Journal of Modern Applied Statistical Methods

Volume 5 | Issue 1 Article 14

5-1-2006

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Journal of Modern Applied Statistical Methods
May, 2006, Vol. 5, No. 1, 152-170

1538 – 9472/06/$95.00

Analysis Of Type-II Progressively Hybrid Censored Competing Risks Data

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A Type-II progressively hybrid censoring scheme for competing risks data is introduced, where the experiment terminates at a pre-specified time. The likelihood inference of the unknown parameters is derived under the assumptions that the lifetime distributions of the different causes are independent and exponentially distributed. The maximum likelihood estimators of the unknown parameters are obtained in exact forms. Asymptotic confidence intervals and two bootstrap confidence intervals are also proposed. Bayes estimates and credible intervals of the unknown parameters are obtained under the assumption of gamma priors on the unknown parameters. Different methods have been compared using Monte Carlo simulations. One real data set has been analyzed for illustrative purposes.

Key words: Competing risk; maximum likelihood estimator; Type-I and Type-II censoring; Fisher information matrix; asymptotic distribution; bayesian inference; exponential distribution; gamma distribution; Type-II progressive censoring scheme.

Introduction

In medical studies or in reliability analysis, it is quite common that more than one cause or risk factor may be present at the same time. In analyzing the competing risks model, it is assumed that data consists of a failure time and an indicator denoting the cause of failure. Several studies have been carried out under this assumption for both the parametric and the non-parametric set up. For the parametric set up it is assumed that different lifetime distributions follow some special parametric distribution, namely exponential, Weibull or gamma. Several authors, for example Berkson and Elveback (1960), Cox (1959), David and Moeschberger (1978) considered the problem from the parametric point of view. In the non-parametric set up, no specific lifetime distribution is assumed. Kaplan and Meier (1958), Efron (1967) and Peterson (1991) analyzed the non-parametric version of this model.

The two most common censoring schemes, namely Type-I and Type-II censoring schemes, are widely used in practice. Briefly, they can be described as follows. Consider n items are under observations in a particular experiment. In the conventional Type-I censoring scheme, the experiment continues up to a pre-specified time T. On the other hand, the conventional Type-II censoring scheme requires the experiment to continue until a pre-specified number of failures $m \leq n$ occurs. In this scenario, only the smallest lifetimes are observed. The mixture of Type-I and Type-II censoring schemes is known as the hybrid censoring scheme. This hybrid censoring scheme was first introduced by Epstein (1954; 1960). But, recently it becomes quite popular in the reliability and life-testing experiments. See for example the work of Chen and Bhattacharya (1988), Childs, Chandrasekhar, Balakrishnan, and Kundu (2003), Draper and Guttman (1987),

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Fairbanks, Madasan and Dykstra (1982), Gupta and Kundu (1998), and Jeong, Park and Yum (1996).

One of the drawbacks of the conventional Type-I, Type-II, or hybrid censoring schemes is that they do not allow for removal of units at points other than the terminal point of the experiment. When the items are highly reliable it might be necessary to know the causes for which the items are failed and also necessary to remove items in between the experiment (at the time of each failure) for efficient estimation of the parameters. Because of this, one censoring scheme known as progressive censoring scheme under competing risks becomes very popular for the last few years. It can be described as follows: Consider n items in a study and assume that there is K causes of failure, which are known. Suppose m < n is fixed before the experiment. Moreover, m other integers, $R_1, \ldots, R_m$ are also fixed before so that $R_1 + \ldots + R_m + m = n$. At the time of the first failure $X_{1:m:n}$, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure $X_{2:m:n}$, R_2 of the remaining units are randomly removed and so on. Finally, at the time of the m^th^ failure $X_{m:m:n}$, the rest of the R_m units are removed. It is also known that the first failure takes place due to cause δ_1, similarly the second failure takes place due to cause δ_2 and so on, finally the m^{th} failure takes place due to cause δ_m. For an exhaustive list of references and further details on Type-II progressive censoring, the readers may refer to the book by Balakrishnan and Aggarwala (2000).

In this article, a Type-II progressively hybrid censoring scheme under competing risk is introduced. As the name suggests, it is a mixture of Type-II progressive and hybrid censoring schemes under the competing risk data. In this new censoring scheme, the likelihood inference of the unknown parameters is obtained, under the assumptions that the lifetime distributions of the different causes are independent identically distributed (i.i.d.) exponential random variables. It is observed that the maximum likelihood estimators of the unknown parameters always exist and one obtains the explicit form of the maximum likelihood estimators (MLEs) of the unknown parameters. One also obtains the asymptotic confidence intervals and proposed two bootstrap confidence intervals. Bayes estimates and credible intervals are also obtained under the assumption of the gamma priors on the unknown parameters. Different methods are compared using Monte Carlo simulations and for illustrative purposes, one real data set is analyzed.

Model Description and Notation

Suppose n identical items are put on a test and the lifetime distributions of the n items are denoted by $X_1, \ldots, X_n$. The integer m < n is pre-fixed and also $R_1, \ldots, R_m$ are m pre-fixed integers satisfying $R_1 + \ldots + R_m + m = n$. T is a pre-fixed time point. At the time of first failure R_1 of the remaining units are randomly removed. Similarly at the time of the second failure R_2 of the remaining units are removed and so on. If the m^{th} failure occurs before the time point T, the experiment stops at the time point $X_{m:m:n}$. On the other hand, suppose the m^{th} failure does not occur before time point T and only J failures occur before the time point T, where $0 \leq J < m$, then at the time point T all the remaining R_J^* units are removed and the experiment terminates at the time point T. Note that $R_J^* = n - (R_1 + \ldots + R_J) - J$. The two cases are denoted as Case I and Case II respectively and this censoring scheme is referred to as the Type-II progressively hybrid censoring scheme under competing risk data. In the presence of Type-II progressively hybrid censoring scheme under competing risks data, the following is a type of observation:

Case I: $\{(X_{1:m:n}, \delta_1, R_1), \ldots, (X_{m:m:n}, \delta_m, R_m)\}$; if $X_{m:m:n} < T$, or Case II: $\{(X_{1:m:n}, \delta_1, R_1), \ldots, (X_{J:m:n}, \delta_J, R_J), (T, R_J^*)\}$; if $X_{J:m:n} < T < X_{J+1:m:n}$.

Note that for Case II, $X_{J:m:n} < T < X_{J+1:m:n} < \ldots < X_{m:m:n}$ and $X_{J+1:m:n} < \ldots < X_{m:m:n}$ are not observed.

The conventional Type-I progressive censoring scheme needs the pre-specification of $R_1, \ldots, R_m$ and also $T_1, \ldots, T_m$, see Cohen (1963; 1966) for details. The choices of $T_1, \ldots, T_m$ are not trivial. For the conventional Type-II progressive censoring scheme the experimental time is unbounded. In the proposed censoring scheme, the choice of T depends upon how much maximum experimental time the experimenter can afford to spend. Moreover, the experimental time is bounded.

