The Efficiency Of OLS In The Presence Of Auto-Correlated Disturbances In Regression Models

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The ordinary least squares (OLS) estimates in the regression model are efficient when the disturbances have mean zero, constant variance, and are uncorrelated. In problems concerning time series, it is often the case that the disturbances are correlated. Using computer simulations, the robustness of various estimators are considered, including estimated generalized least squares. It was found that if the disturbance structure is autoregressive and the dependent variable is nonstochastic and linear or quadratic, the OLS performs nearly as well as its competitors. For other forms of the dependent variable, rules of thumb are presented to guide practitioners in the choice of estimators.

Key words: Autocorrelation, autoregressive, ordinary least squares, generalized least squares, efficiency

Introduction

Let the relationship between an observable random variable y and k explanatory variables $X_1, X_2, \ldots, X_k$ in a T-finite system be specified in the following linear regression model:

$$y = X\beta + u \qquad (1)$$

where y is a $(T \times 1)$ vector of observations on a response variable, X is a $(T \times k)$ design matrix, β is a $(k \times 1)$ vector of unknown regression parameters, and u is a $(T \times 1)$ random vector of disturbances. For convenience, it is assumed that

X is full column rank $k < T$ and its first column is 1's. *The ordinary least squares* (OLS) estimator of β in the regression model (1) is

$$\hat{\beta} = (X'X)^{-1} X' y \qquad (2)$$

In problems concerning time series, it is often the case that the disturbances are, in fact, correlated. Practitioners are then faced with a decision, use OLS anyway, or try to fit a more complicated disturbance structure. The problem is difficult because the properties of the estimators depend highly on the structure of the independent variables in the model. For more complicated disturbance structures, many of the properties are not well understood. If the disturbance term has mean zero, i.e. $E(u) = 0$, but is in fact, autocorrelated, i.e. $Cov(u) = \sigma_u^2 \Sigma$, where Σ is a $T \times T$ positive definite matrix and the variance σ_u^2 is either known or unknown positive and finite scalar, then the OLS parameter estimates will continue to be unbiased, i.e. $E(\hat{\beta}) = \beta$. But it has a different covariance matrix;

$$Cov_\Sigma(\hat{\beta}) = \sigma_u^2 (X'X)^{-1} X' \Sigma X (X'X)^{-1}. \qquad (3)$$

The most serious implication of autocorrelated disturbances is not the resulting inefficiency of OLS, but the misleading inference when standard tests are used. The autocorrelated nature of disturbances is accounted for in the generalized least squares (GLS) estimator given by:

$$\widetilde{\beta} = \left(X'\Sigma^{-1}X\right)^{-1}X'\Sigma^{-1}y \qquad (4)$$

which is unbiased, i.e. $E\left(\widetilde{\beta}\right) = \beta$, with covariance matrix

$$Cov\left(\widetilde{\beta}\right) = \sigma_u^2\left(X'\Sigma^{-1}X\right)^{-1}. \qquad (5)$$

The superiority of GLS over OLS is due to the fact that GLS has a smaller variance. According to the Generalized Gauss Markov Theorem, the GLS estimator provides the *Best Linear Unbiased Estimator* (BLUE) of β. But the GLS estimator requires prior knowledge of the matrix correlation structure, Σ. The OLS estimator $\hat{\beta}$ is simpler from a computational point of view and does not require a prior knowledge of Σ.

A common approach for modeling univariate time series is the autoregressive model. The general finite order *autoregressive process of order p* or briefly, AR(p), is

$$u_t = \phi_1 u_{t-1} + \phi_2 u_{t-2} + \cdots + \phi_p u_{t-p} + \varepsilon_t, \ \varepsilon_t \sim$$
i.i.d. $N\left(0, \sigma_\varepsilon^2\right)$ $\qquad (6)$

There are numerous articles describing the efficiency of the OLS coefficient estimator $\hat{\beta}$, which ignore the correlation of the error, relative to the GLS estimator $\widetilde{\beta}$, which takes this correlation into account. One strand is concerned with conditions on regressors and error correlation structure, which guarantee that OLS is asymptotically as efficient as GLS (e.g. Chipman, 1979; Krämer, 1980). The efficiency of the OLS estimators in a linear regression containing an autocorrelated error term depends on the structure of the matrix of observations on the independent variables (e.g. Anderson, 1948; 1971; Grenander & Rosenblatt, 1957).

