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A Combined Individuals and Moving Range Control Chart

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An individuals control chart is usually used to monitor shifts in the process mean when it is not possible to form subgroups. The moving range of two successive process measures is used as the basis for estimating the process variability. Similar to the case of the $\overline{X}-R$ and $\overline{X}-S$ charts, the individuals-moving range (I-MR) charts are used simultaneously in the monitoring of the process mean and variance respectively for individual observations, requiring maintaining two different charts. In this article, a new approach is suggested where the measurements of both the process mean and variance are plotted on one chart. It is referred to as the combined I-MR chart. An average run length (ARL) study is conducted to evaluate its performance with respect to shifts in the process mean and variance. Examples are provided.

Key words: Individuals charts; moving range charts; average run length (ARL); process mean; process variance

Introduction

There are many situations in which the sample size used for process monitoring is one (Montgomery, 2001). Some of these are in situations involving the use of automated inspection and measurement technology where every unit manufactured is analyzed. Situations where the production rate is slow and monitoring of the process is required before the time needed to form subgroups may also call for process monitoring involving individual observations. The monitoring of individual observations is also important in situations where repeat measurements on a process differ only because of laboratory or analysis error, as in many chemical processes.

Traditionally, individuals control charts are used in the monitoring of processes involving individual observations. For such cases, the moving range charts are employed in the monitoring of the process variability. Here, the moving range of two successive observations is defined as (Montgomery, 2001):

$$MR_i = \left| X_i - X_{i-1} \right|, \quad i = 2, 3, \ldots \tag{1}$$

A moving range chart is established by plotting the moving ranges computed from eq. (1) based on the limits

$$UCL = D_4 \overline{MR} \tag{2a}$$

$$CL = \overline{MR} \tag{2b}$$

$$LCL = D_3 \overline{MR} \tag{2c}$$

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where $\overline{MR}$ is the average of the moving range computed from a preliminary set of data. After establishing an in-control state for the process variability, the individuals chart is set up by plotting the individual observations, X_i, on a chart with limits (Montgomery, 2001):

$$UCL = \overline{X} + 3\frac{\overline{MR}}{d_2} \quad (3a)$$

$$CL = \overline{X} \quad (3b)$$

$$LCL = \overline{X} - 3\frac{\overline{MR}}{d_2} \quad (3c)$$

Note that in eqs. (2a), (2c), (3a) and (3c), D_3, D_4 and d_2 are control chart constants for $n = 2$ whose values are given in most quality control textbooks.

A Combined I-MR Chart

Let X_i, $i = 1, 2, \ldots$, represent individual observations from a process for a quality characteristic of interest. It is assumed that $X_i \sim N(\mu + a\sigma, b^2\sigma^2)$, where $a = 0$ and $b = 1$ indicate that the process is in-control; otherwise, the process is out-of-control. Here, μ and σ denote the on-target mean and standard deviation. Define

$$M_i = \frac{X_i - \mu}{\sigma} \sim N(0,1), \quad i = 1, 2, \ldots \quad (4)$$

and

$$V_i = \Phi^{-1}\left\{H_1\left[\frac{1}{2\sigma^2}(X_i - X_{i-1})^2\right]\right\} \sim N(0,1), \quad i = 2, 3, \ldots \quad (5)$$

where $\Phi^{-1}(\cdot)$ and $H_1(\cdot)$ are the inverse of the standard normal distribution function and the chi-square distribution function with one degree of freedom respectively. Because the value of X_i is unavailable when $i = 0$, V_1 is computed using $\Phi^{-1}\left\{H_1\left[\frac{1}{2\sigma^2}(X_1 - \mu)^2\right]\right\}$. It is found that V_i follows a standard normal distribution (Appendix).

Due to the transformation of the V_i statistic in (5), $\text{cov}(M_i, V_i)$ is intractable. Thus, in finding the correlation of M_i and V_i to determine the extent of the relationship between the two statistics, 500 individual observations from a $N(0,1)$ distribution are generated, the M_i and V_i statistics computed and the sample correlation coefficient of M_i and V_i is calculated using the Pearson correlation procedure from SPSS version 11. The output is shown in Figure 1. Note that the individual observations can also be generated from other normal distributions. From Figure 1, the correlation of M_i and V_i is insignificant at the 1% significance level because its associated p-value is 0.657. Here, the sample correlation coefficient is –0.02. Based on this result, it can be concluded that the correlation of M_i and V_i is negligible if the underlying distribution of the individual observations is normal.

