Journal of Modern Applied Statistical Methods

Volume 4 | Issue 2 Article 21

11-1-2005

Statistical Model And Estimation Of The Optimum Price For A Chain Of Price Setting Firms

Chengjie Xiong
Washington University in St. Louis, chengjie@wubios.wustl.edu

Kejun Zhu
China University of Geosciences (Wuhan)

Journal of Modern Applied Statistical Methods
November, 2005, Vol. 4, No. 2, 553-566

1538 – 9472/05/$95.00

Statistical Model And Estimation Of The Optimum Price For A Chain Of Price Setting Firms

Chengjie Xiong
Division of Biostatistics
Washington University in St. Louis

Kejun Zhu
College of Management
China University of Geosciences (Wuhan)

A stochastic approach is used to model the economics of a chain of price setting firms. It is assumed that these firms have fixed capacities in their products, but random demands for their products. The optimum price, the optimum revenue, and the expected marginal revenue at a given price are investigated. The method of maximum likelihood is used to provide both point and confidence interval estimates. The coverage probabilities of confidence interval estimates based on a simulation study are presented.

Key words: Asymptotic confidence interval; capacity; gamma distribution; marginal revenue; maximum likelihood estimate (MLE); optimum revenue; Poisson distribution.

Introduction

Fixed capacity is very common in businesses. For example, an established hotel must operate with a fixed number of rooms; and an established restaurant has a fixed number of seats. While the capacity is fixed for many firms, the demand for their products is uncertain. By their very nature, the hotel and the restaurant cannot respond to the uncertain demand by inventory adjustments, nor for that matter, by using high priced resources to temporarily increase production when demand is high. The most important goal for these firms is to choose a price that maximizes their expected profits under random demand for a fixed capacity. Many authors have studied the problem of firm decision making when demand for the product is uncertain. Epstein (1978) and Turnovsky (1973), provided the classic approach to the problem. Scott, Highfill, and Sattler (1988) and Balvers and Miller (1992) studied several production side questions such as the derived factor demand with capacity constraints. Flacco and Kroetch (1986) and Booth (1990), investigated the production levels and/or inventory adjustments in the decision making.

In this article, it is assumed that these firms operate as monopolies and are risk neutral. It is also assumed that capacity is a strict upper bound on the provision of service and must be set before the demand is arriving. With these same assumptions, Scott, Sattler, and Highfill (1995) studied the optimum price for a single firm when the demand is random. Highfill, Quigg, Sattler, and Scott (2000) investigated the problem of capacity decision for a single firm when the product demand is uncertain. Here, a chain of price setting firms with random demands are considered and the optimum price and its estimation applicable to a population of firms is studied. There are two levels of uncertainty in the demand side now: one is the demand uncertainty for any given firm in the chain, the other is the demand uncertainty from firm to firm in the chain. Therefore, two statistical models are needed to model the demand at two different stages, one for a given

Financial support for this study was provided in part to Kejun Zhu by the National Natural Science Foundation, Grant #70273044, of the People's Republic of China. The funding agreement ensured the authors' independence in designing the study, interpreting the data, writing and publishing the report. Address correspondence to the first author at: chengjie@wubios.wustl.edu.

firm and the other for across the firms in the chain.

A simple example of the kind of problem under consideration in this article is a chain of hotels which operates with the number of rooms as the strict upper bound for the service. The variability in demand will cause the hotels to experience excess capacity and excess demand. Both excesses will depend on the capacity of the firms and the probability distribution for the demand. The question answered in this article is, for a randomly selected hotel in the chain, how the price should be set and estimated so that the maximum profit can be achieved.

In the following section, the statistical model is proposed and the optimum price is studied by assuming that all the parameters are known in the model. Also, the effect of capacity on the optimum price is considered. Next, the estimation for the model parameters is provided and asymptotic confidence intervals for the optimum price, the optimum revenue, and the expected marginal revenue at a given price are presented.

It is convenient to use a chain of hotels as the economic reference of a chain of firms in this article. The results in this article apply to all businesses where capacity is a strict upper bound on the provision of service and the demand is random.

The Model and the Optimum Price

For a given hotel H in a chain of hotels, let $Y|_H$ be the number of people to rent a room. The uncertain number of people to rent a room is treated as a standard queuing problem with the quantity demanded a random variable distributed as Poisson whose mean is λ_H, i.e.,

$$P(Y|_H = y) = e^{-\lambda_H}\frac{\lambda_H^y}{y!},$$

for y=0,1,2,....

In order to model the demand variability from hotel to hotel in the chain, it is assumed that the population of demand mean λ_H of $Y|_H$ from the hotels follows a Gamma distribution with index $\alpha>0$ and scale parameter $\theta>0$, i.e., λ_H is distributed according to the probability density function

$$f(\lambda) = \frac{1}{\Gamma(\alpha)\theta^a}\lambda^{a-1}e^{-\frac{\lambda}{\theta}}.$$

It is also assumed that α is independent of price and θ is linearly and inversely related to unit price p, i.e.,

$$\theta = a + bp,$$

where $a>0$ and $b<0$ are two constants.

Let Y denote the number of people to rent a room from a hotel randomly sampled from the chain. The probability distribution of Y is then given by

$$P(Y = y) = \int_0^\infty e^{-\lambda}\frac{\lambda^y}{y!}f(\lambda)d\lambda$$
$$= \frac{\alpha(\alpha+1)...(\alpha+y-1)}{y!(1+\theta)^\alpha}\left(\frac{\theta}{\theta+1}\right)^y,$$

for y=0,1,2,....

The distribution of Y is the well known negative binomial distribution when α is a positive integer. The index parameter α in the model allows for the flexibility to choose different densities in the Gamma family to model the demand variability across the hotels. Let P_α denote the probability of events instead of just P to indicate the dependence of the probabilities on the parameter α. Notice that $EY = E(EY|_H) = E\lambda_H = \alpha\theta = \alpha a + \alpha bp$, $a>0$, $b<0$. The expected number of people to rent a room from this randomly selected hotel in the chain is also linearly and inversely related to price p.

