Journal of Modern Applied Statistical Methods

Volume 4 | Issue 1 Article 18

5-1-2005

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Vincent A. R. Camara
University of South Florida, gvcamara@ij.net

Chris P. Tsokos
University of South Florida, profcpt@cas.usf.edu

Recommended Citation
Camara, Vincent A. R. and Tsokos, Chris P. (2005) "Bayesian Reliability Modeling Using Monte Carlo Integration," *Journal of Modern Applied Statistical Methods*: Vol. 4 : Iss. 1 , Article 18.
DOI: 10.22237/jmasm/1114906680

Journal of Modern Applied Statistical Methods
May, 2005, Vol. 4, No.1, 172-186

1538 – 9472/05/$95.00

Bayesian Reliability Modeling Using Monte Carlo Integration

Vincent A. R. Camara Chris P. Tsokos
Department of Mathematics
University of South Florida

The aim of this article is to introduce the concept of Monte Carlo Integration in Bayesian estimation and Bayesian reliability analysis. Using the subject concept, approximate estimates of parameters and reliability functions are obtained for the three-parameter Weibull and the gamma failure models. Four different loss functions are used: square error, Higgins-Tsokos, Harris, and a logarithmic loss function proposed in this article. Relative efficiency is used to compare results obtained under the above mentioned loss functions.

Key words: Estimation, loss functions, Monte Carlo Integration, Monte Carlo Simulation, reliability functions, relative efficiency.

Introduction

In this article, the concept of Monte Carlo Integration (Berger, 1985) is used to obtain approximate estimates of the Bayes rule that is ultimately used to derive estimates of the reliability function. Monte Carlo Integration is used to first obtain approximate Bayesian estimates of the parameter inherent in the failure model, and using this estimate directly, obtain approximate Bayesian estimates of the reliability function. Secondly, the subject concept is used to directly obtain Bayesian estimates of the reliability function.

In the present modeling effort, the three-parameter Weibull and the gamma failure models are considered, that are respectively defined as follows:

$$f(x;a,b,c)=\frac{c}{b}(x-a)^{c-1}e^{-\frac{(x-a)^c}{b}},$$
$$x \geq a; b,c > 0 \tag{1}$$

where a, b and c are respectively the location, scale and shape parameters;

and

$$g(x;\alpha,\beta) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{-\frac{x}{\beta}}, \tag{2}$$

where α and β are respectively the shape and scale parameters.

For these two failure models, consider the scale parameters b and β to behave as random variables that follow the lognormal distribution which is given by

$$\pi(\theta) = \frac{1}{\theta\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{Ln(\theta)-\mu}{\sigma}\right]^2}, \tag{3}$$

For each of the above underlying failure models, approximate Bayesian estimates will be obtained for the subject parameter and the reliability function with the squared error, the Higgins-Tsokos, the Harris, and a proposed logarithmic loss functions. The loss functions

Vincent A. R. Camara earned a Ph.D. in Mathematics/Statistics. His research interests include the theory and applications of Bayesian and empirical Bayes analyses with emphasis on the computational aspect of modeling. Chris P. Tsokos is a Distinguished Professor of Mathematics and Statistics. His research interests are in statistical analysis and modeling, operations research, reliability analysis-ordinary and Bayesian, time series analysis.

along with a statement of their key characteristics are given below.

Square error loss function

The popular square error loss function places a small weight on estimates near the true value and proportionately more weight on extreme deviation from the true value of the parameter. Its popularity is due to its analytical tractability in Bayesian reliability modeling. The squared error loss is defined as follows:

$$L_{SE}(\hat{R}, R) = \left(\hat{R} - R\right)^2 \qquad (4)$$

Higgins-Tsokos loss function

The Higgins-Tsokos loss function places a heavy penalty on extreme over-or underestimation. That is, it places an exponential weight on extreme errors. The Higgins-Tsokos loss function is defined as follows:

$$L_{HT}(\hat{R}, R) = \frac{f_1 e^{f_2(\hat{R}-R)} + f_2 e^{-f_1(\hat{R}-R)}}{f_1 + f_2} - 1,$$

$$f_1, f_2 \succ 0.$$

Harris loss function

The Harris loss function is defined as follows:

$$L_H(\hat{R}, R) = \left| \frac{1}{1-\hat{R}} - \frac{1}{1-R} \right|^k, k \succ 0. \qquad (6)$$

To our knowledge, the properties of the Harris loss function have not been fully investigated. However it is based on the premises that if the system is 0.99 reliable then on the average it should fail one time in 100, whereas if the reliability is 0.999 it should fail one time in 1000. Thus, it is ten times as good.

Logarithmic loss function

The logarithmic loss function characterizes the strength of the loss logarithmically, and offers useful analytical tractability. This loss function is defined as:

$$L_{Ln}(\hat{R}, R) = \left| Ln\left(\frac{\hat{R}}{R}\right) \right|^l, l \succ 0. \qquad (7)$$

It places a small weight on estimates whose ratios to the true value are close to one, and proportionately more weight on estimates whose ratios to the true value are significantly different from one. $R(t)$ and $\hat{R}(t)$ represent respectively the true reliability function and its estimate.