Correlations

		M	V
M	Pearson Correlation	1.000	-.020
	Sig. (2-tailed)	.	.657
	N	500	500
V	Pearson Correlation	-.020	1.000
	Sig. (2-tailed)	.657	.
	N	500	500

Figure 1. The Sample Correlation Coefficient of the M_i and V_i Statistics based on 500 Individual Observations (Output from SPSS)

M_i monitors the process mean while V_i the process variability. These two statistics are combined to form a new statistic given by

$$C_i = \max(|M_i|, |V_i|) \quad (6)$$

The statistic C_i will be large when the process mean has shifted away from its target value and/or when the process variance has increased or decreased.

Because the correlation of M_i and V_i is negligible, it is shown (Appendix) that the approximate density function of C_i for the in-control case is

$$f(c) = 4\phi(c)\{2\Phi(c) - 1\}, \quad c \geq 0 \qquad (7)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the density and distribution functions of a standard normal random variable respectively. The combined I-MR chart only requires an upper control limit (*UCL*) because C_i is nonnegative. Suppose that the desired Type-I error set by management is α, then the *UCL* can be obtained from the following integral:

$$\int_{UCL}^{\infty} f(c)dc = \alpha \qquad (8)$$

Steps for Implementing the Combined I-MR Chart

The following steps serve as guidelines in setting up a combined I-MR chart:

(i) If the process parameter(s) are unknown, then they are estimated as follow: The process mean, μ, is estimated from the formula, $\overline{X} = \frac{\sum_{i=1}^{m} X_i}{m}$, where m is the number of observations in the stable preliminary data set used in the estimation. The process standard deviation, σ, is estimated using $\frac{\overline{MR}}{d_2}$, where

$$\overline{MR} = \frac{MR_2 + MR_3 + ... + MR_m}{m-1}$$

is the average of the moving ranges. Here, d_2 is the value of the control chart constant for sample size, n = 2.

(ii) Compute M_i, V_i and C_i for each observation.

(iii) Determine the *UCL* using eq. (8) based on a desired Type-I error.

(iv) When $C_i \leq UCL$, plot a dot at time *i*. When $C_i > UCL$, check both $|M_i|$ and $|V_i|$ against *UCL*. If only $|M_i|$ is greater than *UCL*, plot "*m*+" at time *i* when $M_i > 0$ to indicate the process mean has increased, and plot "*m*–" at time *i* when $M_i < 0$ to indicate the process mean has decreased.

Similarly, if $|V_i|$ alone is greater than *UCL*, plot "*v*+" at time *i* when $V_i > 0$ to indicate the process variability has increased, and plot "*v*–" at time *i* when $V_i < 0$ to indicate the process variability has decreased. For the case when both $|M_i|$ and $|V_i|$ are greater than the *UCL*, plot "++", "+–", "–+" or "––" if $M_i > 0$ and $V_i > 0$, $M_i > 0$ and $V_i < 0$, $M_i < 0$ and $V_i > 0$, or $M_i < 0$ and $V_i < 0$ respectively.

(v) Investigate the cause(s) for each out-of-control point so that appropriate corrective actions can be taken.

Plots for Determining the *UCL*

Figure 2 gives a plot for approximating the *UCL* based on a desired Type-I error. The plot is based on in-control ARLs (ARL_0s) between 100 and 1000. It is constructed from points (*UCL* , ARL_0) obtained using a simple Mathematica 4.0 program shown in Figure 3 based on eq. (8).