Suppose that c is the capacity number of rooms in the hotel. Let X be the unit sales of the hotel. Then

$$X = \begin{cases} Y, & Y \le c \\ c, & Y > c \end{cases}.$$

Therefore

$$P_a(X=x) = \begin{cases} P(Y=x), & x<c \\ 1-\sum_{x=0}^{c-1} P(Y=x), & x=c \end{cases}$$

$$= \begin{cases} \frac{\alpha(\alpha+1)\ldots(\alpha+x-1)}{x!(1+\theta)^{\alpha}}\left(\frac{\theta}{\theta+1}\right)^{x}, & x<c \\ 1-\sum_{k=0}^{c-1}\frac{\alpha(\alpha+1)\ldots(\alpha+k-1)}{k!(1+\theta)^{\alpha}}\left(\frac{\theta}{\theta+1}\right)^{k}, & x=c\cdot \end{cases}$$

When the demand is random and the capacity is fixed, there are positive probabilities that excess demand (denoted by *ED*) and excess capacity (denoted by *EC*) occurs. It is straightforward to find the probability of excess demand and the probability of excess capacity as

$$P_{\alpha}(ED)=\sum_{x=c+1}^{\infty}\frac{\alpha(\alpha+1)\ldots(\alpha+x-1)}{x!(1+\theta)^{\alpha}}\left(\frac{\theta}{\theta+1}\right)^{x}$$

and

$$P_{\alpha}(EC)=\sum_{x=0}^{c-1}\frac{\alpha(\alpha+1)\ldots(\alpha+x-1)}{x!(1+\theta)^{\alpha}}\left(\frac{\theta}{\theta+1}\right)^{x},$$

respectively. Two integral representations of these probabilities and their derivatives are given, which will be used later in the article:

$$P_{\alpha}(ED)=\frac{1}{B(\alpha,c+1)}\theta^{c+1}\int_{1}^{\infty}\frac{t^{c}}{(1+t\theta)^{\alpha+c+1}}dt \qquad (1)$$

$$P_{\alpha}(EC)=\frac{1}{B(\alpha,c)}\theta^{c}\int_{1}^{\infty}\frac{t^{c-1}}{(1+t\theta)^{\alpha+c}}dt \qquad (2)$$

$$\frac{dP_{\alpha}(ED)}{d\theta}=\frac{\theta^{c}}{B(\alpha,c+1)(1+\theta)^{\alpha+c+1}}, \qquad (3)$$

$$\frac{dP_{\alpha}(EC)}{d\theta}=-\frac{\theta^{c-1}}{B(\alpha,c)(1+\theta)^{\alpha+c}}, \qquad (4)$$

where $B(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the Beta function. (1) and (2) can be obtained by using equation (6) and (7) from Highfill, Quigg, Sattler and Scott (2000) and applying Fubini's Theorem for the exchange of integrals. (3) and (4) can be obtained by directly taking derivatives from (1) and (2), respectively. Combining (3) and (4) further gives

$$\theta\alpha\frac{dP_{\alpha+1}(EC)}{dp}+c\frac{dP_{\alpha}(ED)}{dp}=0. \qquad (5)$$

The expected unit sales of the hotel is then

$$EX=\sum_{x=0}^{c}x\frac{\alpha(\alpha+1)\ldots(\alpha+x-1)}{x!(1+\theta)^{\alpha}}\left(\frac{\theta}{\theta+1}\right)^{x}+cP_{\alpha}(ED)$$
$$=\theta\alpha P_{\alpha+1}(EC)+cP_{\alpha}(ED).$$

Therefore, the expected unit sales of the hotel contain two parts, one is the expected demand $\theta\alpha$ multiplied by the probability of excess capacity at index $\alpha+1$, and the other is the capacity *c* multiplied by the probability of excess demand.

For any hotel in the chain, the problem is to find the optimum price that maximizes the expected profit based on the fixed capacity. It is assumed that these hotels have a constant non-stochastic marginal cost function. Therefore, as pointed out by Highfill, Quigg, Sattler and Scott (2000), the constant can be set at zero since the analysis is not materially affected by the choice of this constant (i.e., one can concentrate on the expected revenue). Let *R* be the revenue for a randomly selected hotel, i.e., $R=Xp$. The expected revenue is

$$ER=p\theta\alpha P_{\alpha+1}(EC)+pcP_{\alpha}(ED).$$

Therefore, the expected revenue of the hotel contains two parts too, one is the expected revenue for all demand multiplied by the probability of excess capacity at index $\alpha+1$, the other is the revenue at capacity multiplied by the probability of excess demand. The following theorem gives the optimum price which maximizes the expected revenue.

Theorem 1

The optimum price p^* is the unique solution to the equation:

$$(\theta\alpha + pb\alpha)P_{\alpha+1}(EC) + cP_\alpha(ED) = 0. \qquad (6)$$

In addition, $p^* > -\frac{a}{2b}$ and $\lim_{c\to\infty} p^* = -\frac{a}{2b}$. Refer to the Appendix for the proof.

Let $\theta^* = a + bp^*$. Denote the optimum expected revenue, the probability of excess capacity and the probability of excess demand at optimum price p^* by ER^*, $P_\alpha(EC^*)$ and $P_\alpha(ED^*)$, respectively. Recall that the expected demand is $EY=\alpha(a+bp)$ and the expected unit sales is $EX = \theta\alpha P_{\alpha+1}(EC) + cP_\alpha(ED)$. It is always true that $EX<EY$, because $X<Y$. If the capacity is hypothetically infinity, then $X=Y$ and $ER=p\alpha(a+bp)$. Therefore ER attains the maximum $-\alpha a^2/(4b)$ when price $p = -a/(2b)$. Theorem 1 indicates that in real world business applications where the capacity c is always a finite number, the optimum price for the hotel is always larger than that in the limiting capacity situation, and the optimum revenue for the hotel is always smaller than that in the limiting capacity situation. But, as the capacity increases, the optimum price and the optimum revenue approach their limiting values respectively.

Scott, Sattler and Highfill (1995) defined the expected marginal revenue (*EMR*) as $EMR = dER/dEX$. The expected marginal revenue measures the change in expected revenue for a given change in expected unit sales. Notice that $dEX/dp = b\alpha P_{\alpha+1}(EC)$.

Therefore,

$$EMR = \frac{dER}{dp} \Big/ \frac{dEX}{dp}$$

$$= \frac{(\theta\alpha + pb\alpha)P_{\alpha+1}(EC) + cP_\alpha(ED)}{b\alpha P_{\alpha+1}(EC)}$$

$$= 2p + \frac{a}{b} + \frac{cP_\alpha(ED)}{b\alpha P_{\alpha+1}(EC)}.$$

As the capacity approaches infinity, $P_\alpha(ED)$ approaches 0 and $P_{\alpha+1}(EC)$ approaches 1. Therefore, the expected marginal revenue approaches the standard marginal revenue under linear demand.