Without loss of generality, it is assumed that there are only two independent causes of failure i.e. K = 2. It may be extended to the case of K > 2. Before progressing further, the following notations are introduced/ reviewed:

X_{ji} : lifetime of the i[th] individual under cause j; for j = 1, 2 and i = 1, . . . , n

$X_{i:m:n}$: i[th] observed failure time; i = 1, . . . ,m

f(.) : probability density function (PDF) of X_i

F(.) : cumulative distribution function (CDF) of X_i

$F_j(.)$: cumulative distribution function (CDF) of X_{ji}

m_1 : the number of failures observed before termination due to cause 1 for Case I

m_2 : the number of failures observed before termination due to cause 2 for Case I

m : total number of failures observed before termination for Case I; *i.e.* $m = m_1 + m_2$

J_1 : the number of failures observed before termination due to cause 1 for Case II

J_2 : the number of failures observed before termination due to cause 2 for Case II

J : total number of failures observed before termination for Case II; *i.e.* $J = J_1 + J_2$

D_1 : the number of failures due to cause 1, *i.e.* $D_1 = m_1$ for Case I and $D_1 = J_1$ for Case II

D_2 : the number of failures due to cause 2, *i.e.* $D_2 = m_2$ for Case I and $D_2 = J_2$ for Case II

D : total number of failures, *i.e.* $D = m = m_1 + m_2$ for Case I and $D = J = J_1 + J_2$ for Case II

R_i : the number of units removed at the time of i[th] failure; $R_i \geq 0$

R_J^* : the number of remaining units left at the time point T for Case II

δ_i : indicator variable denoting the cause of failure of the i[th] individual

e(λ) : exponential random variable with PDF $\lambda e^{-\lambda x}$

gamma(α, λ) : gamma random variable with PDF $\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$

It is assumed that (X_{1i}, X_{2i}); i = 1, . . ., n are n *i.i.d.* exponential random variables. Further, X_{1i} and X_{2i} are independent for all i = 1, . . ., n and $X_i = \min(X_{1i}, X_{2i})$. Now, the MLEs of the unknown parameters are provided when X_{ji}'s (for I = 1, . . ., n) are *i.i.d.* exp(λ_j), for j= 1, 2.

Maximum Likelihood Estimator

Based on the observations as discussed in the previous subsection, the log-likelihood function (without the constant term) can be written as;

$$L(\lambda_1, \lambda_2) = D_1 \ln \lambda_1 + D_2 \ln \lambda_2 - (\lambda_1 + \lambda_2) W, \quad (1)$$

where

$$D_1 = m_1,\ D_2 = m_2,\ W = \sum_{i=1}^{m} (1+R_i) x_{i:m:n}$$

for Case I and

$$D_1 = J_1,\ D_2 = J_2,\ W = \sum_{i=1}^{J} (1+R_i) x_{i:m:n} + TR_J^*$$

for Case II. From (1), it is clear that the MLEs of λ_1 and λ_2 always exists and they are

$$\hat{\lambda}_1 = \frac{D_1}{W} \quad \text{and} \quad \hat{\lambda}_2 = \frac{D_2}{W}. \tag{2}$$

It is not possible to obtain the exact distribution of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ because of the complicated nature of the conditional distributions of $X_{1:m:n}$, . . ., $X_{m:m:n}$ given $X_{m:m:n} < T$. Interestingly, the distribution of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the mixture of discrete and continuous distributions. They have positive masses at the point 0 and have the bounded supports. Since, the exact distributions of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are not known, the exact confidence intervals also cannot be obtained.

Confidence Intervals

In this section, three different confidence intervals are proposed. One is based on the asymptotic distribution of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ and two different bootstrap confidence intervals.

Asymptotic Confidence Interval

In this section, we present the Fisher Information matrix of λ_1 and λ_2. Let $I(\lambda_1, \lambda_2) = (I_{ij}(\lambda_1, \lambda_2))$; i, j =1, 2, denote the Fisher Information matrix of the parameters λ_1 and λ_2, where

$$I_{ij}(\lambda_1, \lambda_2) = -E\left[\frac{\partial^2 L(\lambda_1, \lambda_2)}{\partial \lambda_i \partial \lambda_j}\right] \tag{3}$$

From (1) it follows that

$$I_{11}(\lambda_1, \lambda_2) = \frac{E(D_1)}{\lambda_1^2},$$

$$I_{12}(\lambda_1, \lambda_2) = I_{21}(\lambda_1, \lambda_2) = 0$$

and

$$I_{22}(\lambda_1, \lambda_2) = \frac{E(D_2)}{\lambda_2^2}.$$

Simple calculation shows that

$$E(D_1) = \sum_{i=1}^{m_1} P(X_{i:m:n} < T)$$

and

$$E(D_2) = \sum_{i=1}^{m_2} P(X_{i:m:n} < T).$$

It is not easy to compute $P(X_{i:m:n} < T)$ for general i, because $X_{i:m:n}$ is a sum of i independent, but not identically distributed exponential random variables. Therefore, for $D_1 > 0$ and $D_2 > 0$, the following approximate $100(1-\alpha)\%$ confidence interval for λ_1 and λ_2 are proposed,

$$\hat{\lambda}_1 \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{\lambda}_1^2}{D_1}}$$

and

$$\hat{\lambda}_2 \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{\lambda}_2^2}{D_2}} \tag{4}$$

respectively.

Bootstrap Confidence Intervals

In this subsection, two confidence intervals based on the bootstrapping are proposed. The two bootstrap methods that are widely used in practice are:

(1) The percentile bootstrap (Boot-p) proposed by Efron (1982), and

(2) The bootstrap-t method (Boot-t) proposed by Hall (1988).

It is observed that in this type of situations (Kundu, Kannan, & Balakrishnan, 2004), the non-parametric bootstrap method does not work well. Hence, the following two parametric bootstrap confidence intervals for λ_1 and λ_2 are proposed. The procedure is illustrated for the parameter λ_1. For the other parameter (λ_2), a confidence interval may be constructed in an analogous manner.

Boot-p Method

1. Estimate $\hat{\lambda}_1$ and $\hat{\lambda}_2$ from the sample using (2).
2. Generate a bootstrap sample $\{X^*_{1:m:n},\ldots,X^*_{D^*:m:n}\}$, using $\hat{\lambda}_1$ and $\hat{\lambda}_2$, $R_1, \ldots, R_m$ and T. Obtain the bootstrap estimate of λ_1 say, $\hat{\lambda}_1^*$ using the bootstrap sample.
3. Repeat Step 2 NBOOT times.
4. Let $\hat{CDF}(x) = P(\hat{\lambda}_1^* \le x)$, be the cumulative distribution function of $\hat{\lambda}_1^*$. Define $\hat{\lambda}_{1Boot-p}(x) = \hat{CDF}^{-1}(x)$ for a given x. The approximate 100(1-α)% confidence interval for λ_1 is given by:

$$\left(\hat{\lambda}_{1Boot-p}\left(\frac{\alpha}{2}\right), \hat{\lambda}_{1Boot-p}\left(1-\frac{\alpha}{2}\right)\right).$$

Boot-t Method

1. Estimate $\hat{\lambda}_1$ and $\hat{\lambda}_2$ from the sample using (2) as before.
2. Generate a bootstrap sample $\{X^*_{1:m:n},\ldots,X^*_{D^*:m:n}\}$, using $\hat{\lambda}_1$ and $\hat{\lambda}_2$, $R_1; \ldots; R_m$ and T. Also compute $\hat{V}(\hat{\lambda}_1^*) = \frac{\hat{\lambda}_1^{*2}}{D_1^*}$ for $D_1^* > 0$.
3. Determine the T_1^* statistic

$$T_1^* = \frac{(\hat{\lambda}_1^* - \hat{\lambda}_1)}{\sqrt{\hat{V}(\hat{\lambda}_1^*)}}$$