For a linear regression model with first order autocorrelated disturbances, several alternative estimators for the regression coefficients have been discussed in the literature, and their efficiency properties have been investigated with respect to the OLS and GLS estimators (e.g. Kadiyala, 1968; Maeshiro, 1976; 1979; Ullah et al., 1983).

The relative efficiency of GLS to OLS in the important cases of autoregressive disturbances of order one, AR(1), with autoregressive coefficient ρ and second order, AR(2), with autoregressive coefficients $\left(\phi_1, \phi_2\right)$ for specific choices of the design vector have been investigated.

Building on work on the economics and time series literature, the price one must pay for using OLS under suboptimal conditions required investigation. Different designs are being explored, under which relative efficiency of the OLS estimator to that of GLS estimator approaches to one or zero, determining ranges of first-order autoregressive coefficient, ρ, in AR(1) disturbance and second order of autoregressive coefficients, $\left(\phi_1, \phi_2\right)$ in AR(2) for which OLS is efficient and quantifying the effect of the design on the efficiency of the OLS estimator. Furthermore, a simulation study has been conducted to examine the sensitivity of estimators to model misspecification. In particular, how do estimators perform when an AR(2) process is appropriate and the process is incorrectly assumed to be an AR(1) or AR(4)?

Performance Comparisons

In this section, numerical results are presented using the formulas in (3) and (5). Focus will be placed on two issues; first, the relative efficiency of GLS estimator as compared with the OLS estimator when the structure of the design vector, X, is nonstochastic. For example, linear, quadratic, and exponential design vectors with an intercept term included in the design vector. Secondly, the relative efficiency of the GLS estimator as compared with the OLS for a stochastic design vector. In the example considered here, a standard Normal stochastic design vector of length 1000 was generated. The three finite sample sizes used are 50, 100, and 200 for

selected values of the autoregressive coefficients. Both AR(1) and AR(2) error processes are considered to discuss the behavior of OLS as compared to GLS.

Performance Comparisons for AR (1) Process

The relative efficiencies of OLS to GLS are discussed when the disturbance term follows an AR(1) process, $u_t = \rho u_{t-1} + \varepsilon_t, t = 1, 2, \ldots, T$, assuming that the autoregressive coefficient, ρ, is known priori. The three finite sample sizes used are 50, 100, and 200 for the elected values of $|\rho| \leq .9$, evaluated in steps of .2.

Table (1) shows the relative efficiencies of the variances of GLS to OLS for a regression coefficient on linear trend with an intercept term included in the design. For estimating an intercept term, the relative efficiency of the OLS estimator as compared to the GLS estimator

decreases with increasing values of $|\rho|$. For small and moderate sample sizes, the efficiency of the OLS estimator appears to be nearly as efficient as the GLS estimator for $|\rho| \leq .7$. In addition, for large size sample data, the OLS estimator performs nearly as efficiently as the GLS estimator for the additional values of $\rho = \pm .9$. Further, the efficiency for estimating the slope mimics the efficiency of the intercept, except for large sample size; the efficiency of the OLS estimator appears to be nearly as efficient as the GLS estimator for $\rho \neq \pm .9$.