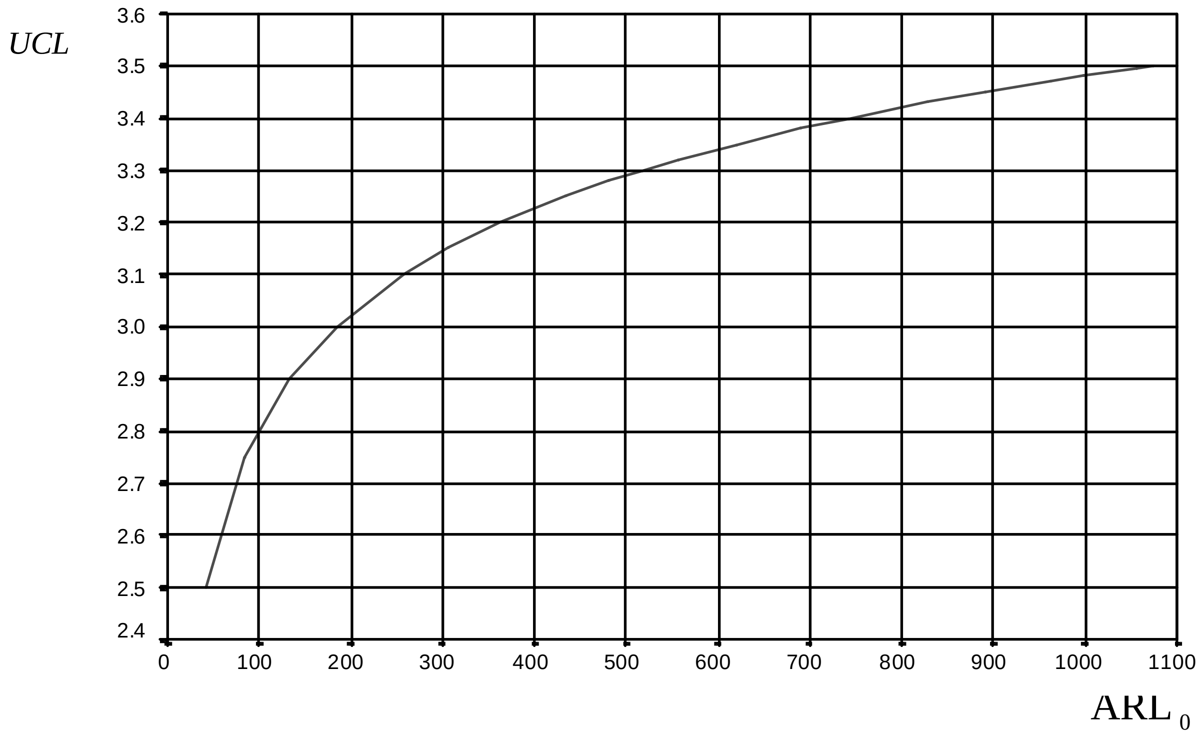

Figure 2. A Plot of *UCL* vs. ARL_0 for the Combined I-MR Chart

UCL =

$$\textbf{NIntegrate}\left[\mathbf{2}\sqrt{\frac{\mathbf{2}}{\pi}}\times\left(e^{-\frac{c^2}{2}}\right)\times\left(\sqrt{\frac{\mathbf{2}}{\pi}}\ \textbf{Integrate}\left[\mathbf{e}^{-\frac{t^2}{2}},\{\mathbf{t},-\infty,\mathbf{c}\}-\mathbf{1}\right]\right),\{\mathbf{c},\textbf{UCL},\infty\}\right]$$

Figure 3. A Mathematica 4.0 Program to Compute the *UCL* for a Combined I-MR Chart

A sensitivity analysis can be performed using the Mathematica 4.0 program in Figure 3 to obtain the exact *UCL* for a desired Type-I error. The following example shows how a sensitivity analysis is performed, assuming that the Type-I error is set at α = 0.004. The corresponding in-control ARL is ARL_0 = 250. The value of *UCL* approximated from the plot in Figure 2 is 3.08. Values of α which correspond to values of *UCL*s close to the one approximated, i.e., 3.08 are computed using the program in Figure 3 and are tabulated in Table 1. From Table 1, it is noticed that the value of *UCL* which produces the closest Type-I error to $\alpha = \frac{1}{250} = 0.004$ is 3.09.