4. Repeat Steps 2 - 3 NBOOT times.
5. Let $\hat{CDF}(x) = P(T_1^* \le x)$, be the cumulative distribution function of T_1^*. For a given x, define $\hat{\lambda}_{1Boot-t}(x) = \hat{\lambda}_1 + \sqrt{\hat{V}(\hat{\lambda}_1^*)}\,\hat{CDF}^{-1}(x)$. The approximate 100(1-$\alpha$)% confidence interval for λ_1 is given by

$$\left(\hat{\lambda}_{1Boot-t}\left(\frac{\alpha}{2}\right), \hat{\lambda}_{1Boot-t}\left(1-\frac{\alpha}{2}\right)\right).$$

Bayesian Analysis

In this section, the problem is approached from the Bayesian point of view. In the context of exponential lifetimes, λ_1 and λ_2 may be reasonably modelled by the gamma priors. It is assumed that λ_1 and λ_2 are independently distributed as *gamma* (a_1, b_1) and *gamma* (a_2, b_2) priors, respectively. The gamma parameters a_1, b_1, a_2 and b_2 are all assumed to be positive. When $a_1 = b_1 = 0$ ($a_2 = b_2 = 0$), one obtains the non-informative priors of λ_1 (λ_2). The posterior density of λ_1 and λ_2 based on the gamma priors is given by

$$l(\lambda_1, \lambda_2 | data) \propto \lambda_1^{D_1+a_1-1}\lambda_2^{D_2+a_2-1}e^{-\lambda_1(W+b_1)}e^{-\lambda_2(W+b_2)} \tag{5}$$

From (5), it is clear that the posterior density functions of λ_1 and λ_2, say l(λ_1|*data*) and l(λ_2|*data*), respectively, are independent. Further, l(λ_1|*data*) is the density function of a *gamma*($D_1 + a_1$, $W + b_1$) random variable, and l(λ_2|*data*) is the density function of a *gamma*($D_2 + a_2$, $W + b_2$) random variable.

Therefore, the Bayes estimates of λ_1 and λ_2 under squared error loss functions are

$$\hat{\lambda}_{1Bayes} = \frac{D_1 + a_1}{W + b_1}$$

and

$$\hat{\lambda}_{2Bayes} = \frac{D_2 + a_2}{W + b_2} \quad (6)$$

respectively. Interestingly, when the non-informative priors $a_1 = b_1 = a_2 = b_2 = 0$, the Bayes estimators coincide with the corresponding MLEs.

The credible intervals for λ_1 and λ_2 can be obtained using the posterior distributions of λ_1 and λ_2. Note that *a posteriori* $Z_1 = 2\lambda_1 (W + b_1)$ and $Z_2 = 2\lambda_2 (W + b_2)$ follow χ^2 distributions with $2(D_1 + a_1)$ and $2(D_2 + a_2)$ degrees of freedom respectively, provided both $2(D_1 + a_1)$ and $2(D_2 + a_2)$ are positive integers. Therefore, $100(1-\alpha)\%$ credible intervals for λ_1 and λ_2 are

$$\left[\frac{\chi^2_{2(D_1+a_1),1-\frac{\alpha}{2}}}{2(W+b_1)}, \frac{\chi^2_{2(D_1+a_1),\frac{\alpha}{2}}}{2(W+b_1)}\right]$$

and

$$\left[\frac{\chi^2_{2(D_2+a_2),1-\frac{\alpha}{2}}}{2(W+b_2)}, \frac{\chi^2_{2(D_2+a_2),\frac{\alpha}{2}}}{2(W+b_2)}\right] \quad (7)$$

respectively for $(D_1 + a_1) > 0$ and $(D_2 + a_2) > 0$. Here $\chi^2_{k,\frac{\alpha}{2}}$ and $\chi^2_{k,1-\frac{\alpha}{2}}$ denote the lower and upper $\frac{\alpha}{2}$-th percentile points of a χ^2 distribution with k degrees of freedom. Note that if $2(D_1 + a_1)$ and $2(D_2 + a_2)$ are not integer values, then gamma distribution can be used to construct the credible intervals. If no prior information is available, then non-informative priors can be used to compute the credible intervals for λ_1 and λ_2. Alternatively, using the suggestion of Congdon (2001), very small positive values of a_1, b_1, a_2 and b_2 can be used to construct the Bayes estimates or the corresponding credible intervals.

Numerical Results and Discussions

Since the performance of the different methods cannot be compared theoretically, Monte Carlo simulations are used to compare different methods for different parameter values and for different sampling schemes. The term different sampling schemes means for different sets of R_i's and for different T values. All the computations are performed using Pentium IV processor and using the random number generation algorithm RAN2 of Press, Flannery, Teukolsky, & Vetterling.(1991). All the programs are written in FORTRAN and they can be obtained from the authors on request.

Before progressing further, first a description of how the Type-II progressively hybrid censored competing risk data was generated for a given set n, m, $R_1, \ldots, R_m$ and T. The following transformation as suggested in Balakrishnan and Aggarwala (2000) is used.

$$Z_1 = nX_{1:m:n}$$
$$Z_2 = (n - R_1 - 1)(X_{2:m:n} - X_{1:m:n})$$
$$\vdots$$
$$Z_m = (n - R_1 - \ldots - R_{m-1} - m + 1)(X_{m:m:n} - X_{m-1:m:n}). \quad (8)$$

It is known that if X_i's are *i.i.d.* *exp*$(\lambda_1 + \lambda_2)$, then the spacings Z_i's are also *i.i.d.* *exp*$(\lambda_1 + \lambda_2)$ random variables. From (8) it follows that

$$X_{1:m:n} = \frac{1}{n} Z_1$$
$$X_{2:m:n} = \frac{1}{n - R_1 - 1} Z_2 + \frac{1}{n} Z_1$$
$$\vdots$$
$$X_{m:m:n} = \frac{1}{n - R_1 - \ldots - R_{m-1} - m + 1} Z_m + \ldots + \frac{1}{n} Z_1. \quad (9)$$

Using (9), Type-II progressively hybrid censored competing risk data can be easily generated as follows. For a given n, m, $R_1,\ldots,R_m$, $X_{1:m:n},\ldots,X_{m:m:n}$ is generated using (9). Again using the random number generation algorithm RAN2 of Press *et al.* (1991), a new random variable U(i), for i = 1…m is generated. Now if U(i) < $\frac{\lambda_1}{\lambda_1+\lambda_2}$, then assign δ_i = 1 otherwise, δ_i = 2. If $X_{m:m:n}$ < T. Then, one has Case I and the corresponding sample is $\{(X_{1:m:n},\delta_1,R_1),\ldots,(X_{m:m:n},\delta_m,R_m)\}$ otherwise, one has Case II and J, such that $X_{J:m:n} < T < X_{J+1:m:n}$ is found. The corresponding sample is $\{(X_{1:m:n},\delta_1,R_1),\ldots,(X_{m:m:n},\delta_m,R_m),(T,R^*{}_J)\}$, where R^*_J is same as defined before.

Different n, m, T, λ_1, λ_2 and R_i's are considered. In all of the simulation experiments, λ_1 = 1.0 and λ_2 = 0.8 is taken. The following are taken n = 15, 25, 50, 100, m = 5, 10, 15, T = 0.25, 0.50, 1.00, 2.00 and three different sampling schemes. Scheme 1: $R_1 = \ldots = R_{m-1} = 0$ and R_m = n - m. Scheme 2: R_1 = n - m and $R_1 = \ldots = R_m = 0$. Scheme 3: $R_1 = \ldots = R_{m-1} = 1$ and R_m = n -2m + 1. For each case, the MLEs and the 95% confidence intervals of λ_1 and λ_2 are computed using all three of the proposed methods. For comparison purposes, the 95% credible intervals are computed using non-informative prior. The process is replicated 1000 times in each case and the average bias, mean squared errors, and the coverage percentages are reported. The results are reported in Tables 1 - 9.