The efficiency of GLS estimator to the OLS estimator for the quadratic design agrees with the behavior for the linear design vector. In contrast, the gain in efficiency of the GLS estimator for different design vectors such as exponential and 1000 standard Normal, N(0,1)

Table 1: Relative Efficiency of GLS to OLS for Linear Design

ρ	Intercept			Slope		
	T = 50	T = 100	T = 200	T = 50	T = 100	T = 200
-0.9	0.7097	0.8276	0.9047	0.6739	0.8012	0.8881
-0.7	0.9162	0.9552	0.9768	0.9024	0.9471	0.9724
-0.5	0.9694	0.9840	0.9918	0.9640	0.9810	0.9903
-0.3	0.9908	0.9952	0.9976	0.9891	0.9943	0.9971
-0.1	0.9991	0.9995	0.9998	0.9989	0.9994	0.9997
0.1	0.9991	0.9995	0.9998	0.9989	0.9994	0.9997
0.3	0.9911	0.9953	0.9976	0.9894	0.9944	0.9971
0.5	0.9717	0.9846	0.9920	0.9662	0.9816	0.9904
0.7	0.9288	0.9585	0.9777	0.9147	0.9503	0.9732
0.9	0.8359	0.8691	0.9164	0.8000	0.8418	0.8993

compared to the OLS estimator is substantial for moderate and large values of ρ. However, for small values of ρ the OLS appears to be nearly as efficient as GLS.

Performance Comparisons for AR (2) Process

The relative efficiencies of OLS to GLS are discussed for linear, quadratic, and exponential design vectors when the disturbance term follows an AR(2) process, $u_t = \phi_1 u_{t-1} + \phi_2 u_{t-2} + \varepsilon_t$, $t = 1, 2, \ldots, T$, assuming that the autoregressive coefficients ϕ_1 and ϕ_2 are known priori. The three finite sample sizes used are 50, 100, and 200 for the selected 45 pairs of the autoregressive coefficients. These coefficients were chosen according to stationary conditions $\left(\phi_1 + \phi_2 < 1, \ \phi_2 - \phi_1 < 1, \text{ and} |\phi_2| < 1\right)$ and so that $\rho_1 = \phi_1 (1 - \phi_2)^{-1}$ is positive. This second condition was chosen since this is the case in most econometric studies.

To demonstrate the efficiency of OLS, consider the linear design vector. When the disturbance term follows an AR(2) process for the linear design with small sample size, OLS performs nearly as efficiently as GLS for estimating the slope for all AR(2) parametrizations except when ϕ's are close to the stationary boundary. As the sample size increases, the difference between the performance of OLS and GLS decreases. Only when $\phi_2 = -.9$, does OLS perform badly regardless of the sample size. The efficiency of GLS to OLS for the quadratic design mimics the behavior for the linear design. Finally, for exponential and 1000 standard Normal design vectors, the efficiency of OLS appears to be nearly as efficient as GLS for $\phi_1 = .2$ and small values of $\phi_{2\,2}$ for all sample sizes. Otherwise, OLS performs poorly.

Simulation Study

In this section, the robustness of various estimators are considered, including estimated generalized least squares (EGLS). These simulations examine the sensitivity of estimators to model misspecification. In particular, how do estimators perform when an AR(2) process is appropriate and it is incorrectly assumed that the process is an AR(1)? The finite sample efficiencies of the OLS estimator relative to four GLS estimators are compared: the GLS based on the correct disturbance model structures and known AR(2) coefficients denoted as GLS-AR(2); the GLS based on the correct disturbance model structures, but with estimated AR(2) coefficients denoted as EGLS-AR(2); the GLS based on AR(1) incorrect disturbance model structures with an estimated AR(1) coefficient denoted as EIGLS-AR(1); and the GLS based on AR(4) incorrect disturbance model structures with estimated AR(4) coefficients denoted as EIGLS-AR(4). This study focuses only on AR(p) GLS corrections disturbances which are widely used in econometric studies.