Methodology

Considering the fact that the reliability of a system at a given time t is the probability that the system fails at a time greater or equal to t, the reliability function corresponding to the three-parameter Weibull failure model is given by

$$R(t) = e^{-\frac{(t-a)^c}{b}}, \qquad (8)$$

and for the gamma failure model

$$R(t) = 1 - \frac{\gamma(\alpha, \frac{t}{\beta})}{\Gamma(\alpha)}, t > 0, \alpha, > 0 . \qquad (9)$$

where $\gamma(l_1, l_2)$ denotes the incomplete gamma function. When α is an integer, equation (9) becomes

$$R(t) = \left(\sum_{i=0}^{\alpha-1} \frac{1}{i!}\left(\frac{t}{\beta}\right)^i\right) e^{-\frac{t}{\beta}},$$

and in particular when $\alpha = 1$

$$R(t) = e^{-\frac{t}{\beta}}, \quad t > 0 .$$

Consider the situation where there are *m* independent random variables $X_1, X_2, \ldots, X_m$ with the same probability density function $dF(x|\theta)$, and each of them having *n* realizations, that is,

$X_1 : x_{11}, x_{21}, \ldots, x_{n1}$;
$X_2 : x_{12}, x_{22}, \ldots, x_{n2}$; ……. ;
$X_m : x_{1m}, x_{2m}, \ldots, x_{nm}$

The minimum variance unbiased estimate, MVUE, $\hat{\theta}_j$ of the parameter θ_j is obtained from the *n* realizations $x_{1j}, x_{2j}, \ldots x_{nj}$, where j = 1,...m.

Repeating this independent procedure *k* times, a sequence of MVUE is obtained for the θ_j's, that is, $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m$. Using the $\hat{\theta}_j$'s and their common probability density function, approximate Bayesian reliability estimates are obtained.

Let $L(x;\theta)$, $g(\theta)$, $\pi(\theta)$ and $h(\theta)$ represent respectively the likelihood function, a function of θ, a prior distribution of θ and a probability density function of θ called the importance function. Using the strong law of large numbers, [7], write

$$\int_{\Theta} g(\theta)L(x;\theta)\pi(\theta)d\theta = E^h\left[\frac{g(\theta)L(x;\theta)\pi(\theta)}{h(\theta)}\right]$$

$$= \lim_{m\to\infty} \sum_{i=1}^{m} \frac{g(\hat{\theta}_i)L(x;\hat{\theta}_i)\pi(\hat{\theta}_i)}{h(\hat{\theta}_i)} \quad (10).$$

Note that E^h represents the expectation with respect to the probability density function *h*, and $g(\theta)$ is any function of θ which assures convergence of the integral; also, $h(\theta)$ mimics the posterior density function.

For the special case where $g(\theta) = 1$, equation (10) yields

$$\int_{\Theta} L(x;\theta)\pi(\theta)d\theta = \lim_{m\to\infty} \sum_{i=1}^{m} \frac{L(x;\hat{\theta}_i)\pi(\hat{\theta}_i)}{h(\hat{\theta}_i)} \quad (11)$$

Equations (10) and (11) imply that the posterior expected value of $g(\theta)$ is given by

$$E\left(g(\theta)\,|\,x\right) = \frac{\int_{\Theta} g(\theta)L(x;\theta)\pi(\theta)d\theta}{\int_{\Theta} L(x;\theta)\pi(\theta)d\theta}$$

$$= \lim_{m\to\infty} \frac{\sum_{i=1}^{m} \frac{g(\hat{\theta}_i)L(x;\hat{\theta}_i)\pi(\hat{\theta}_i)}{h(\hat{\theta}_i)}}{\sum_{i=1}^{m} \frac{L(x;\hat{\theta}_i)\pi(\hat{\theta}_i)}{h(\hat{\theta}_i)}} \quad (12)$$

This approach is used to obtain approximate Bayesian estimates of $g(\theta)$, for the different loss functions under study. Approximate Bayesian estimates of the parameter θ and the reliability are then obtained by replacing $g(\theta)$ by θ and *R(t)* respectively in the derived expressions corresponding to the approximate Bayesian estimates of $g(\theta)$.

The Bayesian estimates used to obtain approximate Bayesian estimates of the function $g(\theta)$ are the following when the squared error, the Higgins-Tsokos, the Harris and the proposed logarithmic loss functions are used:

$$\hat{g(\theta)}_{B(SE)} = \frac{\int_{\Theta} g(\theta)L(x;\theta)\pi(\theta)d\theta}{\int_{\Theta} L(x;\theta)\pi(\theta)d\theta}$$

$$\hat{g(\theta)}_{B(HT)} = \frac{1}{f_1 + f_2} Ln\left(\frac{\int_{\Theta} e^{f_1 g(\theta)} L(x;\theta)\pi(\theta)d\theta}{\int_{\Theta} e^{-f_2 g(\theta)} L(x;\theta)\pi(\theta)d\theta}\right),$$

$f_1, f_2 \succ 0.$

$$\hat{g(\theta)}_{B(H)} = \frac{\int_{\Theta} \frac{g(\theta)}{1-g(\theta)} L(x;\theta)\pi(\theta)d\theta}{\int_{\Theta} \frac{1}{1-g(\theta)} L(x;\theta)\pi(\theta)d\theta}$$

$$\hat{g}(\theta)_{B(Ln)} = e^{\frac{\int_{\Theta} Ln(g(\theta))L(x;\theta)\pi(\theta)d\theta}{\int_{\Theta} L(x;\theta)\pi(\theta)d\theta}} \quad (13).$$

Using equation (12) and the above Bayesian decision rules, approximate Bayesian estimates of $g(\theta)$ corresponding respectively to the squared error, the Higgins-Tsokos, the Harris and the proposed logarithmic loss functions are respectively given by the following expressions when *m* replicates are considered.