Some of the important observations are as follows. For fixed n as m increases the biases and MSEs of both λ_1 and λ_2 decrease for all cases as expected. But, interestingly for fixed m as n increases the biases increase and the MSEs decrease for both λ_1 and λ_2. This phenomenon is quite counter intuitive and a proper explanation cannot be found for this. Now, comparing different confidence intervals in terms of their average lengths and coverage percentages, it is observed that the MLEs, BOOT-T confidence intervals and Bayes credible intervals behave quite satisfactory unless the T is very small.

Otherwise, most of the cases of these three confidence intervals maintain the nominal coverage probabilities. Since BOOT-T method is involved numerically and the confidence intervals based on the asymptotic distributions are slightly larger than the Bayes credible intervals, it is recommended to use the Bayes credible intervals for all cases. Among the different schemes, it is observed that scheme 1 produces the smallest confidence intervals, followed by scheme 3 and scheme 2.

Data Analysis

In this section, one real-life dataset originally analyzed by Hoel (1972) is considered. The data arose from a laboratory experiment in which male mice received a radiation dose of 300 roentgens at 5 to 6 weeks of age. The cause of death for each mouse was determined by autopsy to be thymic lymphoma, reticulum cell sarcoma, or other causes. For the purpose of analysis, reticulum cell sarcoma is considered as cause 1 and the other causes of death are combined as cause 2. There were n = 77 observations in the data. A progressively type-II censored sample was generated from the original measurements.

Table 1: n = 15, m = 5*.

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
1	MLE	λ_1	0.2406 (1.2953)	0.2834 (1.2330)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1422 (0.6589)	0.1754 (0.6266)	0.1759 (0.6258)	0.1759 (0.6258)
		λ_1	2.8876 (86.4)	2.9185 (93.3)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4473 (90.5)	2.4790 (89.6)	2.4801 (89.6)	2.4801 (89.6)
	Boot-P	λ_1	4.0095 (88.3)	4.0829 (91.1)	4.0721 (91.6)	4.0717 (91.6)
		λ_2	3.2510 (87.0)	3.3224 (89.1)	3.3175 (89.4)	3.3172 (89.4)
	Boot-T	λ_1	2.6389 (87.7)	2.8758 (90.7)	2.9050 (90.6)	2.9055 (90.6)
		λ_2	2.1035 (89.8)	2.3166 (88.7)	2.3436 (88.7)	2.3438 (88.7)
	Bayes	λ_1	2.7977 (93.1)	2.8322 (93.8)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3545 (88.9)	2.3885 (91.6)	2.3895 (91.6)	2.3895 (91.6)
2	MLE	λ_1	0.2280 (1.7153)	0.2247 (1.3883)	0.2417 (1.2802)	0.2759 (1.2423)
		λ_2	0.1689 (1.0298)	0.1461 (0.7663)	0.1475 (0.6577)	0.1706 (0.6320)
		λ_1	3.6133 (79.0)	3.1929 (88.3)	2.9571 (90.7)	2.9142 (92.8)
		λ_2	3.0330 (69.5)	2.6902 (81.5)	2.5017 (87.5)	2.4762 (89.2)
	Boot-P	λ_1	4.1914 (77.3)	4.0090 (85.5)	4.0136 (90.7)	4.0654 (89.9)
		λ_2	3.3645 (67.7)	3.2375 (79.9)	3.2395 (86.2)	3.3093 (88.9)
	Boot-T	λ_1	3.3581 (78.7)	2.9655 (87.4)	2.8422 (91.3)	2.8636 (90.8)
		λ_2	2.6215 (69.4)	2.3683 (80.9)	2.2597 (88.1)	2.3070 (89.0)
	Bayes	λ_1	3.4450 (77.3)	3.0707 (87.1)	2.8612 (92.9)	2.8273 (93.6)
		λ_2	2.8805 (67.8)	2.5721 (80.6)	2.4046 (88.0)	2.3851 (91.0)
3	MLE	λ_1	0.2199 (1.3079)	0.2804 (1.2382)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1269 (0.6734)	0.1725 (0.6300)	0.1759 (0.6258)	0.1759 (0.6258)
		λ_1	2.9090 (89.5)	2.9144 (92.6)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4540 (87.9)	2.4755 (89.3)	2.4801 (89.6)	2.4801 (89.6)
	Boot-P	λ_1	3.9577 (89.2)	4.0778 (90.5)	4.0734 (91.6)	4.0717 (91.6)
		λ_2	3.2041 (85.2)	3.3183 (88.9)	3.3180 (89.4)	3.3172 (89.4)
	Boot-T	λ_1	2.6347 (91.1)	2.8461 (90.7)	2.9038 (90.6)	2.9055 (90.6)
		λ_2	2.0913 (88.2)	2.2907 (88.6)	2.3413 (88.7)	2.3438 (88.7)
	Bayes	λ_1	2.8142 (92.0)	2.8282 (93.7)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3580 (86.2)	2.3848 (91.1)	2.3895 (91.6)	2.3895 (91.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 2: n = 25, m = 5*.

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
1	MLE	λ_1	0.2825 (1.2347)	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1741 (0.6284)	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)
		λ_1	2.9170 (93.1)	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4770 (89.6)	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)
	Boot-P	λ_1	4.0845 (90.8)	4.0726 (91.6)	4.0717 (91.6)	4.0717 (91.6)
		λ_2	3.3214 (89.3)	3.3178 (89.4)	3.3172 (89.4)	3.3172 (89.4)
	Boot-T	λ_1	2.8529 (90.8)	2.9056 (90.6)	2.9055 (90.6)	2.9055 (90.6)
		λ_2	2.2954 (88.9)	2.3428 (88.7)	2.3437 (88.7)	2.3438 (88.7)
	Bayes	λ_1	2.8308 (93.6)	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3864 (91.2)	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)
2	MLE	λ_1	0.2370 (1.6967)	0.2279 (1.3813)	0.2414 (1.2803)	0.2759 (1.2423)
		λ_2	0.1712 (1.0103)	0.1482 (0.7633)	0.1483 (0.6561)	0.1715 (0.6314)
		λ_1	3.6058 (80.1)	3.1899 (88.8)	2.9538 (90.9)	2.9139 (92.8)
		λ_2	3.0232 (70.7)	2.6895 (81.9)	2.5017 (87.7)	2.4777 (89.3)
	Boot-P	λ_1	4.2070 (78.3)	4.0052 (85.3)	4.0114 (90.8)	4.0654 (90.0)
		λ_2	3.3690 (68.8)	3.2410 (79.5)	3.2438 (86.4)	3.3097 (88.9)
	Boot-T	λ_1	3.4596 (79.9)	2.9826 (87.5)	2.8495 (90.8)	2.8646 (90.7)
		λ_2	2.6999 (69.9)	2.3953 (81.5)	2.2670 (88.0)	2.3073 (89.0)
	Bayes	λ_1	3.4403 (78.2)	3.0685 (87.7)	2.8583 (93.0)	2.8271 (93.6)
		λ_2	2.8724 (69.2)	2.5718 (81.3)	2.4047 (88.2)	2.3866 (91.1)
3	MLE	λ_1	0.2812 (1.2368)	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1718 (0.6308)	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)
		λ_1	2.9159 (92.4)	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4744 (89.3)	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)
	Boot-P	λ_1	4.0860 (90.7)	4.0736 (91.6)	4.0717 (91.6)	4.0717 (91.6)
		λ_2	3.3216 (89.1)	3.3181 (89.4)	3.3172 (89.4)	3.3172 (89.4)
	Boot-T	λ_1	2.8364 (90.4)	2.9047 (90.6)	2.9055 (90.6)	2.9055 (90.6)
		λ_2	2.2802 (88.8)	2.3412 (88.7)	2.3437 (88.7)	2.3438 (88.7)
	Bayes	λ_1	2.8297 (94.2)	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3838 (90.8)	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 3: n = 25, m = 10*.