A Study on the Performance of the Combined I-MR Chart

A simulation study is conducted using SAS version 8 to compute the ARL values of the combined I-MR chart based on ARL_0 = 250 and 500. Each ARL reading is based on 5000 simulation trials. The *UCL*s are determined, using the approach discussed in the previous section, to be 3.09 and 3.29 for ARL_0 = 250 and 500 respectively. Shifts in both the process mean and variance are considered. The process mean shifts from μ to $\mu + a\sigma$ while the process variance from σ to $b\sigma$, where $a = 0$ and $b = 1$ represent the in-control case. The values of $a \in$ {0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 2, 3} and $b \in$ {1, 1.05, 1.1, 1.2, 1.25, 1.5, 2, 2.5, 3, 4, 5} are considered. The simulation results for ARL_0 = 250 and 500 are given in Tables 2 and 3 respectively.

Both Tables 2 and 3 show that as the magnitude of the shift (either in the process mean or variance or both) increases, the value of ARL decreases. For example, consider the case of $b = 1$ and a $\in$ {0, 0.25, 0.5, …, 3} in Table 2, where only the process mean shifts. The ARL values for this case are 275.71, 234.75, 153.14, …, 2.13 for a = 0, 0.25, 0.5, …, 3 respectively, where the values show a declining trend as the magnitude of the shift in the mean increases. A similar trend is observed when only the process variance shifts. For example, from Table 2, when $a = 0$ and $b \in$ {1, 1.05, 1.1, …, 5}, the ARL values of 275.71, 192.88, 133.86, …, 1.84 show a declining trend as b increases, i.e., as the magnitude of the shift in the variance increases. Note that the ARL values will also show a decreasing trend if the magnitude of shifts in both the mean and variance increase simultaneously. It is shown in Tables 2 and 3 that the computed ARL_0 values are 275.71 and 546.38 respectively, where they differ only slightly from the desired values of 250 and 500. This shows that the *UCL* computed from the approximate density function, $f(c)$ in eq. (7) is reliable, which indicates that the correlation between M_i and V_i is negligible. The difference in the estimated versus intended Type-I errors is very little, i.e., $\frac{1}{275.71} = 0.00363$ vs. $\frac{1}{250} = 0.004$ and $\frac{1}{546.38} = 0.00183$ vs. $\frac{1}{500} = 0.002$.

Table 1. Values of the Type-I Error (α) Computed from Corresponding *UCL*s

UCL	α
3.07	0.00427659
3.08	0.00413573
3.09	0.00399912

Table 2. ARL Profiles of the Combined I-MR Chart for $ARL_0 = 250$ with $UCL = 3.09$

	a								
b	0	0.25	0.5	0.75	1	1.25	1.5	2	3
1	275.71	234.75	153.14	87.97	50.43	29.15	17.39	7.16	2.13
1.05	192.88	163.40	108.28	67.63	40.59	24.28	15.00	6.59	2.12
1.1	133.86	116.37	81.44	52.36	32.71	20.65	13.13	6.13	2.12
1.2	71.61	64.18	49.05	34.37	23.12	15.33	10.62	5.44	2.11
1.25	55.41	50.93	39.89	28.31	19.80	13.52	9.82	5.21	2.10
1.5	20.70	19.46	16.98	13.70	10.90	8.51	6.59	4.23	2.08
2	7.21	7.04	6.72	6.13	5.51	4.84	4.26	3.27	2.04
2.5	4.29	4.28	4.12	3.94	3.72	3.47	3.21	2.75	1.99
3	3.10	3.10	3.07	3.01	2.93	2.80	2.65	2.39	1.92
4	2.22	2.21	2.19	2.17	2.13	2.10	2.06	1.97	1.77
5	1.84	1.84	1.83	1.82	1.81	1.79	1.78	1.74	1.64

Table 3. ARL Profiles of the Combined I-MR Chart for $ARL_0 = 500$ with $UCL = 3.29$

	a								
b	0	0.25	0.5	0.75	1	1.25	1.5	2	3
1	546.38	457.34	283.45	156.09	83.45	46.63	26.52	10.25	2.57
1.05	358.51	302.16	196.48	110.97	64.00	37.53	22.57	9.08	2.54
1.1	240.74	205.22	137.70	83.06	50.35	30.53	19.04	8.27	2.51
1.2	115.87	101.70	75.92	50.99	33.47	23.77	14.34	7.01	2.46
1.25	85.47	76.72	59.06	41.48	27.85	18.86	12.85	6.52	2.44
1.5	27.48	26.38	22.79	18.12	14.00	10.76	8.26	5.10	2.34
2	8.67	8.42	7.93	7.26	6.40	5.64	4.88	3.69	2.23
2.5	4.89	4.81	4.65	4.44	4.18	3.87	3.60	3.01	2.13
3	3.41	3.41	3.37	3.28	3.16	3.04	2.90	2.58	2.03
4	2.33	2.33	2.33	2.30	2.27	2.22	2.18	2.08	1.85
5	1.94	1.94	1.92	1.90	1.89	1.88	1.85	1.80	1.71