$$\hat{g}(\theta)_{E(SE)} = \frac{\sum_{i=1}^{m} \frac{g(\hat{\theta}_i)L(x;\hat{\theta}_i)\pi(\hat{\theta}_i)}{h(\hat{\theta}_i)}}{\sum_{i=1}^{m} \frac{L(x;\hat{\theta}_i)\pi(\hat{\theta}_i)}{h(\hat{\theta}_i)}} \quad (14)$$

$$\hat{g}(\theta)_{E(HT)} = \frac{1}{f_1 + f_2} Ln\left(\frac{\sum_{i=1}^{m} e^{f_1 g(\hat{\theta}_i)} \frac{L(x;\hat{\theta}_i)\pi(\hat{\theta}_i)}{h(\hat{\theta}_i)}}{\sum_{i=1}^{m} e^{-f_2 g(\hat{\theta}_i)} \frac{L(x;\hat{\theta}_i)\pi(\hat{\theta}_i)}{h(\hat{\theta}_i)}} \right) \quad (15)$$

$f_1, f_2 \succ 0.$

$$\hat{g}(\theta)_{E(H)} = \frac{\sum_{i=1}^{m} \frac{g(\hat{\theta}_i)}{1-g(\hat{\theta}_i)} \frac{L(x;\hat{\theta}_i)\pi(\hat{\theta}_i)}{h(\hat{\theta}_i)}}{\sum_{i=1}^{m} \frac{1}{1-g(\hat{\theta}_i)} \frac{L(x;\hat{\theta}_i)\pi(\hat{\theta}_i)}{h(\hat{\theta}_i)}}, \; g(\hat{\theta}_i) \neq 1 \quad , \quad (16)$$

and

$$\hat{g}(\theta)_{E(Ln)} = e^{\frac{\sum_{i=1}^{m} \frac{Ln(g(\hat{\theta}_i))L(x;\hat{\theta}_i)\pi(\hat{\theta}_i)}{h(\hat{\theta}_i)}}{\sum_{i=1}^{m} \frac{L(x;\hat{\theta}_i)\pi(\hat{\theta}_i)}{h(\hat{\theta}_i)}}} . \quad (17)$$

First, use the above general functional forms of the Bayesian estimates of $g(\theta)$ to obtain approximate Bayesian estimates of the random parameter inherent in the underlying failure model. Furthermore, these estimates are used to obtain approximate Bayesian reliability estimates. Second, use the above functional forms to directly obtain approximate Bayesian estimates of the reliability function.

Three-parameter Weibull underlying failure model

In this case the parameter θ, discussed above, will correspond to the scale parameter *b*. The location and shape parameters *a* and *c* are considered fixed. The likelihood function corresponding to *n* independent random variables following the three-parameter Weibull failure model is given by

$$L_1(x, a, c; b) = e^{-\frac{1}{b}S_n - nLn(b)} \bullet e^{(c-1)\sum_{i=1}^{n} Ln(x_i - a) + nLn(c)} \quad (18)$$

where $S_n = \sum_{i=1}^{n} (x_i - a)^c$.

Furthermore, it can be shown that S_n is a sufficient statistic for the parameter *b*, and a minimum variance unbiased estimator of *b* is given by

$$\hat{b} = \frac{\sum_{i=1}^{n} (x_i - a)^c}{n} .$$

The probability density function of $Y=(X-a)^c$, where X follows the Weibull probability density function, is

$$p(y\mid b)=\frac{c}{b}\left(y^{\frac{1}{c}}\right)^{c-1} e^{-\frac{1}{b}y}\left(\frac{1}{c}\,y^{\frac{1}{c}-1}\right)$$

$$= \quad \frac{1}{b}e^{-\frac{1}{b}y}\,,\ y>0\,,b>0\ . \qquad (19)$$

The moment generating function of Y is given by

$$E(e^{\mu y})=\frac{1}{b}\int_0^{\infty} e^{-y(\frac{1}{b}-\mu)}dy$$

$$= \quad (1-\mu b)^{-1} \qquad (20)$$

Using equation (20) and the fact that the X_i's are independent, the moment generating function of the minimum variance unbiased estimator of the parameter b is

$$E(e^{\mu\hat{b}})=\prod_{i=1}^{n} E\left(e^{\frac{\mu}{n}(x_i-a)^c}\right).$$

$$= \quad \left(1-\mu\frac{b}{n}\right)^{-n} \qquad (21)$$

Equation (21) corresponds to the moment generating function of the gamma distribution $G(n,\frac{b}{n})$. Thus, the conditional probability density function of the MVUE of b is given by

$$h_1(\hat{b},a,c\mid b)=\frac{n^n}{\Gamma(n)b^n}\left(\hat{b}\right)^{n-1} e^{-\frac{n\hat{b}}{b}}\,,\hat{b}>0\,,b>0\ . \qquad (22)$$

Approximate Bayesian estimates for the scale parameter b and the reliability function $R(t)$ are obtained, with the use of equations (18) and (22), by replacing respectively $g(b)$ by b and $R(t)$ in equations (14), (15), (16) and (17). The $\hat{b}_i$'s that are minimum variance unbiased estimates of the scale parameter b will play the role of the $\hat{\theta}_i$'s.