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.0812 (0.3105)	0.1225 (0.2790)	0.1225 (0.2789)	0.1225 (0.2789)
		λ_2	0.0560 (0.2404)	0.0882 (0.2188)	0.0891 (0.2182)	0.0891 (0.2182)
	MLE	λ_1	1.8802 (90.8)	1.8411 (94.0)	1.8406 (93.9)	1.8406 (93.9)
		λ_2	1.6573 (92.5)	1.6259 (92.7)	1.6261 (92.7)	1.6261 (92.7)
1	Boot-P	λ_1	2.1524 (91.4)	2.1440 (94.0)	2.1319 (94.1)	2.1317 (94.1)
		λ_2	1.8623 (88.6)	1.8597 (91.8)	1.8537 (91.8)	1.8536 (91.8)
	Boot-T	λ_1	1.7514 (92.6)	1.8218 (93.7)	1.8341 (93.7)	1.8340 (93.7)
		λ_2	1.5029 (89.7)	1.5810 (90.8)	1.5951 (91.2)	1.5950 (91.2)
	Bayes	λ_1	1.8460 (92.8)	1.8120 (94.3)	1.8116 (94.1)	1.8116 (94.1)
		λ_2	1.6194 (91.1)	1.5932 (93.6)	1.5935 (93.6)	1.5935 (93.6)
		λ_1	0.0753 (0.5199)	0.0778 (0.3620)	0.0984 (0.3136)	0.1181 (0.2821)
		λ_2	0.0400 (0.4258)	0.0497 (0.2902)	0.0733 (0.2355)	0.0828 (0.2208)
	MLE	λ_1	2.5991 (90.3)	2.1705 (91.5)	1.9260 (92.9)	1.8488 (93.7)
		λ_2	2.2059 (85.2)	1.8888 (87.7)	1.7022 (91.6)	1.6304 (92.7)
2	Boot-P	λ_1	2.7334 (91.7)	2.3661 (92.2)	2.1893 (93.5)	2.1398 (93.9)
		λ_2	2.2943 (85.3)	2.0360 (92.0)	1.8917 (89.8)	1.8541 (91.3)
	Boot-T	λ_1	2.4446 (91.5)	2.0895 (91.9)	1.8889 (93.4)	1.8255 (93.8)
		λ_2	2.0044 (85.7)	1.7540 (91.0)	1.6192 (89.9)	1.5852 (91.1)
	Bayes	λ_1	2.5100 (90.7)	2.1177 (92.9)	1.8908 (93.4)	1.8191 (94.4)
		λ_2	2.1189 (83.9)	1.8330 (92.0)	1.6633 (92.9)	1.5971 (93.4)
		λ_1	0.0752 (0.3272)	0.1142 (0.2855)	0.1226 (0.2788)	0.1225 (0.2789)
		λ_2	0.0445 (0.2500)	0.0823 (0.2222)	0.0890 (0.2182)	0.0891 (0.2182)
	MLE	λ_1	1.9918 (90.5)	1.8449 (94.0)	1.8407 (93.9)	1.8406 (93.9)
		λ_2	1.7386 (88.3)	1.6301 (92.3)	1.6261 (92.7)	1.6261 (92.7)
3	Boot-P	λ_1	2.2036 (92.2)	2.1502 (93.5)	2.1335 (94.1)	2.1317 (94.1)
		λ_2	1.9051 (89.8)	1.8606 (91.3)	1.8547 (91.8)	1.8536 (91.8)
	Boot-T	λ_1	1.8715 (92.3)	1.8015 (93.6)	1.8326 (93.7)	1.8340 (93.7)
		λ_2	1.5931 (89.6)	1.5596 (91.0)	1.5940 (91.2)	1.5950 (91.2)
	Bayes	λ_1	1.9504 (92.7)	1.8152 (94.0)	1.8117 (94.1)	1.8116 (94.1)
		λ_2	1.6939 (90.7)	1.5968 (93.7)	1.5935 (93.6)	1.5935 (93.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 4: n = 50, m = 5*.

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
1		λ_1	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)
	MLE	λ_1	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)
	Boot-P	λ_1	4.0723 (91.6)	4.0717 (91.6)	4.0717 (91.6)	4.0717 (91.6)
		λ_2	3.3176 (89.4)	3.3172 (89.4)	3.3172 (89.4)	3.3172 (89.4)
	Boot-T	λ_1	2.9049 (90.6)	2.9055 (90.6)	2.9055 (90.6)	2.9055 (90.6)
		λ_2	2.3430 (88.7)	2.3437 (88.7)	2.3438 (88.7)	2.3438 (88.7)
	Bayes	λ_1	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)
2		λ_1	0.2378 (1.6791)	0.2302 (1.3733)	0.2427 (1.2795)	0.2757 (1.2485)
		λ_2	0.1761 (1.0055)	0.1494 (0.7596)	0.1493 (0.6548)	0.1716 (0.6312)
	MLE	λ_1	3.5945 (80.7)	3.1875 (89.5)	2.9530 (90.8)	2.9136 (92.8)
		λ_2	3.0208 (71.5)	2.6866 (82.2)	2.5029 (87.8)	2.4777 (89.3)
	Boot-P	λ_1	4.2231 (78.9)	4.0181 (85.7)	4.0113 (90.4)	4.0653 (90.1)
		λ_2	3.3637 (69.2)	3.2376 (79.8)	3.2436 (86.2)	3.3096 (88.9)
	Boot-T	λ_1	3.4955 (80.4)	2.9977 (87.6)	2.8515 (90.9)	2.8656 (90.7)
		λ_2	2.7151 (70.4)	2.3951 (81.7)	2.2697 (87.8)	2.3087 (89.0)
	Bayes	λ_1	3.4304 (78.9)	3.0669 (88.0)	2.8577 (92.8)	2.8267 (93.6)
		λ_2	2.8714 (70.1)	2.5696 (81.4)	2.4060 (88.5)	2.3866 (91.0)
3		λ_1	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)
	MLE	λ_1	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)
	Boot-P	λ_1	4.0726 (91.6)	4.0717 (91.6)	4.0717 (91.6)	4.0717 (91.6)
		λ_2	3.3178 (89.4)	3.3172 (89.4)	3.3172 (89.4)	3.3172 (89.4)
	Boot-T	λ_1	2.9056 (90.6)	2.9055 (90.6)	2.9055 (90.6)	2.9055 (90.6)
		λ_2	2.3428 (88.7)	2.3437 (88.7)	2.3438 (88.7)	2.3438 (88.7)
	Bayes	λ_1	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 5: n = 50, m = 10*.