In order to understand the dependence of p^* on capacity c, the effect of an additional unit of capacity on the optimum price p^* is analyzed. Suppose that the hotel capacity is increased from c to $c+1$. Assume that the optimum price is changed from p^* to $p^* + \Delta p^*$ and the optimum expected revenue is changed from ER^* to $ER^* + \Delta ER^*$ accordingly. The following theorem presents the effect of an additional unit of capacity on p^* and ER^*.

Theorem 2

(1) There exists a constant C depending only on a and α such that if $c>C$ then $\Delta p^* < 0$. In addition, $\lim_{c\to\infty} \Delta p^* = 0$.

(2) $\Delta ER^* > 0$ for every $c\geq 1$. In addition, $\lim_{c\to\infty} \Delta ER^* = 0$. Refer to the appendix for the proof.

Theorem 2 indicates that the optimum price will decrease after the capacity increases to a certain level, but the drop in optimum price for each unit increase of capacity approaches 0 when the capacity approaches infinity. On the other hand, when the capacity increases, there is always a positive probability that the extra unit will be taken by customers. Therefore, the optimum expected revenue will always increase. But the increase in the optimum revenue for each unit increase of capacity also approaches 0 when the capacity approaches infinity.

Estimation and Inference

In the previous section, the optimum price and optimum revenue were discussed when all model parameters are assumed known. In this section, it is first assumed that the index parameter α is known in the model and the estimate of the unknown parameters *a* and *b* is discussed using data collected from the hotels in the chain. Suppose that hotels operate independently and *n* hotels in the chain have been observed, resulting in the data $(p_i, c_i, x_i, \delta_i)$, $i=1,2,...,n$, where p_i, c_i, x_i are the price, the capacity, and the unit sales of the *i*-th hotel, respectively, and

$$\delta_i = \begin{cases} 1, & y_i \le c_i \\ 0, & y_i > c_i \end{cases},$$

where y_i is the demand of the *i*-th hotel. The maximum likelihood estimators for *a* and *b* maximize the likelihood function:

$$L(\alpha, a, b) \propto \prod_{i=1}^{n} \left\{ \frac{\theta_i^{\delta_i x_i}}{(\theta_i + 1)^{\delta_i (x_i + \alpha)}} [P_\alpha(ED)_i]^{1-\delta_i} \right\},$$

where

$$P_\alpha(ED)_i = \sum_{x=0}^{c_i+1} \frac{\alpha(\alpha+1)...(\alpha+x-1)}{x!(1+\theta_i)^\alpha} \left(\frac{\theta_i}{\theta_i+1}\right)^x,$$

and

$$\theta_i = a + bp_i, i = 1,2,..., n.$$

Because

$$\ln L \propto \left[\sum_{i=1}^{n} \delta_i x_i \ln \theta_i - \delta_i(\alpha + x_i)\ln(\theta_i + 1) \right] + \sum_{i=1}^{n} \left[(1-\delta_i) \ln P_\alpha(ED)_i \right],$$

the maximum likelihood estimators of *a* and *b* solve the following system of equations:

$$\frac{\partial \ln L}{\partial a} = \sum_{i=1}^{n} \left\{ \frac{\delta_i x_i}{\theta_i} - \frac{\delta_i(\alpha + x_i)}{\theta_i + 1} \right\} + \sum_{i=1}^{n} \left\{ \frac{(1-\delta_i)\theta_i^{c_i}}{B(\alpha, c_i + 1)(1+\theta_i)^{\alpha + c_i + 1} P_\alpha(ED)_i} \right\} = 0,$$

$$\frac{\partial \ln L}{\partial b} = \sum_{i=1}^{n} p_i \left\{ \frac{\delta_i x_i}{\theta_i} - \frac{\delta_i(\alpha + x_i)}{\theta_i + 1} \right\} + \sum_{i=1}^{n} p_i \left\{ \frac{(1-\delta_i)\theta_i^{c_i}}{B(\alpha, c_i + 1)(1+\theta_i)^{\alpha + c_i + 1} P_\alpha(ED)_i} \right\} = 0.$$

It is assumed that there are at least two different prices in the data and *n* is large enough so that not all δ_i are 0. Then, the maximum likelihood estimates uniquely exist. However, except for trivial situations, the solutions to the system cannot be found in a close form. But numerical methods as discussed in Press, Flannery, Teukolsky, and Vetterling (1986) such as Newton-Raphson method can be easily implemented to find the solutions. The symbols $\hat{a}$ and $\hat{b}$ are used to denote the maximum likelihood estimator for *a* and *b*, respectively. Let

$$\Sigma^{-1} = \begin{pmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{12} & \sigma'_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}^{-1},$$

where

$$\sigma_{11} = E\left(-\frac{\partial^2 \ln L}{\partial a^2} \right)$$

$$= \sum_{i=1}^{n} \left\{ \frac{\alpha P_{\alpha+1}(EC)_i}{\theta_i} - \frac{\alpha[(1-P_\alpha(ED)_i) + \theta_i P_{\alpha+1}(EC)_i]}{(\theta_i + 1)^2} \right\} - \sum_{i=1}^{n} \left\{ \frac{\theta_i^{c_i - 1}}{B(\alpha, c_i + 1)} \left[\frac{c_i - (\alpha+1)\theta_i}{(1+\theta_i)^{\alpha + c_i + 2}} \right] \right\} + \sum_{i=1}^{n} \left\{ \frac{\theta_i^{2c_i}}{B^2(\alpha, c_i + 1)(\theta_i + 1)^{2(\alpha + c_i + 1)} P_\alpha(ED)_i} \right\},$$

$$\sigma_{12} = E\left(-\frac{\partial^2 \ln L}{\partial a \partial b} \right)$$

$$=\sum_{i=1}^{n}\left\{\frac{p_i \alpha P_{\alpha+1}(EC)_i}{\theta_i}\right\}$$

$$-\sum_{i=1}^{n}\left\{\frac{p_i \alpha[(1-P_\alpha(ED)_i)+\theta_i P_{\alpha+1}(EC)_i]}{(\theta_i+1)^2}\right\}$$