Considering the lognormal prior, equations (14), (15), (16) and (17) yield the following approximate Bayesian estimates of the scale parameter b corresponding respectively to the squared error, the Higgins-Tsokos, the Harris and our proposed lognormal loss functions, after replacing b_i by $\hat{b}_i$ in the expression of $h_1(\hat{b}_i)$:

$$\hat{b}_{E(SE)} = \frac{\sum_{j=1}^{m}\hat{b}_j\, e^{-\frac{S_n}{\hat{b}_j}-nLn(\hat{b}_j)+(c-1)\sum_{i=1}^{n}Ln(x_i-a)-\frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} e^{-\frac{S_n}{\hat{b}_j}-nLn(\hat{b}_j)+(c-1)\sum_{i=1}^{n}Ln(x_i-a)-\frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}\,, \qquad (23)$$

$$\hat{b}_{E(HT)} = \frac{1}{f_1+f_2}\,Ln\left(\frac{\sum_{j=1}^{m} e^{f_1\hat{b}_j-\frac{S_n}{\hat{b}_j}-nLn(\hat{b}_j)+(c-1)\sum_{i=1}^{n}Ln(x_i-a)-\frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} e^{-f_2\hat{b}_j-\frac{S_n}{\hat{b}_j}-nLn(\hat{b}_j)+(c-1)\sum_{i=1}^{n}Ln(x_i-a)-\frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}\right),$$

$$f_1,\ f_2 \succ 0 \qquad (24)$$

$$\hat{b}_{E(H)} = \frac{\sum_{j=1}^{m}\frac{\hat{b}_j}{1-\hat{b}_j}\, e^{-\frac{S_n}{\hat{b}_j}-nLn(\hat{b}_j)+(c-1)\sum_{i=1}^{n}Ln(x_i-a)-\frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m}\frac{1}{1-\hat{b}_j}\, e^{-\frac{S_n}{\hat{b}_j}-nLn(\hat{b}_j)+(c-1)\sum_{i=1}^{n}Ln(x_i-a)-\frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}\,,$$

$$\hat{b}_{j\neq 1} \qquad (25)$$

and

$$\hat{b}_{E(Ln)} = e^{\frac{\sum_{j=1}^{m} Ln(\hat{b}_j) e^{-\frac{S_n}{\hat{b}_j} - nLn(\hat{b}_j) + (c-1)\sum_{i=1}^{n} Ln(x_i - a) - \frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} e^{-\frac{S_n}{\hat{b}_j} - nLn(\hat{b}_j) + (c-1)\sum_{i=1}^{n} Ln(x_i - a) - \frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}} \quad (26)$$

The approximate Bayesian estimates of the reliability corresponding to the first method are therefore given by

$$\tilde{\tilde{R}}_{Eb}(t, a, c | \hat{b}_E) = e^{-\frac{1}{\hat{b}_E}(t-a)^c} \quad t > a, \quad (27)$$

where $\hat{b}_E$ stands respectively for the above approximate Bayesian estimates of the scale parameter *b*.

Approximate Bayesian reliability estimates corresponding to the second method are also derived by replacing $g(\theta)$ by *R(t)* in equations (14), (15), (16) and (17). The obtained estimates corresponding respectively to the squared error, the Higgins-Tsokos, the Harris and the proposed logarithmic loss functions are respectively given by the following expressions, after replacing b_i by $\hat{b}_i$ in the expression of $h_1(\hat{b}_i)$:

$$\hat{R}_{E(SE)}(t) = \frac{\sum_{j=1}^{m} e^{-\frac{(t-a)^c}{\hat{b}_j} - \frac{S_n}{\hat{b}_j} - nLn(\hat{b}_j) + (c-1)\sum_{i=1}^{n} Ln(x_i - a) - \frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} e^{-\frac{S_n}{\hat{b}_j} - nLn(\hat{b}_j) + (c-1)\sum_{i=1}^{n} Ln(x_i - a) - \frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}, \quad (28)$$

$$\hat{R}_{E(HT)}(t) = \frac{1}{f_1 + f_2} Ln \left(\frac{\sum_{j=1}^{m} e^{f_1 e^{-\frac{(t-a)^c}{\hat{b}_j}} - \frac{S_n}{\hat{b}_j} - nLn(\hat{b}_j) + (c-1)\sum_{i=1}^{n} Ln(x_i - a) - \frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} e^{-f_2 e^{-\frac{(t-a)^c}{\hat{b}_j}} - \frac{S_n}{\hat{b}_j} - nLn(\hat{b}_j) + (c-1)\sum_{i=1}^{n} Ln(x_i - a) - \frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}} \right),$$

$$f_1, f_2 \succ 0, \quad (29)$$

$$\hat{R}_{E(H)}(t) = \frac{\sum_{j=1}^{m} \frac{e^{-\frac{(t-a)^c}{\hat{b}_j}}}{1 - e^{-\frac{(t-a)^c}{\hat{b}_j}}} e^{-\frac{S_n}{\hat{b}_j} - nLn(\hat{b}_j) + (c-1)\sum_{i=1}^{n} Ln(x_i - a) - \frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} \frac{1}{1 - e^{-\frac{(t-a)^c}{\hat{b}_j}}} e^{-\frac{S_n}{\hat{b}_j} - nLn(\hat{b}_j) + (c-1)\sum_{i=1}^{n} Ln(x_i - a) - \frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}, \quad (30)$$

and

$$\hat{R}_{E(Ln)}(t) = e^{\frac{\sum_{j=1}^{m} -\frac{(t-a)^c}{\hat{b}_j} e^{-\frac{S_n}{\hat{b}_j} - nLn(\hat{b}_j) + (c-1)\sum_{i=1}^{n} Ln(x_i - a) - \frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} e^{-\frac{S_n}{\hat{b}_j} - nLn(\hat{b}_j) + (c-1)\sum_{i=1}^{n} Ln(x_i - a) - \frac{1}{2}\left(\frac{Ln(\hat{b}_j)-\mu}{\sigma}\right)^2}}}. \quad (31)$$

Gamma underlying failure model

The likelihood function corresponding to *n* independent random variables following the two-parameter gamma underlying failure model can be written under the following form:

$$L_2(x, \alpha; \beta) = e^{-\frac{1}{\beta} S_n' - n\alpha Ln(\beta)} e^{(\alpha-1)\sum_{i=1}^{n} Ln(x_i) - nLn(\Gamma(\alpha))}, \quad (32)$$

where $S_n' = \sum_{i=1}^{n} x_i$.