The Simulation Setup

Three finite sample sizes (50, 100, and 200) and three nonstochastic design vectors of the independent variable are used; linear, quadratic, and exponential. A standard Normal stochastic design vector of length 1000 is also generated (Assuming that the variance of the error term in AR(2) process $\sigma_\varepsilon^2 = 1$). Further, 1000 observations for each of the AR(2) error disturbances with four pairs of autoregressive coefficients; (.2,-.9), (.8,-.9), (.2,-.7), and (.2,-.1) were also generated. Table (2) shows the values of autocorrelation coefficients ρ_1, ρ_2, disturbance variances, σ_u^2, $\sigma_u^2 = \left[\left(1 - \phi_2^2\right)\left(1 - \rho_1^2\right)\right]^{-1}$ and the relative efficiencies for estimating an intercept β_0, and the slope, β_1 of GLS to OLS for linear design with T=50, denoted $RE(\beta_0)$, and $RE(\beta_1)$. Looking at the table, it may be seen that the choices (.2, -.9) and (.8, -.9) give the worst performance of OLS as compared to GLS for estimating (β_0, β_1) of the regression coefficients and the largest values of σ_u^2. However, the choices (.2, -.7) and (.2, -.1) give the moderate and best performance of OLS as compared to GLS and the smallest values of σ_u^2. Results for other sample sizes and designs demonstrate a similar pattern as in Table (2).

The regression coefficients β_0, and β_1 for an intercept and the slope were each chosen to be equal one. Breusch (1980) has shown that for a fixed design, the distribution of $\dfrac{\hat{\beta}_{EGLS} - \beta}{\sigma_u^2}$ does not depend on the choice for β and σ_u^2, and the result holds even if the covariance matrix Σ is misspecified. When the design vector is stochastic, the assumption of a fixed design can be constructed as conditioning upon a given realization of the design, provided that the design is independent of u_t, Koreisha et al. (2002).

Definition
       The efficiency of the GLS estimates relative to that of OLS in terms of the mean squared error of the regression coefficient, $\hat{\zeta}_{\beta_j}$, is given by:

$$\hat{\zeta}_{\beta_j} = \frac{\sum\limits_{i=1}^{k}\left(\tilde{\beta}_{ij,GLS} - \beta_j\right)^2}{\sum\limits_{i=1}^{k}\left(\hat{\beta}_{ij,OLS} - \beta_j\right)^2} \qquad (7)$$

where $j = 0,1$, for four GLS estimates, and k is the number of simulations. A ratio less than one indicates that the GLS estimates is more efficient than OLS, and if $\hat{\zeta}_{\beta_j}$ is close to one,

then the OLS estimate is nearly as efficient as GLS estimates.

The Simulation Results for $\hat{\zeta}_{\beta_j}$
       Tables (3) through (6) show the complete simulation results of the ratios of the GLS estimators relative to the OLS estimator in terms of the mean squared error of the regression coefficients, $\hat{\zeta}_{\beta_0}$ and $\hat{\zeta}_{\beta_1}$ in (7), when the serially correlated disturbance follows an AR(2) process. Each table presents the results for the three sample sizes considered, as well as all four selected pairs of AR(2) parametrizations. Each of the different designs is presented in a separate table.
       Note that regardless of the sample size, selected design vectors, and AR(2) parametrizations the efficiency in estimating an intercept, β_0, and the slope, β_1, of the regression coefficients is higher for the GLS-AR(2) estimator than OLS. This result emphasizes that GLS is the BLUE. However, OLS performs nearly as efficiently as GLS for all selected sample sizes and designs when $\Phi = (.2, -.1)$. This result is not surprising since the choice of $\Phi = (.2, -.1)$ gives the highest performance of OLS as compared to GLS, in addition, it gives the smallest values of ρ_1, ρ_2, and σ_u^2.

Table 2: Autocorrelation Coefficients, Disturbance Variances and the Relative Efficiencies of GLS to OLS for Standardized Linear Design with T = 50

(ϕ_1, ϕ_2)	ρ_1	ρ_2	σ_u^2	$RE(\beta_0)$	$RE(\beta_1)$
(.2, -.9)	.1053	-.8789	5.3221	.7656	.5645
(.8, -.9)	.4211	-.5632	6.3973	.8325	.6026
(.2, -.7)	.1176	-.6765	1.9883	.9414	.8531
(.2, -.1)	.1818	-.0636	1.0446	.9993	.9980

When the order of the disturbance term is under estimated, i.e. EIGLS-AR(1), the GLS estimate performs poorly. In fact, OLS is more efficient for nearly every situation considered here. For example, when Φ = (.8, -.9) for quadratic design with T = 50, $\left(\hat{\zeta}_{\beta_0}, \hat{\zeta}_{\beta_1} \right)$ = $(1.4179, 1.7296)$ as shown in Table (3).