$$-\sum_{i=1}^{n}\left\{\frac{\theta_i^{c_i-1}p_i}{B(\alpha,c_i+1)}\left[\frac{c_i-(\alpha+1)\theta_i}{(1+\theta_i)^{\alpha+c_i+2}}\right]\right\}$$

$$+\sum_{i=1}^{n}\left\{\frac{\theta_i^{2c_i}p_i}{B^2(\alpha,c_i+1)(\theta_i+1)^{2(\alpha+c_i+1)}P_\alpha(ED)_i}\right\},$$

$$\sigma_{22}=E\left(-\frac{\partial^2 \ln L}{\partial b^2}\right)$$

$$=\sum_{i=1}^{n}\left\{\frac{p_i^2 \alpha P_{\alpha+1}(EC)_i}{\theta_i}\right\}$$

$$-\sum_{i=1}^{n}\left\{\left[\frac{\alpha[(1-P_\alpha(ED)_i)+\theta_i P_{\alpha+1}(EC)_i]}{p_i^{-2}(\theta_i+1)^2}\right]\right\}$$

$$-\sum_{i=1}^{n}\left\{\frac{\theta_i^{c_i-1}p_i^2}{B(\alpha,c_i+1)}\left[\frac{c_i-(\alpha+1)\theta_i}{(1+\theta_i)^{\alpha+c_i+2}}\right]\right\}$$

$$+\sum_{i=1}^{n}\left\{\left[\frac{\theta_i^{2c_i}p_i^2}{B^2(\alpha,c_i+1)(\theta_i+1)^{2(\alpha+c_i+1)}P_\alpha(ED)_i}\right]\right\},$$

and

$$P_{\alpha+1}(EC)_i=\sum_{x=0}^{c_i-1}\frac{(\alpha+1)\ldots(\alpha+x)}{x!(\theta_i+1)^{1+\alpha}}\left(\frac{\theta_i}{\theta_i+1}\right)^x.$$

These equations are obtained by using equation (3) in the previous section and

$$E(1-\delta_i)=P_\alpha(ED)_i,$$
$$E(\delta_i X_i)=\theta_i \alpha P_{\alpha+1}(EC)_i,\quad i=1,2,\ldots,n.$$

A randomly selected hotel from the chain is considered and the estimate for the optimum price and the optimum revenue for the hotel is given. Also, the expected marginal revenue at a given price p is estimated. Again, it is assumed that c is the capacity of the hotel and similar notations are used. Let $p^*=p^*(a,b)$ be the solution to

$$(\theta\alpha+pb\alpha)P_{\alpha+1}(EC)+cP_\alpha(ED)=0.$$

A direct application of the chain rule when taking the derivative from both sides of the equation gives

$$\frac{\partial p^*}{\partial a}=\frac{p^*b\theta^{*(c-1)}-B(\alpha+1,c)(1+\theta^*)^{\alpha+1+c}P_{\alpha+1}(EC^*)}{b[2B(\alpha+1,c)(1+\theta^*)^{\alpha+1+c}P_{\alpha+1}(EC^*)-p^*b\theta^{*(c-1)}]} \quad (7)$$

$$\frac{\partial p^*}{\partial b}=\frac{p^*[p^*b\theta^{*(c-1)}-2B(\alpha+1,c)(1+\theta^*)^{\alpha+1+c}P_{\alpha+1}(EC^*)]}{b\left[2B(\alpha+1,c)(1+\theta^*)^{\alpha+1+c}P_{\alpha+1}(EC^*)-p^*b\theta^{*(c-1)}\right]} \quad (8)$$

Notice that

$$ER^*=p^*[\theta^*\alpha P_{\alpha+1}(EC^*)+cP_\alpha(ED^*)]$$
$$=-p^{*2}b\alpha P_{\alpha+1}(EC^*).$$

Applying the chain rule again,

$$\frac{\partial ER^*}{\partial a}$$

$$=-b\alpha p^*\left[2P_{\alpha+1}(EC^*)\frac{\partial p^*}{\partial a}\right.$$

$$\left.+\left[\left(1+b\frac{\partial p^*}{\partial a}\right)\frac{b\alpha p^{*2}\,\theta^{*(c-1)}}{B(\alpha+1,c)(1+\theta^*)^{\alpha+1+c}}\right]\right],$$

$$\frac{\partial ER*}{\partial b}$$

$$= -\alpha p*\left[\left(2b\frac{\partial p*}{\partial b} + p*\right)P_{\alpha+1}(EC*)\right]$$

$$+\left[\left(p* + b\frac{\partial p*}{\partial b}\right)\frac{\alpha p*^2\, b\theta*^{(c-1)}}{B(\alpha+1,c)(1+\theta*)^{\alpha+1+c}}\right],$$

where $\partial p*/\partial a$ and $\partial p*/\partial b$ are given by (7) and (8). Recall that at a given price *p*, the expected marginal revenue

$EMR = 2p + a/b + cP_{\alpha}(ED)/[b\alpha P_{\alpha+1}(EC)]$

is a function of *a* and *b*. Another application of the chain rule yields

$$\frac{\partial EMR}{\partial a} = \frac{1}{b} + \frac{c\theta^{c-1}[B(\alpha+1,c)\theta P_{\alpha+1}(EC) + B(\alpha,c+1)P_{\alpha}(ED)]}{b\alpha B(\alpha,c+1)B(\alpha+1,c)(1+\theta)^{\alpha+1+c}P_{\alpha+1}(EC)^2}$$

$$\frac{\partial EMR}{\partial b} = -\frac{a}{b^2} - \frac{cP_{\alpha}(ED)}{b^2\alpha P_{\alpha+1}(EC)} + \frac{cp\theta^{c-1}[B(\alpha+1,c)\theta P_{\alpha+1}(EC) + B(\alpha,c+1)P_{\alpha}(ED)]}{b\alpha B(\alpha,c+1)B(\alpha+1,c)(1+\theta)^{\alpha+1+c}P_{\alpha+1}(EC)^2}.$$