The combined I-MR chart has an advantage over the traditional individuals and moving range charts because the former allows practitioners to set the joint Type-I error of the two charts. Conversely, the Type-I error of the traditional moving range chart cannot be set by practitioners because it is based on fixed limits given in eqs. (2a) – (2c). Another advantage of using the combined chart is practitioners do not have to plot two charts separately, i.e, one each for individual measurements and moving ranges. Due to the advent of modern computers, the computation of the combined I-MR chart's statistics in eqs. (4), (5) and (6) is only a trivial problem.

Applications

Two examples will be given to illustrate how the combined I-MR chart is used in real situations. They are based on observations generated from SAS version 8. The in-control observations are assumed to follow a standard normal distribution. Out-of-control observations are generated from a $N(\mu + a\sigma, b^2\sigma^2)$ distribution with $a > 0$, $b > 1$, $\mu = 0$ and $\sigma = 1$. The first example deals with a shift in the process mean while the second a shift in the process variance. The Type-I error for the two examples is set as $\alpha = 0.004$ which corresponds to $\text{ARL}_0 = 250$. Thus, the *UCL* is determined to be 3.09.

Example 1

The first 5 observations are generated from a *N*(0,1) distribution to represent the in-control situation. Observations 6 to 20 which represent the out-of-control situation involving a shift in the mean are generated from a *N*(3,1) distribution. Here, the magnitude of the shift in the mean in multiples of standard deviation is $a = 3$. The individual observations generated, X_i, $i = 1, 2, \ldots, 20$ together with the computed M_i, V_i and C_i statistics are shown in Table 4. The combined I-MR chart is plotted in Figure 4.

The chart shows that out-of-control signals due to a shift in the mean are detected at observations 7, 9, 12, 13, 15, 19 and 20. Following the first out-of-control signal at observation 7, an investigation needs to be made to search for the assignable cause(s) so that the process can return to an in-control state again.

Example 2

This example involves a shift in the variance. The first 5 observations which represent the in-control case are generated from a standard normal distribution. This is followed by generating the next 15 observations from a *N*(0,4) distribution, where the magnitude of the shift in the standard deviation is $b = 2$. Table 5 summarizes the values of the individual observations, X_i, and their corresponding computed M_i, V_i and C_i statistics. The combined I-MR chart is given in Figure 5.

Conclusion

The proposed combined I-MR chart extends the work of Chen, Cheng and Xie (2001) where they suggested a joint monitoring of the process mean and variance of subgrouped data with one EWMA chart. The combined I-MR chart enables a simultaneous monitoring of the process mean and variance involving individual measurements. It combines the usual individuals chart and the moving range chart into a single chart. The advantages of the combined chart discussed in an earlier section serve as basis for practitioners to use the chart in place of its traditional counterparts.

Table 4. An Example of Application for a Shift in the Process Mean based on a = 3 and UCL = 3.09

Obs. No., i	X_i	M_i	V_i	C_i
1	0.7508	0.7508	–0.2416	0.7508
2	0.7835	0.7835	–2.0870	2.0869
3	0.6009	0.6009	–1.2660	1.2660
4	0.1087	0.1087	–0.6063	0.6063
5	–0.1614	–0.1614	–1.0300	1.0300
6	2.4860	2.4860	1.5447	2.4860
7	4.2386	4.2386*	0.7884	4.2386*
8	2.9663	2.9664	0.3363	2.9664
9	3.2089	3.2089*	–1.0978	3.2089*
10	1.1256	1.1256	1.0771	1.1256
11	2.9149	2.9149	0.8211	2.9149
12	3.4370	3.4370*	–0.5592	3.4370*
13	3.2020	3.2020*	–1.1171	3.2020*
14	2.9880	2.9880	–1.1737	2.9880
15	4.3715	4.3715*	0.4456	4.3715*
16	3.0377	3.0377	0.3972	3.0377
17	2.6764	2.6764	–0.8357	2.6764
18	2.1498	2.1498	–0.5523	2.1498
19	4.6574	4.6574*	1.4311	4.6574*
20	3.2859	3.2859*	0.4340	3.2859*