This shows that EIGLS-AR(1) can be much less efficient than OLS. The poor performance of EIGLS-AR(1) relative to OLS is most marked when the sample size is relatively

estimation is smaller than an appropriate estimated AR structure. This suggests the small (i.e. T = 50) and the order of the autoregressive process used in the GLS surprising result that OLS may often be better than assuming an AR(1) when the actual process is AR(2). However, for the choice of Φ = (.2, -.1) there is little difference between OLS and EIGLS-AR(1). For example, for linear design with T=200, $\left(\hat{\zeta}_{\beta_0}, \hat{\zeta}_{\beta_1} \right)$ = (.9998, .9984) as presented in Table (4).

Table 3: Efficiency for MSEs of the Regression Coefficients of the GLS Estimators Relative to OLS Estimator for Quadratic Design

(Φ_1, Φ_2)

Size	Estimator	(.2, -.9)		(.8, -.9)		(.2, -.7)		(.2, -.1)	
		$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$
50	GLS-AR(2)	0.7929	0.5540	0.8321	0.6174	0.9435	0.8349	1.0002	0.9954
	EGLS-AR(2)	0.7934	0.5567	0.8342	0.6172	0.9453	0.8409	1.0079	1.0094
	EIGLS-AR(1)	1.0935	1.1973	1.4179	1.7296	1.0355	1.0861	1.0063	1.0050
	EIGLS-AR(4)	0.7968	0.5623	0.8399	0.6182	0.9564	0.8500	1.0385	1.0332
100	GLS-AR(2)	0.8660	0.6950	0.8849	0.7104	0.9638	0.9287	1.0003	0.9993
	EGLS-AR(2)	0.8661	0.6957	0.8844	0.7089	0.9676	0.9319	0.9993	0.9980
	EIGLS-AR(1)	1.0453	1.0963	1.2136	1.4127	1.0207	1.0348	0.9989	1.0001
	EIGLS-AR(4)	0.8651	0.6974	0.8861	0.7093	0.9723	0.9342	1.0078	1.0091
200	GLS-AR(2)	0.9410	0.8331	0.9700	0.8269	0.9628	0.9400	1.0004	0.9984
	EGLS-AR(2)	0.9409	0.8326	0.9702	0.8265	0.9637	0.9400	1.0016	0.9990
	EIGLS-AR(1)	1.0180	1.0417	1.0453	1.2683	1.0094	1.0187	1.0014	1.0023
	EIGLS-AR(4)	0.9418	0.8338	0.9707	0.8290	0.9627	0.9407	1.0018	1.0021

Table 4: Efficiency for MSEs of the Regression Coefficients of the GLS Estimators Relative to OLS Estimator for Linear Design