Let

$$\sigma^2 = \begin{pmatrix}\frac{\partial p^*}{\partial a} & \frac{\partial p^*}{\partial b}\end{pmatrix}\Sigma^{-1}\begin{pmatrix}\frac{\partial p^*}{\partial a} & \frac{\partial p^*}{\partial b}\end{pmatrix}^t,$$

$$\delta^2 = \begin{pmatrix}\frac{\partial ER^*}{\partial a} & \frac{\partial ER^*}{\partial b}\end{pmatrix}\Sigma^{-1}\begin{pmatrix}\frac{\partial ER^*}{\partial a} & \frac{\partial ER^*}{\partial b}\end{pmatrix}^t,$$

and

$$\tau^2 = \begin{pmatrix}\frac{\partial EMR}{\partial a} & \frac{\partial EMR}{\partial b}\end{pmatrix}\Sigma^{-1}\begin{pmatrix}\frac{\partial EMR}{\partial a} & \frac{\partial EMR}{\partial b}\end{pmatrix}^t,$$

where *t* stands for the transpose. Finally, let

$$\hat{\Sigma}^{-1} = \begin{pmatrix}\hat{\sigma}_{11}' & \hat{\sigma}_{12}' \\ \hat{\sigma}_{12}' & \hat{\sigma}_{22}'\end{pmatrix} = \begin{pmatrix}\hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{12} & \hat{\sigma}_{22}\end{pmatrix}^{-1}$$

be the MLE of Σ^{-1}. Let $\hat{p}*$, $\hat{\sigma}^2$, $E\hat{R}^*$, $\hat{\delta}^2$, $E\hat{M}R$, and $\hat{\tau}$ be the MLEs of p^*, σ^2, ER^*, δ^2, *EMR*, and τ, respectively. Since p^*, σ^2, ER^*, δ^2, *EMR*, and τ are functions of *a* and *b*. Their MLEs are obtained by replacing *a* and *b* by $\hat{a}$ and $\hat{b}$ in their functions, respectively.

For 0<γ<1, let *Z* be the standard normal distribution and $z_{\gamma/2}$ be such that $\Pr(Z \geq z_{\gamma/2}) = \gamma/2$. The following theorem gives the confidence interval estimations for *a*, *b*, p^*, ER^*, and *EMR*.

Theorem 3

If there exist two constants D_1 and D_2 not dependent on *n* such that $p_i < D_1$, $a + bD_1 > 0$, and $c_i \leq D_2$ for *i*=1,2,...,*n*, then the following statements are correct (refer to the appendix for the proofs):

(1) An asymptotic 100(1-γ)% confidence interval for *a* is $\hat{a} \pm z_{\gamma/2}\sqrt{\hat{\sigma}_{11}'}$,

(2) An asymptotic 100(1-γ)% confidence interval for *b* is $\hat{b} \pm z_{\gamma/2}\sqrt{\hat{\sigma}_{22}'}$,

(3) An asymptotic 100(1-γ)% confidence interval for p^* is $\hat{p}* \pm z_{\gamma/2}\hat{\sigma}$,

(4) An asymptotic 100(1-γ)% confidence interval for ER^* is $\hat{ER}^* \pm z_{\gamma/2}\hat{\delta}$,

(5) An asymptotic 100(1-γ)% confidence interval for *EMR* at a given price *p* is $\hat{EMR} \pm z_{\gamma/2}\hat{\tau}$.

In the more realistic situation when none of parameter α, *a* and *b* are known, a stepwise procedure to find the maximum likelihood estimators of α, *a* and *b* is proposed. The traditional approach of maximizing a likelihood function is simply by setting the derivative of the likelihood function with respect to each parameter to 0 simultaneously and then solving the system of equations. This approach becomes very complicated in this case because the derivative of the likelihood function with respect to the index parameter α is rather complicated.

It is proposed that the maximum likelihood estimators $(\hat{\alpha}, \hat{a}, \hat{b})$ should be obtained by first using the method described above to get the maximum likelihood estimators $\hat{a}(\alpha)$ and $\hat{b}(\alpha)$ for specified α values, and then combining with a search procedure to obtain $\hat{\alpha}$, the value of α that maximizes $L_{\max}(a) = L(\alpha, \hat{a}(\alpha), \hat{b}(\alpha))$. The simplex search method of Nelder and Mead (1965) has proved successful in many problems, particularly when there are not too many parameters present. Other search procedures such as those of Powell (1964) and Fletcher and Reeves (1964) are also widely used. After the maximum likelihood estimators $(\hat{\alpha}, \hat{a}, \hat{b})$ are obtained, Theorem 3 can still be used to obtain the asymptotic confidence intervals for model parameters when α is replaced by $\hat{\alpha}$. These asymptotic confidence intervals are still valid based on the fact that $\hat{\alpha}$ is a strongly consistent estimator to α.

Notice that all confidence intervals given by Theorem 3 are asymptotic confidence intervals whose coverage probability approaches 100(1-γ)% when the sample size *n* approaches infinity. In order to assess how these confidence intervals perform with a limited sample size, a simulation study was carried out to compare the empirical coverage to the nominal coverage probability for a selected set of sample size *n*. The following values were chosen α=2, *c*=50, *a*=100, *b*=-1. For each selected sample size for *X*, one third of the sample comes from each unit price of *p*=40, 65, 90. For a given unit price *p*, the one third of the sample for *X* are simulated by using the distribution of *X* as given in Section 2.

In order to generate these samples, random samples on the integer set {1, 2,...,51} based on the 51 probabilities of *X* from *X*=0 to *X*=50 as given in Section 2 are first generated using the random number generating function RANTBL from Statistical Analysis System (1999). One is then subtracted from the samples to give the random samples for *X*. Table 1 presents the empirical coverage probability of the true parameter values. Each empirical coverage probability reported by Table 1 is computed from a simulation of 500 independent confidence intervals based on 500 independent samples of *X* for parameters *a*, *b*, p^*, ER^*, and *EMR* at *p*=60. The optimum price p^* as the solution to (6) is computed using the Newton-Raphson method. All confidence intervals are computed based on Theorem 3 when the index parameter α is replaced by the maximum likelihood estimator α. The maximum likelihood estimators $(\hat{\alpha}, \hat{a}, \hat{b})$ are obtained by the stepwise procedure described above using the simplex search method of Nelder and Mead (1965) when $L_{\max}(a) = L(\alpha, \hat{a}(\alpha), \hat{b}(\alpha))$ is maximized. All the nominal confidence levels in Table 1 are 95% (γ=5%).