Note: * indicates the out-of-control points

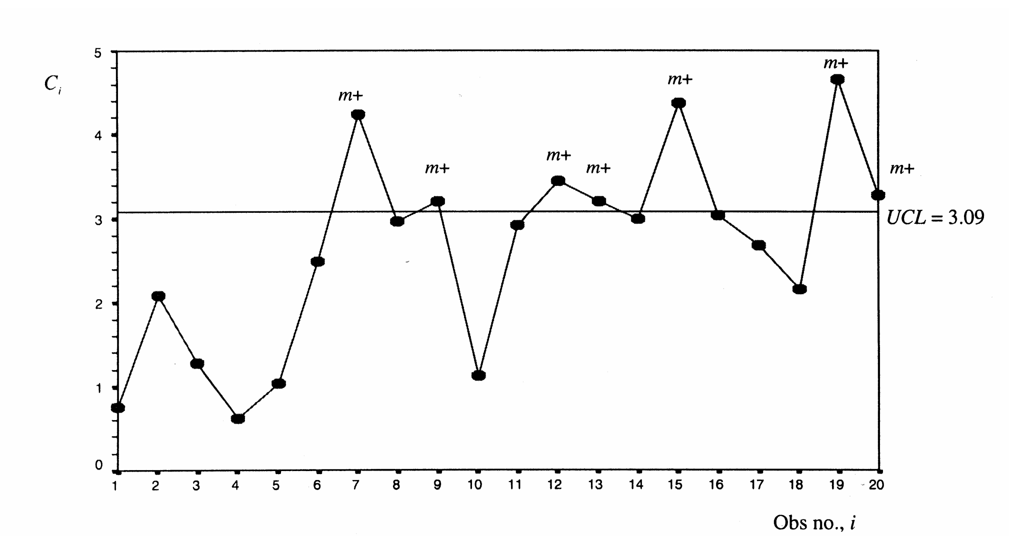

Figure 4. A Combined I-MR Chart for a Shift in the Process Mean

Table 5. An Example of Application for a Shift in the Process Variance based on b = 2 and UCL = 3.09

Obs. No., i	X_i	M_i	V_i	C_i
1	–0.3487	–0.3487	–0.8605	0.8605
2	–1.2907	–1.2907	–0.0134	1.2907
3	1.0317	1.0317	1.2784	1.2784
4	0.0442	0.0442	0.0376	0.0442
5	–0.1895	–0.1895	–1.1207	1.1207
6	–2.0778	–2.0778	0.9086	2.0778
7	–0.1000	–0.1000	0.9864	0.9864
8	0.4558	0.4558	–0.5081	0.5081
9	–0.3241	–0.3241	–0.2053	0.3241
10	3.0338	3.0338	2.1065	3.0338
11	0.4064	0.4064	1.5286	1.5286
12	1.8603	1.8603	0.5132	1.8603
13	2.3679	2.3679	–0.5818	2.3679
14	–2.7172	–2.7172	3.4111*	3.4111*
15	1.8373	1.8373	3.0162	3.0162
16	–1.4168	–1.4168	2.0258	2.0258
17	–0.7237	–0.7237	–0.3162	0.7237
18	0.9509	0.9509	0.7180	0.9509
19	–0.5085	–0.5085	0.5184	0.5184
20	–1.6768	–1.6768	0.2308	1.6768