Size	Estimator	(Φ_1, Φ_2)							
		(.2, -.9)		(.8, -.9)		(.2, -.7)		(.2, -.1)	
		$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$
50	GLS-AR(2)	0.7472	0.5740	0.8214	0.6193	0.9740	0.8511	1.0012	0.9964
	EGLS-AR(2)	0.7485	0.5771	0.8219	0.6200	0.9773	0.8548	1.0091	1.0073
	EIGLS-AR(1)	1.1004	1.1893	1.3624	1.6995	1.0181	1.0895	1.0055	1.0079
	EIGLS-AR(4)	0.7490	0.5824	0.8255	0.6220	0.9868	0.8595	1.0122	1.0448
100	GLS-AR(2)	0.8756	0.6641	0.8992	0.7340	0.9724	0.9204	1.0005	0.9996
	EGLS-AR(2)	0.8766	0.6632	0.8992	0.7323	0.9718	0.9219	1.0025	1.0003
	EIGLS-AR(1)	1.0349	1.0995	1.1826	1.4783	1.0156	1.0266	1.0025	1.0023
	EIGLS-AR(4)	0.8782	0.6654	0.8992	0.7391	0.9758	0.9285	1.0133	1.0021
200	GLS-AR(2)	0.9127	0.8137	0.9584	0.8662	0.9623	0.9262	0.9990	0.9977
	EGLS-AR(2)	0.9127	0.8135	0.9586	0.8662	0.9621	0.9271	1.0000	0.9980
	EIGLS-AR(1)	1.0252	1.0464	1.0666	1.2104	1.0092	1.0175	0.9998	0.9984
	EIGLS-AR(4)	0.9117	0.8123	0.9584	0.8668	0.9618	0.9255	1.0022	1.0032

This result is expected because the choice of $\phi_2 = -.1$ indicates that the serially correlated disturbance very nearly AR(1) since ϕ_2 is close to zero.

To further demonstrate the efficiency of OLS, consider the quadratic and linear designs. OLS is nearly as efficient or more efficient in estimating (β_0, β_1) than the GLS estimators; EGLS-AR(2), and EIGLS-AR(4), for moderate and large sample sizes (i.e. T=100 and 200) with AR(2) parametrizations $\Phi = (.2, -.7)$ and (.2, -.1) Tables (3) and (4). However, there are examples where OLS performs poorly as well. For the exponential design, OLS is nearly as efficient as EGLS-AR(2), and EIGLS-AR(4) for all sample sizes only when $\Phi = (.2, -.1)$.

Otherwise, OLS performs poorly as shown in Table (5). For example, when T = 50 with $\Phi = (.2, -.9)$, $\hat{\zeta}_{\beta_1} = (.2035, .2108)$. However, even in this case, the performance of the OLS estimator for estimating the intercept is not bad, $\hat{\zeta}_{\beta_0} = (.7561, .7606)$. In fact, the performance of OLS is always better for estimating the intercept than the slope.

For the standard Normal stochastic design model, OLS fares more poorly. Only for $\Phi = (.2, -.1)$ does the efficiency of OLS match GLS as shown in Table (6). However, regardless of the sample size, OLS performs as nearly as efficiently or better than EIGLS-AR(1) for all selected autoregressive coefficients for estimating β_0.

Table 5: Efficiency for MSEs of the Regression Coefficients of the GLS Estimators
Relative to OLS Estimator for Exponential Design

		(Φ_1, Φ_2)							
		(.2, -.9)		(.8, -.9)		(.2, -.7)		(.2, -.1)	
Size	Estimator	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$
50	GLS-AR(2)	0.7529	0.1951	0.8208	0.2160	0.9394	0.5576	0.9986	0.9706
	EGLS-AR(2)	0.7561	0.2035	0.8256	0.2241	0.9464	0.5642	0.9987	1.0030
	EIGLS-AR(1)	1.0451	1.1030	1.1649	1.1683	1.0167	1.0656	0.9969	1.0019
	EIGLS-AR(4)	0.7606	0.2108	0.8293	0.2322	0.9473	0.5815	1.0042	1.0775
100	GLS-AR(2)	0.8922	0.1979	0.9139	0.2311	0.9668	0.5461	1.0009	0.9830
	EGLS-AR(2)	0.8895	0.2021	0.9163	0.2353	0.9682	0.5467	0.9993	0.9980
	EIGLS-AR(1)	1.0115	1.0803	1.0893	1.1383	1.0077	1.0575	0.9991	0.9965
	EIGLS-AR(4)	0.8904	0.2068	0.9149	0.2357	0.9692	0.5578	0.9997	1.0187
200	GLS-AR(2)	0.9168	0.2139	0.9771	0.2084	1.0162	0.5293	1.0022	0.9877
	EGLS-AR(2)	0.9164	0.2143	0.9782	0.2132	1.0150	0.5303	1.0008	0.9990
	EIGLS-AR(1)	1.0053	1.0645	1.0492	1.2352	0.9999	1.0390	1.0012	0.9900
	EIGLS-AR(4)	0.9171	0.2161	0.9802	0.2151	1.0149	0.5425	1.0006	1.0062