Table 1. Empirical Coverage Probability of Confidence Intervals

	α=2, *c*=50, *a*=100, *b*=-1				
Sample size	*a*	*b*	p^*	ER^*	*EMR* at *p*=60
18	0.910	0.912	0.924	0.896	0.906
24	0.924	0.940	0.970	0.936	0.924
30	0.962	0.932	0.938	0.936	0.944
36	0.932	0.944	0.938	0.960	0.952
42	0.960	0.964	0.942	0.928	0.958
48	0.932	0.972	0.938	0.942	0.946
60	0.962	0.946	0.932	0.952	0.958
75	0.972	0.948	0.946	0.958	0.946
90	0.958	0.940	0.960	0.946	0.958
105	0.954	0.952	0.944	0.952	0.964
120	0.956	0.946	0.948	0.954	0.952
150	0.958	0.952	0.944	0.948	0.950
180	0.946	0.952	0.954	0.954	0.946
240	0.944	0.958	0.944	0.954	0.946
300	0.948	0.956	0.958	0.944	0.956

Conclusion

This article has proposed a two-stage statistical model to model the demand variability from a chain of price setting firms. The demand variability from within a firm is modeled by a Poisson distribution, and the demand variability from across the firms is modeled by a Gamma distribution. It was shown that the optimum price under a capacity constraint decreases after the capacity increases to a certain level. On the other hand, the optimum expected revenue increases when the capacity increases. The article also provides a stepwise procedure to find the maximum likelihood estimates of model parameters. The proposed method does not require taking the derivative of the likelihood function with respect to the index parameter α.

Asymptotic confidence interval estimates are developed for the optimum price, the optimum revenue, and the expected marginal revenue at a given price based on the asymptotic normality for the maximum likelihood estimates. A limited simulation study seems to suggest that a relatively large sample size (>100) is required for the asymptotic confidence intervals to achieve the nominal coverage probability.

References

Billingsley, P. (1986). *Probability and measure* (2nd ed.). New York: John Wiley & Sons. Turnovsky, S. J. (1973). Production flexibility, uncertainty, and the behavior of the competitive firms. *International Economic Review, 14*, 395-412.

Balvers, R. J., Miller, N. C. (1992). Factor demand under conditions of product demand and supply uncertainty. *Economic Inquiry*, *30*, 544-555.

Booth, L. A. (1990). Note on adjustment to production uncertainty and the theory of the Firm. *Economic Inquiry, 28,* 616-621.

Cox, D. R. & Hinkley, D. V. (1974). *Theoretical statistics*. London: Chapman & Hall.Scott, R. C., Sattler, E. L., Highfill, J. K. (1995). A hotel capacity utilization model. *The Journal of Economics*, *21*, 101-105.

Epstein, L. G. (1978). Production flexibility and behavior of competitive firms under price uncertainty. *Review of Economic Studies*, *45*, 251-261.

Flacco, P. R., Kroetch, B. G. (1986). Adjustment to production uncertainty and the theory of the firm. *Economic Inquiry*, *24*, 485-495.

Fletcher, R. & Reeves, C. M. (1964). Function minimization by conjugate gradients. *Computer Journal*, *7*, 149-154.

Highfill, J. K., Quigg, D., Sattler, E. L., & Scott, R. C. (2000). The capacity decision when product demand is uncertain: A timing approach. *The Journal of Economics*, *26*, *1*, 71-85.

Nelder, J. A. & Mead, R. A. (1965). Simplex method for function minimization. *Computer Journal*, *7*, 380-383.

Powell, M. J. D. (1964). An efficient method for finding the minimum of a function of several variables without calculating derivatives. *Computer Journal*, *1*, 155-162.

Press, W. H., Flannery, B. P., Teukolsky, S. A., & Vetterling, W. T. (1986). *Numerical recipes: The art of scientific computing*. Cambridge: Cambridge University Press.

SAS Institute, Inc. (1999). *SAS Language* (Version 8), Cary, NC.

Scott, R. C., Highfill, J. K., Sattler, E. L. (1988). Advantage to a risk neutral firm of flexible resources under demand uncertainty. *Southern Economic Journal*, *54*, 934-949.

Appendix

Proof of Theorem 1

The optimum price p^* maximizes ER and therefore solves $dER/dp = 0$, i.e.,

$$(\theta\alpha + pb\alpha)P_{\alpha+1}(EC) + cP_\alpha(ED) + p\theta\alpha\frac{dP_{\alpha+1}(EC)}{dp} + pc\frac{dP_\alpha(ED)}{dp} = 0.$$

Thus, using equation (5) in Section 2, it is concluded that p^* satisfies the equation:

$$(\theta\alpha + pb\alpha)P_{\alpha+1}(EC) + cP_\alpha(ED) = 0.$$

In addition,

$$\frac{d^2ER}{dp^2} = 2b\alpha P_{\alpha+1}(EC) + pb\alpha\frac{dP_{a+1}(EC)}{dp}$$

is negative by the fact that b<0 and equation (4). It then follows that p^* is the unique solution to (6). It is clear that the first term in (6) has to be negative to make (6) hold. Therefore, p^* satisfies $\theta\alpha+pb\alpha<0$, i.e., $p^* > -a/(2b)$. Since $\lim_{c\to\infty} P_{\alpha+1}(EC) = 1$, it follows from (6) that $\lim_{c\to\infty}(\theta\alpha + pb\alpha) = 0$, i.e., $\lim_{c\to\infty} p^* = -a/(2b)$.

Proof of Theorem 2

(1): For $0 < p < -a/b$ and $\theta=a+bp$, let

$$f(p,c) = \frac{dER}{dp} = (\theta\alpha + pb\alpha)P_{\alpha+1}(EC) + cP_\alpha(ED).$$

A direct application of equation (6) gives

$$\frac{c!(1+\theta^*)^{\alpha}}{(\alpha+1)\ldots(\alpha+c)}\left(\frac{\theta^*+1}{\theta^*}\right)^{c} f(p^*, c+1)$$

$$=\frac{\theta^*\alpha+p^*b\alpha}{\theta^*+1}-\frac{c\alpha\theta_i}{(c+1)(\theta_i+1)}+I(c),$$

where

$$I(c)=\frac{c!(1+\theta^*)^{\alpha}}{(\alpha+1)\ldots(\alpha+c)}\left(\frac{\theta^*+1}{\theta^*}\right)^{c}$$

$$\sum_{x=c+2}^{\infty}\frac{\alpha(\alpha+1)\ldots(\alpha+x-1)}{x!(1+\theta^*)^{\alpha}}\left(\frac{\theta^*}{\theta^*+1}\right)^{x}.$$

