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Assumed that the distribution of the lifetime of any unit follows a lognormal distribution with parameters μ and σ. Also, assume that the relationship between μ and the stress level V is given by the power rule model. Several types of bootstrap intervals of the parameters were studied and their performance was studied using simulations and compared in term of attainment of the nominal confidence level, symmetry of lower and upper error rates and the expected width. Conclusions and recommendations are given.

Key words: Power rule model, lognormal distribution, bootstrap intervals, accelerated life test.

Introduction

The lognormal distribution has many special features that allowed it to be used as a model in various real life applications. In particular, it is used in analyzing biological data (Koch, 1966), and for analyzing data in workplace exposure to contaminants (Lyles & Kupper, 1997). It is also of importance in modeling lifetimes of products and individuals (Lawless, 1982). Various other motivations and applications of the lognormal distribution may also be found (see Johnson et. al., 1994, Schneider, 1986).

In a life testing experiment, the problem is that most units have a very long life under the normal conditions. Therefore, by the time the experiment is completed and an estimate of the reliability is obtained, the results will be outdated. To overcome this delay, accelerated life testing was introduced (Mann. et. al., 1974).

In an accelerated life testing experiment a certain number of units are subjected to a stress that is higher than the normal stress. The experiment is repeated under different values of stress. In order to do so, some relationship between the parameters of the time to failure distribution of the unit and the corresponding stress level must be postulated.

It is assumed that density function of the time to failure of a unit depends on one parameter say θ, and the environment depends on one stress V and that the relationship between θ and V is given by $\theta = \frac{C}{V^P}$ where C and P are positive constants. This relationship is known as the power rule model.

Consider the interval estimation for the parameters of the lognormal distribution after reparametrizing the location parameter μ as a function of the stress V using power rule model. The performance of the bootstrap and Jacknife intervals (Efron & Tibshirani, 1993) in term of attainment of the nominal confidence level, symmetry of lower and upper error rates and the expected width of the intervals will be compared.

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The Model and The Maximum likelihood Estimation

It is assumed that the lifetime (T) of any unit follows a lognormal distribution with location parameter μ and scale parameter σ. The probability density function of T is given by (Lawless, 1982):

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln t - \mu)^2}{2\sigma^2}\right), \; 0 < t < \infty \quad (1)$$

The location parameter μ was reparameterized as a function of the stress V using the power rule model $\mu = \frac{C}{V^{p_0}}$, therefore c and σ are the new parameters of the model. The unknown parameters c and σ were estimated using k complete samples. The j-th sample is obtained by using n_j units and the value V_j for the stress, j = 1, 2, …., k.

The likelihood function of the k complete samples is given by:

$$L(\mu, \sigma) = \frac{e^{\frac{-1}{2\sigma^2}\sum_{j=1}^{k}\sum_{i=1}^{n_j}(\ln t_{ij} - \mu_j)^2}}{\sigma^{\sum_{j=1}^{k} n_j} (2\pi)^{\frac{\sum_{j=1}^{k} n_j}{2}} \prod_{j=1}^{k}\prod_{i=1}^{n_j} t_{ij}} \quad (2)$$

Using the power rule model $\mu_j = \frac{C}{V_j^{P_0}}$ j = 1, 2, …., k , the likelihood function is given by:

$$L(C, \sigma) = \frac{e^{\frac{-1}{2\sigma^2}\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(\ln t_{ij} - \frac{c}{v_j^{po}}\right)^2}}{\sigma^{\sum_{j=1}^{k} n_j} (2\pi)^{\frac{\sum_{j=1}^{k} n_j}{2}} \prod_{j=1}^{k}\prod_{i=1}^{n_j} t_{ij}} \quad (3)$$

It is easy to show that the Maximum likelihood estimators of C and σ are given by:

$$\hat{C} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{n_j} \ln t_{ij} / v_j^{p_0}}{\sum_{j=1}^{k} n_j / v_j^{2p_0}} \quad (4)$$

and

$$\hat{\sigma} = \sqrt{\frac{\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(\ln t_{ij} - \frac{\hat{c}}{v_j^{p_0}}\right)^2}{\sum_{j=1}^{k} n_j}} \quad (5)$$

It is obvious that $\hat{C}$ is an unbiased estimator of C while $\hat{\sigma}$ is a biased estimator of σ.