Table 6: Efficiency for MSEs of the Regression Coefficients of the GLS Estimators Relative to OLS Estimator Standard Normal Stochastic Design

(Φ_1, Φ_2)

Size	Estimator	(.2, -.9)		(.8, -.9)		(.2, -.7)		(.2, -.1)	
		$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$	$\hat{\zeta}_{\beta_0}$	$\hat{\zeta}_{\beta_1}$
50	GLS-AR(2)	0.6416	0.1127	0.7427	0.0667	0.8851	0.3334	0.9998	0.8838
	EGLS-AR(2)	0.6437	0.1149	0.7442	0.0676	0.8807	0.3428	1.0176	0.9513
	EIGLS-AR(1)	1.0536	0.9591	1.2168	0.5186	1.0129	0.9598	1.0091	0.9244
	EIGLS-AR(4)	0.6509	0.1211	0.7477	0.0737	0.8908	0.3652	1.0385	1.0221
100	GLS-AR(2)	0.7601	0.1055	0.8466	0.0640	0.9350	0.3230	0.9978	0.8902
	EGLS-AR(2)	0.7598	0.1060	0.8472	0.0639	0.9341	0.3274	0.9984	0.9076
	EIGLS-AR(1)	1.0241	0.9568	1.1141	0.5261	1.0121	0.9546	0.9994	0.9158
	EIGLS-AR(4)	0.7611	0.1109	0.8477	0.0668	0.9346	0.3386	1.0033	0.9359
200	GLS-AR(2)	0.8624	0.1002	0.9323	0.0720	0.9715	0.3226	1.0038	0.9194
	EGLS-AR(2)	0.8628	0.1006	0.9319	0.0725	0.9707	0.3245	1.0028	0.9331
	EIGLS-AR(1)	1.0141	0.9581	1.0400	0.5181	1.0021	0.9598	1.0033	0.9418
	EIGLS-AR(4)	0.8630	0.1033	0.9314	0.0748	0.9719	0.3307	1.0058	0.9512

Discussion

In investigating the simulation results in the previous section, the following significant results were observed. First and foremost, it was noticed that regardless of the sample size for all design structures and selected autoregressive coefficients, the efficiency in estimating an intercept, β_0, and the slope, β_1, of the regression model is higher for the GLS estimator based on the correct disturbance model structures and known AR(2) coefficients. This result is expected since GLS is BLUE, but because GLS requires a priori knowledge of Σ, this is not a viable option.

In addition, the relative efficiency of OLS is better than EIGLS-AR(1) in estimating (β_0, β_1) for all sample sizes and nonstochastic design vectors. The relative efficiency of OLS to be superior to that of EIGLS in estimating the slope when T=50 with AR(2) parametrization (.8, -.9) was also observed. This choice of (.8, -.9) gives the highest first-order autoregressive coefficient $(\rho_1 = .4211)$ and largest variance of the error process $(\sigma_u^2 = 6.3973)$ among the other choices of AR(2) parametrizations. This explains the poor relative performance of OLS to GLS for this choice of parameter.

However, from Table (3) through Table (6), it may be seen that the performance of EIGLS-AR(1) is even worse. This appears to occur because AR(2) parametrization (.8, -.9) produces large values of ρ_1, ρ_2 in absolute value ($\rho_2 = -.5632$) and disturbance variance comparing to the other parameter choices. This means using OLS is better than assuming another incorrect error process.

The third general conclusion from the simulation study is that regardless of the sample size, all of the estimators perform equally well with AR(2) parametrization   (.2, -.1). This result is not surprising because the choice of (.2, -.1) gives the smallest variance of the process $\left(\sigma_u^2 = 1.0446\right)$, which is sufficiently close to the variance of standard OLS.