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.1226 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)
		λ_2	0.0890 (0.2183)	0.0891 (0.2182)	0.0891 (0.2182)	0.0891 (0.2182)
	MLE	λ_1	1.8408 (93.9)	1.8406 (93.9)	1.8406 (93.9)	1.8406 (93.9)
		λ_2	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)
1	Boot-P	λ_1	2.1406 (94.0)	2.1318 (94.1)	2.1317 (94.1)	2.1317 (94.1)
		λ_2	1.8576 (91.7)	1.8536 (91.8)	1.8536 (91.8)	1.8536 (91.8)
	Boot-T	λ_1	1.8280 (93.7)	1.8340 (93.7)	1.8340 (93.7)	1.8340 (93.7)
		λ_2	1.5886 (91.1)	1.5950 (91.2)	1.5950 (91.2)	1.5950 (91.2)
	Bayes	λ_1	1.8118 (94.1)	1.8116 (94.1)	1.8116 (94.1)	1.8116 (94.1)
		λ_2	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)
		λ_1	0.0812 (0.5127)	0.0794 (0.3626)	0.1002 (0.3127)	0.1183 (0.2816)
		λ_2	0.0405 (0.4190)	0.0510 (0.2876)	0.0733 (0.2343)	0.0831 (0.2204)
	MLE	λ_1	2.5875 (90.1)	2.1628 (91.3)	1.9254 (93.4)	1.8488 (93.6)
		λ_2	2.1918 (85.7)	1.8825 (87.8)	1.7004 (91.7)	1.6306 (92.9)
2	Boot-P	λ_1	2.7158 (92.1)	2.3613 (92.3)	2.1873 (93.3)	2.1396 (93.8)
		λ_2	2.3004 (86.0)	2.0385 (91.6)	1.8924 (90.2)	1.8550 (91.3)
	Boot-T	λ_1	2.4721 (91.7)	2.0908 (91.5)	1.8900 (93.3)	1.8256 (93.8)
		λ_2	2.0481 (86.1)	1.7653 (90.9)	1.6233 (90.3)	1.5857 (91.1)
	Bayes	λ_1	2.5003 (91.0)	2.1106 (92.4)	1.8904 (93.5)	1.8191 (94.5)
		λ_2	2.1061 (84.8)	1.8274 (91.9)	1.6616 (93.0)	1.5972 (93.6)
		λ_1	0.1225 (0.2790)	0.1225 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)
		λ_2	0.0882 (0.2188)	0.0891 (0.2182)	0.0891 (0.2182)	0.0891 (0.2182)
	MLE	λ_1	1.8411 (94.0)	1.8406 (93.9)	1.8406 (93.9)	1.8406 (93.9)
		λ_2	1.6259 (92.7)	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)
3	Boot-P	λ_1	2.1440 (94.0)	2.1319 (94.1)	2.1317 (94.1)	2.1317 (94.1)
		λ_2	1.8597 (91.8)	1.8537 (91.8)	1.8536 (91.8)	1.8536 (91.8)
	Boot-T	λ_1	1.8218 (93.7)	1.8341 (93.7)	1.8340 (93.7)	1.8340 (93.7)
		λ_2	1.5810 (90.8)	1.5951 (91.2)	1.5950 (91.2)	1.5950 (91.2)
	Bayes	λ_1	1.8120 (94.3)	1.8116 (94.1)	1.8116 (94.1)	1.8116 (94.1)
		λ_2	1.5932 (93.6)	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 6: n = 50, m = 15*.

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
1	MLE	λ_1	0.0800 (0.1570)	0.0862 (0.1520)	0.0862 (0.1520)	0.0862 (0.1520)
		λ_2	0.0336 (0.1174)	0.0366 (0.1150)	0.0366 (0.1150)	0.0366 (0.1150)
		λ_1	1.4553 (93.5)	1.4530 (94.0)	1.4530 (94.0)	1.4530 (94.0)
		λ_2	1.2720 (93.1)	1.2687 (93.7)	1.2687 (93.7)	1.2687 (93.7)
	Boot-P	λ_1	1.6128 (93.6)	1.5828 (94.3)	1.5826 (94.3)	1.5826 (94.3)
		λ_2	1.4223 (93.1)	1.4045 (93.5)	1.4043 (93.5)	1.4043 (93.5)
	Boot-T	λ_1	1.4274 (94.0)	1.4515 (93.9)	1.4516 (93.9)	1.4516 (93.9)
		λ_2	1.2578 (93.0)	1.2819 (93.5)	1.2817 (93.5)	1.2817 (93.5)
	Bayes	λ_1	1.4400 (94.0)	1.4379 (94.4)	1.4379 (94.4)	1.4379 (94.4)
		λ_2	1.2545 (95.9)	1.2515 (94.8)	1.2515 (94.8)	1.2515 (94.8)
2	MLE	λ_1	0.0746 (0.3559)	0.0651 (0.2411)	0.0682 (0.1739)	0.0819 (0.1545)
		λ_2	0.0313 (0.2689)	0.0270 (0.1677)	0.0275 (0.1314)	0.0332 (0.1180)
		λ_1	2.1969 (87.6)	1.7837 (90.7)	1.5448 (93.3)	1.4626 (94.1)
		λ_2	1.8902 (90.7)	1.5599 (92.3)	1.3513 (92.6)	1.2771 (92.9)
	Boot-P	λ_1	2.2113 (91.7)	1.8593 (94.5)	1.6663 (94.0)	1.5974 (94.7)
		λ_2	1.8917 (91.8)	1.6091 (92.0)	1.4683 (94.4)	1.4134 (93.4)
	Boot-T	λ_1	2.0680 (91.0)	1.7434 (94.6)	1.5346 (93.4)	1.4580 (93.9)
		λ_2	1.7138 (91.4)	1.4864 (91.5)	1.3445 (93.0)	1.2842 (93.3)
	Bayes	λ_1	2.1411 (93.0)	1.7534 (92.2)	1.5258 (93.6)	1.4471 (94.3)
		λ_2	1.8314 (92.3)	1.5262 (93.1)	1.3298 (94.4)	1.2594 (95.2)
3	MLE	λ_1	0.0686 (0.1630)	0.0862 (0.1520)	0.0862 (0.1520)	0.0862 (0.1520)
		λ_2	0.0241 (0.1216)	0.0365 (0.1151)	0.0366 (0.1150)	0.0366 (0.1150)
		λ_1	1.4702 (93.2)	1.4530 (94.0)	1.4530 (94.0)	1.4530 (94.0)
		λ_2	1.2846 (93.1)	1.2687 (93.6)	1.2687 (93.7)	1.2687 (93.7)
	Boot-P	λ_1	1.6215 (93.1)	1.5844 (94.3)	1.5826 (94.3)	1.5826 (94.3)
		λ_2	1.4262 (93.3)	1.4056 (93.4)	1.4043 (93.5)	1.4043 (93.5)
	Boot-T	λ_1	1.4336 (94.1)	1.4499 (93.9)	1.4516 (93.9)	1.4516 (93.9)
		λ_2	1.2587 (93.3)	1.2813 (93.5)	1.2817 (93.5)	1.2817 (93.5)
	Bayes	λ_1	1.4539 (93.7)	1.4379 (94.4)	1.4379 (94.4)	1.4379 (94.4)
		λ_2	1.2660 (94.9)	1.2515 (94.8)	1.2515 (94.8)	1.2515 (94.8)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 7: n = 100, m = 5*.

