.015	.000	-.03	-.03	-.07	-.02	.009	.015	-.04	-.01	.000	.036	-.10	.044	.024	.670	.678	.727	.604	.656	1.00		
-.02	-.02	.040	.013	,057	.074	-.02	-.04	.003	.087	.000	.045	-.01	.109	-.01	-.05	.578	.609	.540	.644	.658	.580	1.00	
-.02	.027	.021	.053	.044	.041	-.05	-.05	-.03	.021	.026	.007	.025	.061	.031	-.06	.666	.602	.613	.656	.667	.640	.649	1.00

Appendix F : Population correlation matrix $p{:}m$ = 8:1 (p= 24, m= 3, a_{ij} = .50)

V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11	V12	V13	V14	V15	V16	V17	V18	V19	V20	V21	V22	V23	V24
1.00																							
.271	1.00																						
.281	.303	1.00																					
.184	.297	.275	1.00																				
.271	.271	.276	.243	1.00																			
.233	.293	.287	.303	.239	1.00																		
.280	.226	.250	.213	.250	.199	1.00																	
.190	.236	.229	.296	.233	.242	.218	1.00																
-.03	-.03	-.02	.005	-.03	.000	-.03	.054	1.00															
-.03	-.04	-.04	-.02	.000	-.05	.007	.041	.276	1.00														
-.04	.065	.038	.096	.008	.072	-.09	.039	.262	.224	1.00													
-.04	-.03	-.03	.016	-.05	-.03	.025	.049	.293	.283	.234	1.00												
.018	-.05	-.02	-.06	-.02	-.03	.077	-.05	.228	.244	.149	.253	1.00											
.061	.001	.027	-.04	-.02	.000	.055	-.05	.229	.204	.191	.250	.302	1.00										
.000	.034	.040	.041	.013	.023	.037	-.03	.215	.207	.260	.245	.269	.287	1.00									
-.02	-.02	-.05	-.02	.000	-.02	-.05	.025	.265	.289	.262	.246	.219	.193	.189	1.00								
.029	.002	.017	-.02	.024	-.02	.075	-.05	-.05	-.08	-.06	-.01	.062	.057	.052	-.05	1.00							
-.09	.026	-.01	103	-.01	.052	-.09	.061	.010	.000	.110	.002	-.07	-.09	.000	.050	.190	1.00						
.083	.000	.025	-.07	-.02	.010	.031	-.08	-.01	-.04	-.05	-.02	.052	.088	.006	-.03	.283	.152	1.00					
.000	.027	.000	.016	.026	.011	-.08	.031	.013	.020	.071	-.03	-.09	-.07	-.05	.060	.185	.310	.208	1.00				
.050	-.05	-.01	-.08	-.02	-.06	.105	-.03	.011	.014	-.12	.043	.087	.084	.008	-.05	.308	.115	.321	.164	1.00			
-.04	.005	-.03	.017	.000	.026	-.09	.029	.013	.019	.062	-.02	-.07	-.06	-.05	.075	.185	.338	.204	.326	.153	1.00		
.018	-.06	-.04	-.07	.007	-.06	.040	.008	.018	.045	-.08	.021	.032	.015	-.04	.015	.262	.180	.259	.235	.325	.233	1.00	
-.09	.019	-.02	.106	.000	.025	.057	.072	.008	.009	.085	.027	.058	-.07	.018	.029	.219	.400	.138	.280	.150	.305	.196	1.00

Appendix G: Population correlation matrix $p{:}m$ = 4:1 (p = 24, m = 6, a_{ij} = .8).