Note that $S_n^{'}$ is a sufficient statistic for the scale parameter β. Furthermore,

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i}{n\alpha}$$

is a minimum variance unbiased estimator of β, and its moment generating function is given by

$$E(e^{\mu\hat{\beta}}) = \prod_{i=1}^{n} E(e^{\mu\frac{x_i}{n\alpha}}) = \left(1 - \mu\frac{\beta}{n\alpha}\right)^{-n\alpha} \tag{33}$$

which is the moment generating function of the gamma distribution $G(n\alpha, \frac{\beta}{n\alpha})$. Thus, the conditional density function of the MVUE of β is given by

$$h_2(\hat{\beta},\alpha \mid \beta) = \frac{(n\alpha)^{n\alpha}}{\Gamma(n\alpha)\beta^{n\alpha}}\left(\hat{\beta}\right)^{n\alpha-1} e^{-\frac{n\alpha}{\beta}\hat{\beta}}, \hat{\beta} > 0 \tag{34}$$

Approximate Bayesian estimates for the scale parameter β and the reliability function *R(t)* are obtained, with the use of equations (32) and (34) by replacing respectively $g(\theta)$ by β and $R(t)$ in equations (14), (15), (16) and (17). The $\hat{\beta}_i$'s that are the minimum variance unbiased estimates of the scale parameter β will play the role of the $\hat{\theta}_i$'s.

Considering the lognormal prior, equation (14), (15), (16) and (17) yield the following approximate Bayesian estimates of the scale parameter β corresponding respectively to the squared error, the Higgins-Tsokos, the Harris and the proposed lognormal loss functions, after replacing β_i by $\hat{\beta}_i$ in the expression of $h_2(\hat{\beta}_i)$:

$$\hat{\beta}_{E(SE)} = \frac{\sum_{j=1}^{m} \hat{\beta}_j e^{-\frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} e^{-\frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}}, \tag{35}$$

$$\hat{\beta}_{E(HT)} = \frac{1}{f_1 + f_2} Ln \left(\frac{\sum_{j=1}^{m} e^{f_1\hat{\beta}_j - \frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} e^{--f_2\hat{\beta}_j - \frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}} \right),$$

$$f_1, f_2 \succ 0 \tag{36}$$

$$\hat{\beta}_{E(H)} = \frac{\sum_{j=1}^{m} \frac{\hat{\beta}_j}{1-\hat{\beta}_j} e^{-\frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} \frac{1}{1-\hat{\beta}_j} e^{-\frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}},$$

$$\hat{\beta}_j \neq 1 \tag{37}$$

and

$$\hat{\beta}_{E(Ln)} = e^{\frac{\sum_{j=1}^{m} Ln(\hat{\beta}_j) e^{-\frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} e^{-\frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}}} \tag{38}$$

Approximate Bayesian estimates of the reliability corresponding to the first method are therefore given by

$$\tilde{\tilde{R}}_{E\beta}(t,\alpha \mid \hat{\beta}_E) = 1 - \frac{\gamma\left(\alpha, \dfrac{t}{\hat{\beta}_E}\right)}{\Gamma(\alpha)} \quad , \quad t \succ 0 \tag{39}$$

where $\hat{\beta}_E$ is the approximate Bayesian estimate of the scale parameter β.

The approximate Bayesian reliability estimates corresponding to the second method are obtained by replacing $g(\theta)$ by $R(t)$ in equations(14), (15), (16) and (17). The obtained estimates corresponding respectively to the squared error, the Higgins-Tsokos, the Harris and the proposed logarithmic loss functions are given by the following expressions, after replacing β_i by $\hat{\beta}_i$ in the expression of $h_2(\hat{\beta}_i)$:

$$\hat{R}_{E(SE)}(t) = \frac{\sum_{j=1}^{m}\left(1 - \dfrac{\gamma(\alpha, \frac{t}{\hat{\beta}})}{\Gamma(\alpha)}\right) e^{-\frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} e^{-\frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}} \tag{40}$$

$$\hat{R}_{E(HT)}(t) = \frac{1}{f_1 + f_2} Ln \left(\frac{\sum_{j=1}^{m} e^{f_1\left(1 - \frac{\gamma(\alpha, \frac{t}{\hat{\beta}_j})}{\Gamma(\alpha)}\right) - \frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} e^{-f_2\left(1 - \frac{\gamma(\alpha, \frac{t}{\hat{\beta}_j})}{\Gamma(\alpha)}\right) - \frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}} \right),$$