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)
	MLE	λ_1	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)
1	Boot-P	λ_1	4.0717 (91.6)	4.0717 (91.6)	4.0717 (91.6)	4.0717 (91.6)
		λ_2	3.3172 (89.4)	3.3172 (89.4)	3.3172 (89.4)	3.3172 (89.4)
	Boot-T	λ_1	2.9055 (90.6)	2.9055 (90.6)	2.9055 (90.6)	2.9055 (90.6)
		λ_2	2.3438 (88.7)	2.3438 (88.7)	2.3438 (88.7)	2.3438 (88.7)
	Bayes	λ_1	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)
		λ_1	0.2398 (1.6732)	0.2317 (1.3679)	0.2428 (1.2792)	0.2759 (1.2422)
		λ_2	0.1783 (1.0011)	0.1500 (0.7576)	0.1512 (0.6542)	0.1715 (0.6313)
	MLE	λ_1	3.5902 (80.8)	3.1872 (89.8)	2.9520 (90.7)	2.9141 (92.7)
		λ_2	3.0201 (71.6)	2.6851 (82.3)	2.5047 (87.9)	2.4775 (89.3)
2	Boot-P	λ_1	4.2216 (78.9)	4.0150 (85.8)	4.0098 (90.5)	4.0650 (90.1)
		λ_2	3.3769 (69.5)	3.2425 (79.8)	3.2461 (86.2)	3.3100 (88.9)
	Boot-T	λ_1	3.4957 (80.4)	2.9995 (87.4)	2.8521 (90.9)	2.8666 (90.7)
		λ_2	2.7357 (71.0)	2.4007 (81.6)	2.2715 (87.9)	2.3092 (89.0)
	Bayes	λ_1	3.4270 (78.9)	3.0669 (88.4)	2.8568 (92.8)	2.8272 (93.6)
		λ_2	2.8711 (70.6)	2.5683 (81.5)	2.4079 (88.5)	2.3865 (91.0)
		λ_1	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)	0.2842 (1.2314)
		λ_2	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)	0.1759 (0.6258)
	MLE	λ_1	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)	2.9192 (93.4)
		λ_2	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)	2.4801 (89.6)
3	Boot-P	λ_1	4.0717 (91.6)	4.0717 (91.6)	4.0717 (91.6)	4.0717 (91.6)
		λ_2	3.3172 (89.4)	3.3172 (89.4)	3.3172 (89.4)	3.3172 (89.4)
	Boot-T	λ_1	2.9055 (90.6)	2.9055 (90.6)	2.9055 (90.6)	2.9055 (90.6)
		λ_2	2.3437 (88.7)	2.3438 (88.7)	2.3438 (88.7)	2.3438 (88.7)
	Bayes	λ_1	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)	2.8331 (93.9)
		λ_2	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)	2.3895 (91.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 8: n = 100, m = 10*.

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
		λ_1	0.1225 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)
		λ_2	0.0891 (0.2182)	0.0891 (0.2182)	0.0891 (0.2182)	0.0891 (0.2182)
	MLE	λ_1	1.8406 (93.9)	1.8406 (93.9)	1.8406 (93.9)	1.8406 (93.9)
		λ_2	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)
1	Boot-P	λ_1	2.1318 (94.1)	2.1317 (94.1)	2.1317 (94.1)	2.1317 (94.1)
		λ_2	1.8536 (91.8)	1.8536 (91.8)	1.8536 (91.8)	1.8536 (91.8)
	Boot-T	λ_1	1.8340 (93.7)	1.8340 (93.7)	1.8340 (93.7)	1.8340 (93.7)
		λ_2	1.5950 (91.2)	1.5950 (91.2)	1.5950 (91.2)	1.5950 (91.2)
	Bayes	λ_1	1.8116 (94.1)	1.8116 (94.1)	1.8116 (94.1)	1.8116 (94.1)
		λ_2	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)
		λ_1	0.0833 (0.5097)	0.0795 (0.3643)	0.1005 (0.3126)	0.1182 (0.2817)
		λ_2	0.0418 (0.4155)	0.0512 (0.2890)	0.0729 (0.2342)	0.0830 (0.2204)
	MLE	λ_1	2.5789 (90.0)	2.1578 (91.4)	1.9246 (93.5)	1.8485 (93.6)
		λ_2	2.1851 (86.0)	1.8791 (87.9)	1.6989 (91.7)	1.6303 (92.9)
2	Boot-P	λ_1	2.7055 (91.9)	2.3619 (92.4)	2.1864 (93.3)	2.1397 (93.9)
		λ_2	2.3012 (86.6)	2.0384 (91.4)	1.8924 (90.3)	1.8552 (91.3)
	Boot-T	λ_1	2.4757 (91.7)	2.0947 (91.7)	1.8898 (93.3)	1.8258 (93.9)
		λ_2	2.0653 (86.3)	1.7689 (90.7)	1.6233 (90.5)	1.5857 (91.1)
	Bayes	λ_1	2.4928 (91.4)	2.1060 (92.5)	1.8896 (93.7)	1.8189 (94.5)
		λ_2	2.1004 (85.2)	1.8243 (91.8)	1.6603 (93.0)	1.5970 (93.6)
		λ_1	0.1225 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)	0.1225 (0.2789)
		λ_2	0.0891 (0.2182)	0.0891 (0.2182)	0.0891 (0.2182)	0.0891 (0.2182)
	MLE	λ_1	1.8406 (93.9)	1.8406 (93.9)	1.8406 (93.9)	1.8406 (93.9)
		λ_2	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)	1.6261 (92.7)
3	Boot-P	λ_1	2.1318 (94.1)	2.1317 (94.1)	2.1317 (94.1)	2.1317 (94.1)
		λ_2	1.8536 (91.8)	1.8536 (91.8)	1.8536 (91.8)	1.8536 (91.8)
	Boot-T	λ_1	1.8340 (93.7)	1.8340 (93.7)	1.8340 (93.7)	1.8340 (93.7)
		λ_2	1.5950 (91.2)	1.5950 (91.2)	1.5950 (91.2)	1.5950 (91.2)
	Bayes	λ_1	1.8116 (94.1)	1.8116 (94.1)	1.8116 (94.1)	1.8116 (94.1)
		λ_2	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)	1.5935 (93.6)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Table 9: n = 100, m = 15*.