Note: * indicates the out-of-control point

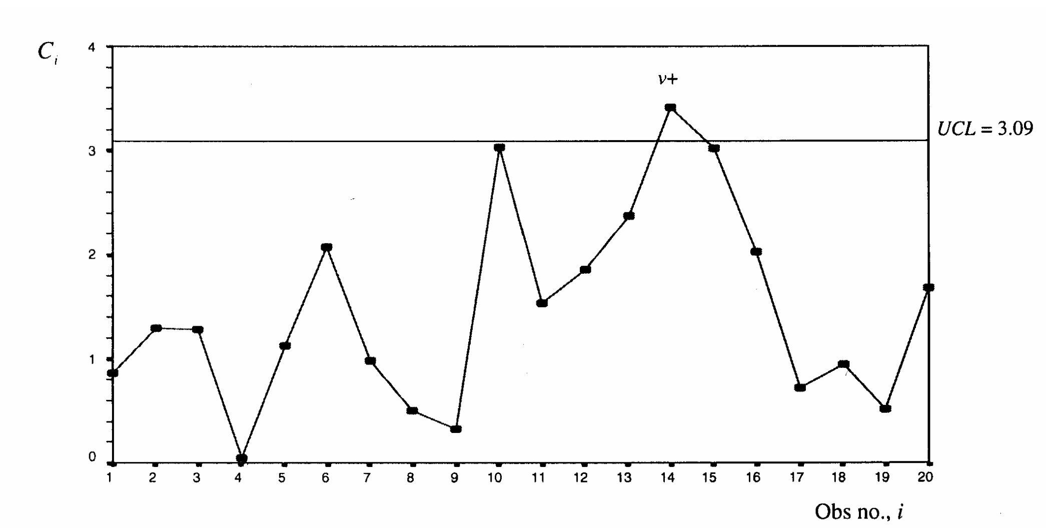

Figure 5. A Combined I-MR Chart for a Shift in the Process Variance

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Appendix

If $X_i \sim N(\mu, \sigma^2)$, $i = 1, 2, \ldots$, then

$$M_i = \frac{X_i - \mu}{\sigma} \sim N(0,1), \quad i = 1, 2, \ldots . \qquad \text{(A1)}$$

Because

$$X_i - X_{i-1} \sim N(0, 2\sigma^2), \quad i = 2, 3, \ldots,$$

it follows that

$$\frac{1}{\sigma\sqrt{2}}(X_i - X_{i-1}) \sim N(0,1), \quad i = 2, 3, \ldots .$$

Since the square of a standard normal statistic is a chi-square statistic with one degree of freedom (Hogg & Craig, 1978), it follows that

$$\frac{1}{2\sigma^2}(X_i - X_{i-1})^2 \sim \chi^2(1), \quad i = 2, 3, \ldots .$$

Then,

$$H_1\left[\frac{1}{2\sigma^2}(X_i - X_{i-1})^2\right] \sim U(0,1), \quad i = 2, 3, \ldots$$

where $H_1(\cdot)$ is the chi-square distribution function with one degree of freedom. Let $\Phi^{-1}(\cdot)$ be the inverse of the standard normal distribution function so that

$$V_i = \Phi^{-1}\left\{H_1\left[\frac{1}{2\sigma^2}(X_i - X_{i-1})^2\right]\right\} \sim N(0,1),$$
$$i = 2, 3, \ldots \qquad \text{(A2)}$$

Note that V_1 is computed using

$$\Phi^{-1}\left\{H_1\left[\frac{1}{2\sigma^2}(X_1 - \mu)^2\right]\right\}$$

because X_0 is unavailable at time i = 1. Define

$$C_i = \max(|M_i|, |V_i|), \quad i = 1, 2, \ldots \qquad \text{(A3)}$$

so that

$$\begin{aligned} F(c) &= P(C_i \le c) \\ &= P(|M_i| \le c, |V_i| \le c) \end{aligned}$$

where $F(\cdot)$ is the distribution function of C_i. Since the correlation of M_i and V_i is negligible, $P(|M_i| \le c, |V_i| \le c)$ can be approximated by

$$\begin{aligned} P(|M_i| \le c, |V_i| \le c) &\approx P(-c \le M_i \le c) \times P(-c \le V_i \le c) \\ &= [P(-c \le Z_i \le c)]^2 \\ &= [2\Phi(c) - 1]^2 \end{aligned}$$

Thus, the approximate density function of C_i is

$$f(c) = 4\phi(c)\{2\Phi(c) - 1\}, \quad c \ge 0 .$$