Replacing c by c+1 in equation (1) of Section 1, provides the following,

$$I(c)=\theta^{*2}(1+\theta^*)^{\alpha-1}\frac{\alpha(c+\alpha+1)}{c+1}$$

$$\int_0^1\frac{[(1+\theta^*)t]^{c+1}}{(1+\theta^* t)^{\alpha+c+2}}dt.$$

For any 1>s>0,

$$\int_0^1\frac{[(1+\theta^*)t]^{c+1}}{(1+\theta^* t)^{\alpha+c+2}}dt$$
$$=\int_0^s\frac{[(1+\theta^*)t]^{c+1}}{(1+\theta^* t)^{\alpha+c+2}}dt$$
$$+\int_s^1\frac{[(1+\theta^*)t]^{c+1}}{(1+\theta^* t)^{\alpha+c+2}}dt.$$

Because

$$\int_0^s\frac{[(1+\theta^*)t]^{c+1}}{(1+\theta^* t)^{\alpha+c+2}}dt<\left[\frac{(1+\theta^*)s}{1+\theta^* s}\right]^{c+1},$$

$\lim_{c\to\infty}\int_0^s\frac{[(1+\theta^*)t]^{c+1}}{(1+\theta^* t)^{\alpha+c+2}}dt=0$ by the fact that $\lim_{c\to\infty}\theta^*=\frac{a}{2}$ and $\frac{(1+a/2)s}{1+as/2}<1$.

Because

$$\int_s^1\frac{[(1+\theta^*)t]^{c+1}}{(1+\theta^* t)^{\alpha+c+2}}dt<1-s,$$

it follows that

$$\lim_{s\to 1^-}\int_s^1\frac{[(1+\theta^*)t]^{c+1}}{(1+\theta^* t)^{\alpha+c+2}}dt=0,$$

where the convergence is uniform on c. Thus, $\lim_{c\to\infty}I(c)=0,$ which further implies that

$$\lim_{c\to\infty}\left[\frac{\theta^*\alpha+p^*b\alpha}{1+\theta^*}-\frac{c\alpha}{c+1}\left(\frac{\theta^*}{\theta^*+1}\right)+I(c)\right]$$
$$=-\frac{\alpha a}{a+2}<0.$$

Therefore, there exists a constant C depending on only a and α such that if c>C then $f(p^*,c+1)<0.$ Because $f(p^*+\Delta p^*,c+1)=0$ and $df(p,c+1)/dp<0,$ it follows that $f(p,c+1)>0$ when $0<p<p^*+\Delta p^*$ and $f(p,c+1)<0$ when $p^*+\Delta p^*<p<-a/b.$ Hence $p^*>p^*+\Delta p^*,$ i.e., $\Delta p^*<0.$ $\lim_{c\to\infty}\Delta p^*=0$ follows from the fact that $\lim_{c\to\infty}p^*=-a/(2b).$

(2): For $0<p<-a/b,$ let $g(p,c)=ER=p[\theta\alpha P_{\alpha+1}(EC)+cP_{\alpha}(ED)].$ Then

$$\Delta ER^*=g(p^*+\Delta p^*,c+1)-g(p^*,c)$$
$$=g(p^*+\Delta p^*,c+1)-g(p^*,c+1)$$
$$+g(p^*,c+1)-g(p^*,c).$$

$g(p^* + \Delta p^*, c+1) - g(p^*, c+1) > 0$ by the fact that $p^* + \Delta p^*$ maximizes $g(p,c+1)$ over p. $\Delta ER^* > 0$ follows from the fact that

$$g(p^*, c+1) - g(p^*, c)$$
$$= p^* \sum_{x=c+1}^{\infty} \frac{\alpha(\alpha+1)...(\alpha+x-1)}{x!(1+\theta^*)^{\alpha}} \left(\frac{\theta^*}{\theta^*+1}\right)^x > 0.$$

Finally, since $ER^* = -p^{*2} b\alpha P_{\alpha+1}(EC^*)$ and $\lim_{c\to\infty} P_{\alpha+1}(EC^*) = 1$, $\lim_{c\to\infty} \Delta ER^* = 0$ follows from the fact that $\lim_{c\to\infty} ER^* = -\alpha a^2/(4b)$.

Proof of Theorem 3

The asymptotic normality is first given for the maximum likelihood estimator $(\hat{a}\ \hat{b})^t$ of $(a\ b)^t$ (t=transpose). Notice that the data come from independent but not identically distributed distributions. Cox and Hinkley (1974) pointed out that the asymptotic normality for the MLEs of such distributions requires two conditions: one is a central limit theorem to $(\partial \ln L/\partial a \quad \partial \ln L/\partial b)^t$ with a nonsingular asymptotic distribution, the other is a weak law of large numbers to insure the convergence in probability of

$$-\frac{1}{n}\begin{pmatrix} \frac{\partial^2 \ln L}{\partial a^2} & \frac{\partial^2 \ln L}{\partial a \partial b} \\ \frac{\partial^2 \ln L}{\partial a \partial b} & \frac{\partial^2 \ln L}{\partial b^2} \end{pmatrix} - \frac{1}{n}\Sigma$$

to zero.

To prove a central limit theorem to $(\partial \ln L/\partial a \quad \partial \ln L/\partial b)^t$, one only needs to do so for

$$= \sum_{i=1}^{n} (t_1 + t_2 p_i)\left\{\frac{\delta_i (X_i - \alpha\theta_i)}{\theta_i(\theta_i+1)}\right\}$$

$$+ \sum_{i=1}^{n}\left\{\frac{(t_1 + t_2 p_i)(1-\delta_i)\theta_i^{c_i}}{B(\alpha, c_i+1)(1+\theta_i)^{\alpha+c_i+1} P_\alpha(ED)_i}\right\}$$

for any choices of t_1 and t_2. For i=1,2,...,n, let

$$T_i = (t_1 + t_2 p_i)\left\{\frac{\delta_i (X_i - \alpha\theta_i)}{\theta_i(\theta_i+1)}\right\}$$

$$+ \frac{(t_1 + t_2 p_i)(1-\delta_i)\theta_i^{c_i}}{B(\alpha, c_i+1)(1+\theta_i)^{\alpha+c_i+1} P_\alpha(ED)_i}.$$