Scheme	Methods		T = 0.25	T = 0.50	T = 1.00	T = 2.00
1		λ_1	0.0862 (0.1520)	0.0862 (0.1520)	0.0862 (0.1520)	0.0862 (0.1520)
		λ_2	0.0366 (0.1150)	0.0366 (0.1150)	0.0366 (0.1150)	0.0366 (0.1150)
	MLE	λ_1	1.4530 (94.0)	1.4530 (94.0)	1.4530 (94.0)	1.4530 (94.0)
		λ_2	1.2687 (93.7)	1.2687 (93.7)	1.2687 (93.7)	1.2687 (93.7)
	Boot-P	λ_1	1.5826 (94.3)	1.5826 (94.3)	1.5826 (94.3)	1.5826 (94.3)
		λ_2	1.4044 (93.5)	1.4043 (93.5)	1.4043 (93.5)	1.4043 (93.5)
	Boot-T	λ_1	1.4516 (93.9)	1.4516 (93.9)	1.4516 (93.9)	1.4516 (93.9)
		λ_2	1.2818 (93.5)	1.2817 (93.5)	1.2817 (93.5)	1.2817 (93.5)
	Bayes	λ_1	1.4379 (94.4)	1.4379 (94.4)	1.4379 (94.4)	1.4379 (94.4)
		λ_2	1.2515 (94.8)	1.2515 (94.8)	1.2515 (94.8)	1.2515 (94.8)
2		λ_1	0.0739 (0.3503)	0.0675 (0.2395)	0.0678 (0.1735)	0.0819 (0.1545)
		λ_2	0.0343 (0.2643)	0.0264 (0.1671)	0.0275 (0.1315)	0.0332 (0.1180)
	MLE	λ_1	2.1841 (87.9)	1.7816 (90.9)	1.5434 (93.3)	1.4625 (94.2)
		λ_2	1.8860 (90.7)	1.5555 (92.0)	1.3503 (92.4)	1.2770 (92.9)
	Boot-P	λ_1	2.2098 (92.0)	1.8572 (94.6)	1.6646 (94.0)	1.5972 (94.7)
		λ_2	1.8977 (91.8)	1.6063 (92.6)	1.4677 (94.4)	1.4136 (93.4)
	Boot-T	λ_1	2.0764 (91.3)	1.7421 (94.2)	1.5339 (93.3)	1.4576 (93.9)
		λ_2	1.7271 (91.6)	1.4871 (91.7)	1.3446 (93.1)	1.2843 (93.3)
	Bayes	λ_1	2.1292 (92.6)	1.7515 (91.8)	1.5245 (93.7)	1.4469 (94.3)
		λ_2	1.8280 (92.5)	1.5221 (93.0)	1.3289 (94.4)	1.2593 (95.2)
3		λ_1	0.0862 (0.1520)	0.0862 (0.1520)	0.0862 (0.1520)	0.0862 (0.1520)
		λ_2	0.0366 (0.1150)	0.0366 (0.1150)	0.0366 (0.1150)	0.0366 (0.1150)
	MLE	λ_1	1.4530 (94.0)	1.4530 (94.0)	1.4530 (94.0)	1.4530 (94.0)
		λ_2	1.2687 (93.7)	1.2687 (93.7)	1.2687 (93.7)	1.2687 (93.7)
	Boot-P	λ_1	1.5828 (94.3)	1.5826 (94.3)	1.5826 (94.3)	1.5826 (94.3)
		λ_2	1.4045 (93.5)	1.4043 (93.5)	1.4043 (93.5)	1.4043 (93.5)
	Boot-T	λ_1	1.4515 (93.9)	1.4516 (93.9)	1.4516 (93.9)	1.4516 (93.9)
		λ_2	1.2819 (93.5)	1.2817 (93.5)	1.2817 (93.5)	1.2817 (93.5)
	Bayes	λ_1	1.4379 (94.4)	1.4379 (94.4)	1.4379 (94.4)	1.4379 (94.4)
		λ_2	1.2515 (94.8)	1.2515 (94.8)	1.2515 (94.8)	1.2515 (94.8)

* In each cell, the first row of λ_1 and λ_2 represents the average biases and the corresponding mean squared errors are reported within brackets for the MLEs. The second, third, fourth and fifth rows of λ_1 and λ_2 represent the average 95% confidence lengths of asymptotic confidence intervals, Boot-p confidence intervals, Boot-t confidence intervals and the credible intervals with respect to the non-informative priors respectively. The corresponding coverage percentages are reported within brackets.

Example 1: In this case, n = 77 and m = 25, T = 700, $R_1 = R_2 = \ldots = R_{24} = 2$ and $R_{25} = 4$ are taken. Thus, the Type II progressively hybrid censored sample is:

(40, 2), (42, 2), (62, 2), (163, 2), (179,2), (206, 2), (222, 2), (228, 2), (252, 2), (259, 2), (318, 1), (385, 2), (407, 2), (420, 2), (462, 2), (507, 2), (517, 2), (524, 2), (525, 1), (528, 1), (536, 1), (605, 1), (612, 1), (620, 2), (621, 1).

In this case, $D_1 = 7$, $D_2 = 18$ and $W = \sum_{i=1}^{25} (1 + R_i) x_{i:m:n} = 28962$. Therefore,

$$\hat{\lambda}_1 = \frac{7}{28962} = 2.41696 \times 10^{-4}$$

and

$$\hat{\lambda}_2 = \frac{18}{28962} = 6.21504 \times 10^{-4}.$$

The 95% asymptotic, Boot-P, Boot-t confidence intervals and also the 95% credible intervals of λ_1 and λ_2 are reported in Table 10.

It is clear that although all of them provided almost similar confidence/credible intervals, but Bayes credible intervals have the smallest lengths. Now, the data using T = 600 instead of T = 700 is generated, while m and R(i)'s are the same as before.

Example 2: In this case the progressively hybrid censored sample obtained as:

(40, 2), (42, 2), (62, 2), (163, 2), (179,2), (206, 2), (222, 2), (228, 2), (252, 2), (259, 2), (318, 1), (385, 2), (407, 2), (420, 2), (462, 2), (507, 2), (517, 2), (524, 2), (525, 1), (528, 1), (536, 1).

Here $D_1 = 4$, $D_2 = 17$ and $W = \sum_{i=1}^{21} (1 + R_i) x_{i:m:n} = 20346$. Therefore, the following is obtained:

$$\hat{\lambda}_1 = \frac{4}{28746} = 1.39150 \times 10^{-4}$$

and

$$\hat{\lambda}_2 = \frac{17}{28746} = 20.23809 \times 10^{-4}.$$

In this case, the 95% asymptotic, Boot-P, Boot-t confidence intervals and also the 95% credible intervals of λ_1 and λ_2 are reported in Table 11.

From Table 11, it is observed that T plays a major role for the estimation of λ's and for the construction of the corresponding confidence intervals. As T decreases, the lengths of the confidence/credible intervals for both the parameters are as expected. It is also important to note that Boot-p and Boot-t are the most affected due to T and the Bayes confidence intervals are the least affected. Therefore, Bayes confidence intervals are quite robust also with respect to T.

Conclusion

In this article, a new censoring scheme is discussed, namely the Type II progressively hybrid censoring scheme under competing risks data. Assuming that the lifetime distributions are exponentially distributed, one may obtain the maximum likelihood estimators of the unknown parameter and propose different confidence intervals using asymptotic distributions as well as using bootstrap methods. Bayesian estimates of the unknown parameters are also proposed and it is observed that the Bayes credible intervals with respect to non-informative prior work quite well in this case and it has several desirable properties. Although it is assumed that the lifetime distributions are exponential, most of the methods may be extended for other distributions also, such as the Weibull or gamma distribution.

Table 10.

Methods	λ_1	λ_2
Asymptotic	$(0.62645\times10^{-4}, 4.20747\times10^{-4})$	$(3.34384\times10^{-4}, 9.08624\times10^{-4})$
Boot-p	$(0.76099\times10^{-4}, 4.52108\times10^{-4})$	$(3.47439\times10^{-4}, 10.52984\times10^{-4})$
Boot-t	$(0.58039\times10^{-4}, 4.26943\times10^{-4})$	$(2.71588\times10^{-4}, 9.46895\times10^{-4})$
Credible	$(0.97174\times10^{-4}, 4.50918\times10^{-4})$	$(3.60913\times10^{-4}, 9.31153\times10^{-4})$

Table 11.

Methods	λ_1	λ_2
Asymptotic	$(0.02783\times10^{-4}, 2.75517\times10^{-4})$	$(10.61752\times10^{-4}, 29.85867\times10^{-4})$
Boot-p	$(0.00000\times10^{-4}, 3.02527\times10^{-4})$	$(14.13159\times10^{-4}, 32.89348\times10^{-4})$
Boot-t	$(0.00000\times10^{-4}, 3.63490\times10^{-4})$	$(11.92432\times10^{-4}, 27.94359\times10^{-4})$
Credible	$(0.37913\times10^{-4}, 3.04992\times10^{-4})$	$(3.37047\times10^{-4}, 8.95152\times10^{-4})$

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