V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11	V12	V13	V14	V15	V16	V17	V18	V19	V20	V21	V22	V23	V24
1.00																							
.662	1.00																						
.686	.688	1.00																					
.593	.659	.606	1.00																				
-.01	-.07	-.03	-.04	1.00																			
.046	.017	.054	-.05	.647	1.00																		
.051	-.08	-.01	-.08	.713	.645	1.00																	
.020	-.09	-.04	-.04	.694	.617	.776	1.00																
-.02	-.03	-.01	-.03	.016	.012	.004	.000	1.00															
.046	.012	.029	-.05	.004	.060	.011	-.01	.658	1.00														
.030	-.03	.035	-.06	.017	.053	.038	.014	.675	.685	1.00													
.011	.098	.053	.046	-.09	.013	-.11	-.09	.591	.636	.612	1.00												
.029	-.07	.006	-.05	.074	.002	.137	.105	-.02	-.02	.033	-.07	1.00											
-.01	.027	.001	.067	-.06	-.03	-.06	-.04	-.04	-.04	-.04	.062	.600	1.00										
.017	.000	.045	-.01	-.03	.033	-.02	-.02	.012	.020	.041	.021	.632	.665	1.00									
-.04	.081	.034	.043	-.06	.025	-.17	-.14	.016	.015	-.01	.097	.510	.676	.676	1.00								
.005	.046	.057	.022	-.08	.038	-.10	-.08	.000	.022	.029	.091	-.06	.050	.073	.110	1.00							
.005	-.07	-.01	-.06	.064	.032	.063	.050	.052	.030	.059	-.08	.048	-.07	.003	-.04	.617	1.00						
-.01	.045	.003	.062	-.08	-.04	-.06	-.03	-.04	-.03	-.04	.070	-.03	.070	.004	.030	.690	.572	1.00					
.006	-.02	-.02	.028	.014	-.01	.002	.022	-.04	.003	-.02	.006	.004	.030	.000	-.02	.644	.634	.655	1.00				
-.06	.010	-.03	.061	-.04	-.03	-.10	-.05	.023	-.02	-.03	.027	-.10	.046	.016	.085	.046	-.03	.028	.007	1.00			
-.06	-.02	-.06	.048	.007	-.04	-.04	-.01	.017	-.03	-.04	-.02	-.05	.027	-.01	.024	-.02	-.01	.006	.017	.706	1.00		
.007	.000	.001	-.04	.008	.000	.054	.034	.028	-.01	.011	-.04	.035	-.05	-.02	-.05	-.05	.017	-.03	-.06	.603	.617	1.00	
-.05	.003	.006	.013	-.01	.015	-.09	-.07	.043	.005	.019	.019	-.06	.004	.033	.100	.065	.015	-.01	-.02	.703	.668	.621	1.00

Appendix H: Population matrix $p{:}\ m$ = 4:1 (p = 24, m = 6, a_{ij} = .50).

V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11	V12	V13	V14	V15	V16	V17	V18	V19	V20	V21	V22	V23	V24
1.00																							
.181	1.00																						
.322	.167	1.00																					
.248	.236	.210	1.00																				
-.04	.069	-.06	.005	1.00																			
.042	-.07	.037	.037	.192	1.00																		
.057	-.04	.077	-.06	.230	.254	1.00																	
.014	-.03	-.03	.061	.243	.301	.214	1.00																
.070	-.06	.045	.005	-.03	.034	.052	.014	1.00															
.049	-.07	.101	-.04	-.06	.039	.074	-.03	.295	1.00														
.063	-.05	.046	-.01	-.03	.031	.054	.023	.314	.297	1.00													
.055	-.06	.030	.011	-.05	.052	.025	.046	.295	.287	.303	1.00												
.011	-.03	-.01	.030	.000	.044	.003	.037	.010	.032	.001	.030	1.00											
-.07	.083	-.10	.004	.076	-.06	-.04	-.02	-.05	-.06	-.06	-.06	.255	1.00										
-.02	-.01	.073	-.06	-.03	-.01	.029	-.07	-.03	.082	-.02	-.02	.237	.231	1.00									
.062	-.07	.077	.001	-.04	.039	.056	-.01	.054	.084	.043	.037	.276	.191	.268	1.00								
-.03	.044	-.01	-.04	.028	-.05	.003	-.03	-.02	-.02	.002	-.03	-.05	.016	.018	-.04	1.00							
-.05	.029	-.07	.054	.048	.000	-.07	.028	-.04	-.04	-.06	-.03	.038	.064	-.02	-.04	.224	1.00						
.066	-.05	.061	-.04	-.05	.017	.075	-.02	.060	.055	.061	.044	-.01	-.06	.004	.056	.243	.171	1.00					
.035	-.04	.003	.065	-.01	.037	-.02	.044	.042	-.01	.025	.026	.010	-.03	-.06	.030	.221	.256	.253	1.00				
.043	-.03	.102	-.08	-.05	-.01	.067	-.06	.022	.07	.042	.017	-.06	-.07	.082	.034	.036	-.10	.074	-.03	1.00			
.026	-.04	.040	.015	-.02	.025	.028	.000	.032	.08	.024	.014	.029	-.02	.023	.067	-.02	-.01	.018	.024	.254	1.00		
-.08	.093	-.15	.063	.075	-.04	-.10	.058	-.06	-.15	-.05	-.04	-.02	.090	-.10	-.10	.032	.073	-.09	.016	.134	.186	1.00	
-.05	.043	-.08	.064	.039	-.01	-.10	.048	-.05	-.10	-.05	-.03	-.01	.038	-.05	-.07	.010	.064	-.08	.025	.178	.207	.375	1.00