It is clear that $ET_i = 0$. Let $\sigma_{T_i}^2 = ET_i^2$. A careful computation using

$$E(\delta_i X_i^2) = \alpha(\alpha+1)\theta_i^2 P_{\alpha+2,c-1}(EC)_i + \alpha\theta_i P_{\alpha+1,c}(ED)_i$$

gives

$$\sigma_{T_i}^2 = \frac{(t_1+t_2p_i)^2\theta_i^{2c_i}}{B(\alpha,c_i+1)^2(1+\theta_i)^{2(\alpha+c_i+1)} P_{\alpha,c}(ED)_i}$$
$$+\frac{(t_1+t_2p_i)^2}{\theta_i^2(\theta_i+1)^2}\left[\alpha(\alpha+1)\theta_i^2 P_{\alpha+2,c-1}(EC)_i\right]$$

$$+\frac{(t_1+t_2p_i)^2\alpha(1-2\alpha\theta_i)P_{\alpha+1,c}(EC)_i}{\theta_i(\theta_i+1)^2}$$

$$+\frac{(t_1+t_2p_i)^2\alpha^2[1-P_{\alpha,c}(ED)_i]}{(\theta_i+1)^2}..$$

Notice that in the above equation, two indices α and c were used in the notation $P_{\alpha,c}(EC)_i$ to indicate the dependence of the probability on these two parameters. Since, for given t_1 and t_2, $\sigma_{T_i}^2$ is a positive continuous function of (θ_i, c) when $0 < a + bD_1 \le \theta_i \le a$ and $1 \le c \le D_2$, $\sigma_{T_i}^2$ has a positive lower bound and a positive upper bound not dependent on i. Thus,

$\sigma_n^2 = \sum_{i=1}^{n} \sigma_{T_i}^2$ approaches infinity when n approaches infinity. Notice that T_i is bounded. Therefore, for any ε >0,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\sigma_n^2} E\{T_i^2 \chi_{\{T_i \geq \varepsilon\sigma_n\}}\} = 0,$$

where $\chi_{\{T_i \geq \varepsilon\sigma_n\}}$ is the indicator of $\{T_i \geq \varepsilon\sigma_n\}$ (i.e., the Lindeberg condition for T_i holds). This proves the central limit theorem for $(\partial \ln L/\partial a \quad \partial \ln L/\partial b)^t$.

To prove a weak law of large numbers to insure the convergence in probability of

$$-\frac{1}{n}\begin{pmatrix} \frac{\partial^2 \ln L}{\partial a^2} & \frac{\partial^2 \ln L}{\partial a \partial b} \\ \frac{\partial^2 \ln L}{\partial a \partial b} & \frac{\partial^2 \ln L}{\partial b^2} \end{pmatrix} - \frac{1}{n}\Sigma$$

to zero, write $\frac{\partial^2 \ln L}{\partial a^2} = \sum_{i=1}^{n} U_i$, $\frac{\partial^2 \ln L}{\partial a \partial b} = \sum_{i=1}^{n} V_i$, and $\frac{\partial^2 \ln L}{\partial b^2} = \sum_{i=1}^{n} W_i$,

where

$$U_i = \frac{\delta_i(X_i+\alpha)}{(\theta_i+1)^2} - \frac{\delta_i X_i}{\theta_i^2} + \frac{(1-\delta_i)\theta_i^{c_i-1} P_\alpha(ED)_i^{-1}}{B(\alpha, c_i+1)(1+\theta_i)^{\alpha+c_i+1}} \left[\frac{c_i - (\alpha+1)\theta_i}{(1+\theta_i)}\right]$$

$$- \frac{(1-\delta_i)\theta_i^{2c_i}}{B^2(\alpha, c_i+1)(1+\theta_i)^{2(\alpha+c_i+1)} P_\alpha^2(ED)_i},$$

and $V_i = p_i U_i$, $W_i = p_i^2 U_i$. Since $\sigma_{U_i}^2$, $\sigma_{V_i}^2$ and $\sigma_{W_i}^2$ are all positive continuous functions of (θ_i, c) when $0 < a + bD_1 \leq \theta_i \leq a$ and $1 \leq c \leq D_2$, they all have positive upper bounds. Because

$$\sigma^2_{\frac{\partial^2 \ln L}{\partial a^2}} = \sum_{i=1}^{n} \sigma^2_{U_i},$$

$$\sigma^2_{\frac{\partial^2 \ln L}{\partial a \partial b}} = \sum_{i=1}^{n} \sigma^2_{V_i},$$

$$\sigma^2_{\frac{\partial^2 \ln L}{\partial b^2}} = \sum_{i=1}^{n} \sigma^2_{W_i},$$

it follows that

$$\lim_{n\to\infty} \frac{\sigma_{\frac{\partial^2 \ln L}{\partial a^2}}}{n} = \lim_{n\to\infty} \frac{\sigma_{\frac{\partial^2 \ln L}{\partial a \partial b}}}{n} = \lim_{n\to\infty} \frac{\sigma_{\frac{\partial^2 \ln L}{\partial b^2}}}{n} = 0.$$

The weak law of large numbers to

$$-\frac{1}{n}\begin{pmatrix} \frac{\partial^2 \ln L}{\partial a^2} & \frac{\partial^2 \ln L}{\partial a \partial b} \\ \frac{\partial^2 \ln L}{\partial a \partial b} & \frac{\partial^2 \ln L}{\partial b^2} \end{pmatrix}$$

follows from Theorem 6.2 of Billingsley (1986). Therefore, as $n\to\infty$,

$$\Sigma^{1/2}\left\{\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}\right\} \to \mathbf{N}\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, I_{2\times 2}\right\}$$

in distribution, where $I_{2\times 2}$ is the 2×2 identity matrix, i.e., asymptotically,

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}$$

is distributed as $\mathbf{N}\left\{\begin{pmatrix}0\\0\end{pmatrix},\Sigma^{-1}\right\}$. (1) and (2) follow directly from the asymptotic normality of $\hat{a}$ and $\hat{b}$, respectively. (3) follows from the fact that as $n \to \infty$, the MLE of $p^* = p^*(a,b)$ satisfies that

$$\frac{\hat{p}^* - p^*}{\sigma} \to N(0,1)$$

in distribution. (4) follows from the fact that as $n \to \infty$, the MLE of ER^* satisfies that

$$\frac{E\hat{R}^* - ER^*}{\delta} \to N(0,1)$$

in distribution. (5) follows from the fact that as $n \to \infty$, the MLE of EMR satisfies that

$$\frac{E\hat{M}R - EMR}{\tau} \to N(0,1)$$

in distribution.