$$f_1, f_2 \succ 0, \tag{41}$$

$$\hat{R}_{E(H)}(t) = \frac{\sum_{j=1}^{m} \dfrac{\Gamma(\alpha) - \gamma(\alpha, \frac{t}{\hat{\beta}_j})}{\gamma(\alpha, \frac{t}{\hat{\beta}_j})} e^{-\frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} \dfrac{\Gamma(\alpha)}{\gamma(\alpha, \frac{t}{\hat{\beta}_j})} e^{-\frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}}, \tag{42}$$

and

$$\hat{R}_{E(Ln)}(t) = \frac{\sum_{j=1}^{m} Ln\left(1 - \dfrac{\gamma(\alpha, \frac{t}{\hat{\beta}_j})}{\Gamma(\alpha)}\right) e^{-\frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}}{\sum_{j=1}^{m} e^{-\frac{S_n^{'}}{\hat{\beta}_j} - n\alpha Ln(\hat{\beta}_j) + (\alpha-1)\sum_{i=1}^{n} Ln(x_i) - \frac{1}{2}\left(\frac{Ln(\hat{\beta}_j)-\mu}{\sigma}\right)^2}}$$

e

(43)

Relative Efficiency with Respect to the Squared Error Loss

To compare our results, the criterion of integrated mean square error, IMSE, of the approximate Bayesian reliability estimate $\tilde{R}_E(t)$ is used. That is,

$$IMSE(\tilde{R}_E(t)) = \int_0^\infty \left(\tilde{R}_E(t) - R(t)\right)^2 dt \quad (44)$$

Define the relative efficiency as the ratio of the IMSE of the approximate Bayesian reliability estimates using a challenging loss function to that of the popular squared error loss. The relative efficiencies of the Higgins-Tsokos, the Harris and the proposed logarithmic loss are respectively defined as follows:

$$Eff(HT) = \frac{IMSE(\tilde{R}_{E(HT)}(t))}{IMSE(\tilde{R}_{E(SE)}(t))} = \frac{\int_0^\infty \left(\tilde{R}_{E(HT)}(t) - R(t)\right)^2 dt}{\int_0^\infty \left(\tilde{R}_{E(SE)}(t) - R(t)\right)^2 dt},$$

$$Eff(H) = \frac{IMSE(\tilde{R}_{E(H)}(t))}{IMSE(\tilde{R}_{E(SE)}(t))} = \frac{\int_0^\infty \left(\tilde{R}_{E(H)}(t) - R(t)\right)^2 dt}{\int_0^\infty \left(\tilde{R}_{E(SE)}(t) - R(t)\right)^2 dt}$$

and

$$Eff(Ln) = \frac{IMSE(\tilde{R}_{E(Ln)}(t))}{IMSE(\tilde{R}_{E(SE)}(t))} = \frac{\int_0^\infty \left(\tilde{R}_{E(Ln)}(t) - R(t)\right)^2 dt}{\int_0^\infty \left(\tilde{R}_{E(SE)}(t) - R(t)\right)^2 dt}.$$

If the relative efficiency is smaller than one, the Bayesian estimate corresponding to the squared error loss is less efficient. The squared error will be more efficient if the relative efficiency is greater than one. If the relative efficiency is approximately equal to one, the Bayesian reliability estimates are equally efficient.

Numerical Simulations

In the numerical simulations, Bayesian and approximate Bayesian estimates of the scale parameter β for the gamma failure model and the lognormal prior will be compared, when the squared error loss is used and the shape parameter α is considered fixed. Second, the new approach will be implemented, and approximate Bayesian reliability estimates will be obtained for the three-parameter Weibull and the gamma failure model under the squared error, the Higgins-Tsokos (with $f_1 = 1, f_2 = 1$), the Harris, and the logarithmic loss functions, respectively.

Comparison between Bayesian estimates and approximate Bayesian estimates of the scale parameter β

Using the square error loss function, the gamma underlying failure model and the lognormal prior, Table 1 gives estimates of the scale parameter β when the shape parameter α is fixed and equal to one.

Table 1.

Lognormal prior	True value of β	Bayesian estimate of β	Approximate Bayesian estimates of β	Number of replicates m
$\mu = 1, \sigma = 0.5$	1	1.1688	0.9795	1
			0.9883	2
			1.0796	3
			1.0625	4
			1.0385	5
			1.0899	6
			1.0779	7
$\mu = 4, \sigma = 9$	1	1.0561	0.9795	1
			0.9880	2
			1.0351	3
			0.9943	4
			0.9665	5
			0.9945	6
			1.0017	7
$\mu = 3, \sigma = 0.8$	2	2.2808	1.9591	1
			1.9766	2
			2.1555	3
			2.1162	4
			2.0658	5
			2.1679	6
			2.1467	7
$\mu = 8, \sigma = 12$	2	2.0376	1.9591	1
			1.9761	2
			2.0704	3
			1.9886	4
			1.9331	5
			1.9892	6
			2.0034	7

The above results show that the obtained approximate Bayesian estimates of the parameter β are as good if not better than the corresponding Bayesian estimates, because they are in general closer to the true state of nature.

Approximate Bayesian Reliability Estimates of the Three-parameter Weibull and the Gamma Failure Models for the different Loss Functions

Using Monte Carlo simulation, information has been respectively generated from the three-parameter Weibull *W(a=1, b=1, c=2) and* the two-parameter gamma $G(\alpha = 1, \beta = 1)$. For each of the above underlying failure models, three different samples are generated of thirty failure times, and three minimum variance unbiased estimates of the scale parameter are obtained.