Fourth, for all stochastic and non-stochastic design vectors, the differences in the relative efficiency of OLS and all GLS estimators in estimating β_0 with a few expected exceptions are negligible. In fact, this is so even when the variance of the process is large, in other words, when AR(2) parametrizations are (.2, -.9) and (.8, -.9).

Similar to results for section 2, when the design vector is linear or quadratic, the relative efficiency of OLS is nearly as good as the EGLS-AR(2) and EIGLS-AR(4) estimators for moderate and large sample sizes for estimating β_1 with small variance of the disturbances.

It is observed that the differences in the relative efficiencies of GLS-AR(2), EGLS-AR(2), and EIGLS-AR(4) in estimating (β_0, β_1) are insignificant. Hence, when confronted with an error with unknown order p, it appears that using AR(4) is the best bet.

Finally, OLS may often be more preferable than assuming an AR(1) process when the actual process is AR(2). In other words, it is sometimes better to ignore the autocorrelation of the disturbance term and use the OLS estimation rather than to incorrectly assume the process is an AR(1).

Future Research

Perhaps, even more important than the efficiency of the different estimation methods in these models, is the effect on forecasting performance. Koreisha et al. (2004) investigated the impact that EIGLS correction may have on forecast performance. They developed a new procedure for generating forecasts for regression models with auto-correlated disturbances based on OLS and a finite AR process. They found that for predictive purposes there is not much gained in trying to identify the actual order and form of the auto-correlated disturbances or using more complicated estimation methods such as GLS or MLE procedures, which often require inversion of large matrices. It is necessary to extend Koreisha et al. (2004) results for different design vectors of the independent variables including both stochastic and nonstochastic designs instead of using one independent variable generated by an AR(1) process as in their investigation.

A second important consideration is the estimation of the standard errors of the estimators. In practice, if one were using a statistical package to compute the OLS estimators the variance estimate produced would be based on $\sigma_u^2\left(X'X\right)^{-1}$, which may be biased for the true variance $\sigma_u^2\left(X'X\right)^{-1}X'\sum X\left(X'X\right)^{-1}$. For GLS estimation ($\Sigma$ known), on the other hand, the variance estimate is unbiased for the true variance of the GLS estimator. It is unclear, however, how the variance estimators for EGLS estimation behave. The impact that the variance estimators may have on inference based on the OLS estimator is currently being investigated.

Finally, the long range goal is the creation of guidelines or rules of thumb which will aid the practitioner when deciding which regression estimation procedure to use.

Conclusion

This article has investigated an important statistical problem concerning estimation of the regression coefficients in the presence of autocorrelated disturbances. In particular, the comparison of efficiency of the ordinary least squares (OLS) estimation to alternative procedures such as generalized least squares (GLS) and estimated GLS (EGLS) estimators in the presence of autocorrelated disturbances was

discussed. Both stochastic and non-stochastic design vectors were used with different sample sizes.

It was found that regardless of the sample size, design vector, and order of the auto-correlated disturbances, the relative efficiency of the OLS estimator generally increases with decreasing values of the disturbance variances. In particular, if the disturbance structure is a first or second order autoregressive and the dependent variable is nonstochastic and linear or quadratic, OLS performs nearly as well as its competitors for small values of the disturbance variances. The gain in efficiency of the GLS estimator for different design vectors such as exponential and standard Normal compared to the OLS estimator is substantial for moderate and large values of the autoregressive coefficient in the case of an AR(1) process and large values of the disturbance variance in the presence of an AR(2) process. However, for small values of the autoregressive coefficient and disturbance variance the OLS estimator appears to be nearly as efficient as the GLS estimator.

It was also found that if the error structure is autoregressive, and the dependent variable is nonstochastic and linear or quadratic, the OLS estimator performs nearly as well as its competitors. When faced with an unknown error structure, however, AR(4) may be the best choice.

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