Three-parameter Weibull W(a=1,b=1,c=2)

A typical sample of thirty failure times that are randomly generated from *W(a=1,b=1,c=2)* is given below:

1.9772260	2.6416950	2.1241180
1.5575370	2.7714080	1.7158910
1.3109790	2.2144780	2.2674890
2.2136030	1.3422820	1.4691720
1.3017910	1.7534080	1.9712720
1.6897900	1.9609470	2.9533880
1.5448060	1.4516050	1.1704900
1.9409150	2.5030900	1.4788690
2.1088060	1.7306430	1.8829980
1.8939380	1.8181710	2.7016010

The obtained minimum variance unbiased estimates of the scale parameter *b* are given below

$$\hat{b}_1 = 1.1408084120$$
$$\hat{b}_2 = 1.0091278197$$
$$\hat{b}_3 = 0.9991267092$$

These minimum variance unbiased estimates will be used along with likelihood function and the lognormal prior $f(b; \mu = 0.34, \sigma = 0.115)$ to obtain approximate Bayesian reliability estimates.

Let $\tilde{\tilde{R}}_{Eb(SE)}(t)$, $\hat{R}_{Eb(SE)}(t)$, $\tilde{\tilde{R}}_{Eb(HT)}(t)$, $\hat{R}_{Eb(HT)}(t)$, $\tilde{\tilde{R}}_{Eb(H)}(t)$, $\hat{R}_{Eb(H)}(t)$, $\tilde{\tilde{R}}_{Eb(Ln)}(t)$ and $\hat{R}_{Eb(Ln)}(t)$

represent, respectively, the approximate Bayesian reliability estimates obtained with the approximate Bayesian reliability estimates of the scale parameter *b*, and the ones obtained by direct computation, when the squared error, the Higgins-Tsokos, the Harris and the proposed logarithmic loss functions are used. These estimates are given below in Table 2. Table 3 gives the approximate Bayesian reliability estimates obtained directly using equations (28), (29), (30) and (31).

Gamma failure model $G(\alpha = 1, \beta = 1)$

A typical sample of thirty failure times that are randomly generated from $G(\alpha = 1, \beta = 1)$ is given below.

0.95497	0.09670	0.09107
2.69516	1.47495	0.56762
1.26364	1.60653	0.94337
0.54999	0.64000	0.62536
1.44922	0.78403	1.08172
0.31084	1.47283	0.47580
3.13788	0.11715	0.92341
0.51249	0.22012	3.81572
0.57911	0.50421	0.14532
0.77497	1.07792	1.08156

The obtained minimum variance unbiased estimates of the scale parameter β are given below.

$$\hat{\beta}_1 = 1.009127916$$
$$\hat{\beta}_2 = 1.140808468$$
$$\hat{\beta}_3 = 0.9991268436$$

Table 2.

	$R(t)$	$\overset{\approx}{R}_{Eb(SE)}(t)$	$\overset{\approx}{R}_{Eb(HT)}(t)$	$\overset{\approx}{R}_{Eb(H)}(t)$	$\overset{\approx}{R}_{Eb(Ln)}(t)$
Approximation	$e^{-(t-1)^2}$	$e^{-\frac{1}{1.1251}(t-1)^2}$	$e^{-\frac{1}{1.1251}(t-1)^2}$	$e^{-\frac{1}{0.9758}(t-1)^2}$	$e^{-\frac{1}{1.1242}(t-1)^2}$
IMSE	0	$2.3810\,10^{-4}$	$3.6764\,10^{-3}$	$1.4820\,10^{-4}$	$3.6301\,10^{-3}$
Relative efficiency with respect to $\overset{\approx}{R}_{Eb(SE)}(t)$	0	1.0	15.44	0.62	15.25

The above approximate Bayesian estimates yield good estimates of the true reliability function.

Table 3.

Time t	$\hat{R}(t)$	$\hat{R}_{Eb(SE)}(t)$	$\hat{R}_{Eb(HT)}(t)$	$\hat{R}_{Eb(H)}(t)$	$\hat{R}_{Eb(Ln)}(t)$
1.00001	1.0000	1.0000	1.0000	1.0000	1.0000
1.25	0.9394	0.9459	0.9459	0.9459	0.9459
1.50	0.7788	0.8005	0.8005	0.8008	0.8005
1.75	0.5698	0.6062	0.6062	0.6066	0.6061
2.00	0.3679	0.4108	0.4108	0.4112	0.4105
2.25	0.2096	0.2492	0.2492	0.2495	0.2488
2.50	0.1054	0.1354	0.1354	0.1355	0.1349
2.75	0.0468	0.0659	0.0659	0.0659	0.0655
3.00	0.0183	0.0287	0.0287	0.0287	0.0284
3.25	0.0063	0.0112	0.0112	0.0112	0.0110
3.50	0.0019	0.0039	0.0039	0.0039	0.0038
3.75	0.0005	0.0012	0.0012	0.0012	0.0012
4.00	0.0001	0.0003	0.0003	0.0003	0.0003

These minimum variance unbiased estimates will be used along with the likelihood function and the lognormal prior $f(x;\mu = 0.0137, \sigma = 0.1054)$ to obtain approximate Bayesian reliability estimates.

Let $\overset{\approx}{R}_{E\beta(SE)}(t)$, $\overset{\Lambda}{R}_{E\beta(SE)}(t)$, $\overset{\approx}{R}_{E\beta(HT)}(t)$, $\overset{\Lambda}{R}_{E\beta(HT)}(t)$, $\overset{\approx}{R}_{E\beta(H)}(t)$, $\overset{\Lambda}{R}_{E\beta(H)}(t)$, $\overset{\approx}{R}_{E\beta(Ln)}(t)$ and $\overset{\Lambda}{R}_{E\beta(Ln)}(t)$ represent respectively the approximate Bayesian reliability estimates obtained with the approximate Bayesian estimate of β, and the ones obtained by direct computation, when the squared error, the Higgins-Tsokos, the Harris and the proposed logarithmic loss functions are used. These estimates are given in Table 5 and Table 6.

For computational convenience, the results presented in Table 3 are used to obtain approximate estimates of the analytical forms of the various approximate Bayesian reliability expressions under study. The results are given in Table 4. Table 6 gives the approximate Bayesian reliability estimates obtained directly by using equations (40), (41), (42) and (43).

For computational convenience, the results presented in Table 6 are used to obtain approximate estimates of the analytical forms of the various approximate Bayesian reliability expressions under study. The results are given in Table 7.

Table 4.

	$R(t)$	$\overset{\Lambda}{R}_{Eb(SE)}(t)$	$\overset{\Lambda}{R}_{Eb(HT)}(t)$	$\overset{\Lambda}{R}_{Eb(H)}(t)$	$\overset{\Lambda}{R}_{Eb(Ln)}(t)$
Approximation	$e^{-(t-1)^2}$	$e^{-\frac{1}{1.1251}(t-1)^2}$	$e^{-\frac{1}{1.1251}(t-1)^2}$	$e^{-\frac{1}{1.1251}(t-1)^2}$	$e^{-\frac{1}{1.1251}(t-1)^2}$
IMSE	0	$2.3813\,10^{-3}$	$2.3813\,10^{-3}$	$2.3813\,10^{-3}$	$2.3813\,10^{-3}$
Relative efficiency with respect to $\overset{\Lambda}{R}_{Eb(SE)}(t)$	0	1	1	1	1

Table 5.

	$R(t)$	$\overset{\approx}{R}_{E\beta(SE)}(t)$	$\overset{\approx}{R}_{E\beta(HT)}(t)$	$\overset{\approx}{R}_{E\beta(H)}(t)$	$\overset{\approx}{R}_{E\beta(Ln)}(t)$
Approximation	e^{-t}	$e^{-\frac{t}{1.0311}}$	$e^{-\frac{t}{1.1250}}$	$e^{-\frac{t}{0.9758}}$	$e^{-\frac{t}{1.1242}}$
IMSE	0.0	$2.381008\,10^{-4}$	$3.676471\,10^{-3}$	$1.482034\,10^{-4}$	$3.630931\,10^{-3}$
Relative efficiency with respect to $\overset{\approx}{R}_{E\beta(SE)}(t)$	0.0	1.0	15.44	0.62	15.25

Table 6.

Time t	$\hat{R}(t)$	$\hat{R}_{E\beta(SE)}(t)$	$\hat{R}_{E\beta(HT)}(t)$	$\hat{R}_{E\beta(H)}(t)$	$\hat{R}_{E\beta(Ln)}$
10^{-100}	1.0000	1.0000	1.0000	1.0000	1.0000
1.00	0.3679	0.3786	0.4108	0.4112	0.4105
2.00	0.1353	0.1437	0.1690	0.1692	0.1685
3.00	0.9498	0.0547	0.0696	0.0697	0.0692
4.00	0.0183	0.0209	0.0287	0.0287	0.0284
5.00	0.0067	0.0080	0.0118	0.0118	0.0117
6.00	0.0025	0.0031	0.0049	0.0049	0.0048
7.00	0.0009	0.0012	0.0020	0.0020	0.0020
8.00	0.0003	0.0005	0.0008	0.0008	0.0008
9.00	0.0001	0.0002	0.0003	0.0003	0.0003
10.00	0.0000	0.0001	0.0001	0.0001	0.0001

Table 7.

	$\hat{R}_{E\beta(SE)}(t)$	$\hat{R}_{E\beta(HT)}(t)$	$\hat{R}_{E\beta(H)}(t)$	$\hat{R}_{E\beta(Ln)}(t)$
Approximation	$e^{-\frac{t}{1.0311}}$	$e^{-\frac{t}{1.1250}}$	$e^{-\frac{t}{1.1250}}$	$e^{-\frac{t}{1.1242}}$
IMSE	2.38100810^{-4}	3.67647110^{-3}	3.67647110^{-3}	3.63093110^{-3}
Relative efficiency with respect to $\hat{R}_{E\beta(SE)}(t)$	1.0	15.44	15.44	15.25

The above approximate Bayesian estimates yield good estimates of the true reliability function.

Conclusion

Using the concept of Monte Carlo Integration, approximate Bayesian estimates of the scale parameter *b* were analytically obtained for the three-parameter Weibull failure model under different loss functions. Using these estimates, approximate Bayesian estimates of the reliability function may be obtained. Furthermore, the concept of Monte Carlo Integration may be used to directly approximate estimates of the Bayesian reliability function.

Second, similar results were obtained for the gamma failure model. Finally, numerical simulations of the analytical formulations indicate:

(1) Approximate Bayesian reliability estimates are in general good estimates of the true reliability function.

(2) When the number of replicates *m* increases, the approximate Bayesian reliability estimates obtained directly converge for each loss function to their corresponding Bayesian reliability estimates.

(3) Approximate Bayesian reliability estimates corresponding to the squared loss function do not always yield the best approximations to the true reliability function. In fact the Higgins-Tsokos, the Harris and the proposed logarithmic loss functions are sometimes equally efficient if not better.

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