The Percentile Interval

The methods of deriving confidence intervals presented in this section and section 4 are based on the parametric bootstrap approach (Efron & Tibshirani, 1993); they are constructed by resampling from the estimated parametric distribution. To construct the percentile interval, a simulation of the bootstrap distribution of $\hat{C}$ and $\hat{\sigma}$ is done by resampling from the parametric model of the original data. That is, a B bootstrap sample is generated and for each sample $\hat{C}$ and $\hat{\sigma}$ are calculated using equation (4) and (5) respectively. The calculated values are denoted by $\hat{C}^*$ and $\hat{\sigma}^*$.

Let $\hat{G}_1$ denotes the cumulative distribution of $\hat{C}^*$, the ($1-\alpha$) % percentile interval of C is $\left(\hat{G}_1^{-1}\left(\frac{\alpha}{2}\right), \hat{G}_1^{-1}\left(1-\frac{\alpha}{2}\right)\right)$, similarly let $\hat{G}_2$ denotes the cumulative distribution of $\hat{\sigma}^*$, the ($1-\alpha$) % percentile interval of σ is $\left(\hat{G}_2^{-1}\left(\frac{\alpha}{2}\right), \hat{G}_2^{-1}\left(1-\frac{\alpha}{2}\right)\right)$.

The Bias Corrected and Accelerated Interval (BCa Interval)

The bias corrected and accelerated interval is constructed by calculating two numbers $\hat{a}$ and $\hat{z}_0$ called the accelerated and the bias correction factor respectively, they are calculated using the following formulas

$$\hat{a} = \frac{\sum_{i=1}^{n}\left(\hat{C}(.) - \hat{C}(i)\right)^3}{6\left(\sum_{i=1}^{n}\left(\hat{C}(.) - \hat{C}(i)\right)^2\right)^{3/2}} \qquad (6)$$

where $\hat{C}(i)$ is the maximum likelihood estimator of C using the original data excluding the i-th observation and $\hat{C}(.) = \frac{\sum_{i=1}^{n}\hat{C}(i)}{n}$, $n = \sum_{j=1}^{k} n_j$.

The value of $\hat{z}_0$ is given by

$$\hat{z}_0 = \Phi^{-1}\left(\frac{\#(\hat{C}^* < \hat{C})}{B}\right) \qquad (7)$$

where $\Phi(.)$ is the standard normal cumulative distribution function. The ($1-\alpha$) % BCa interval of C is $\left(\hat{G}_1^{-1}(\alpha_1), \hat{G}_1^{-1}(\alpha_2)\right)$ where

$$\alpha_1 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{\alpha/2}}{1-\hat{a}(\hat{z}_0 + z_{\alpha/2})}\right)$$

and

$$\alpha_2 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z_{1-\alpha/2}}{1-\hat{a}(\hat{z}_0 + z_{1-\alpha/2})}\right) \qquad (8)$$

where z_α is the α quantile of the standard normal distribution. In the same way, the $(1-\alpha)$ % BCa interval of σ can be constructed.

Jacknife Interval

A ($1-\alpha$) % Jacknife interval of C (Efron and Tibshirani , 1993) is constructed as follows:

$$\hat{C}(.) \pm Z_{(\alpha/2)}\hat{S}_{Jack} ,$$

where

$$\hat{S}^2_{Jack} = \frac{n-1}{n}\sum_{i=1}^{n}\left(\hat{C}(.) - \hat{C}(i)\right)^2 \ , \ \hat{C}(.),\ \hat{C}(i)$$

and n were defined in section 4. Similarly, the $(1-\alpha)$ % Jacknife interval of σ by replacing C by σ in the above interval.

Simulation Study

A simulation study is conducted to investigate the performance of the intervals discussed in sections 3, 4 and 5 above. The indices of the simulation study are:

k : The number of lognormal populations, in this study k = 2.

n_1: Sample size from the first lognormal population, in this study $n_1 =$ 5, 10, 30.

n_2: Sample size from the second lognormal population, in this study $n_2 =$ 5, 10, 30.

C: Parameter of the power rule model, in this study $C = 3$.

P_0: In this study $P_0 = 0.3$.

V_1: The value of stress for the first lognormal population, in this study $V_1 = 100$.

V_2: The value of stress for the second lognormal population, in this study $V_2 = 200$.

σ: In this study $\sigma = 1$.

B: The number of bootstrap samples, in this study B = 1000.

For each combination of n_1 and n_2 2000 samples are generated and a $(1-\alpha)$ % Percentile interval is constructed, BCa interval and Jacknife interval for C and σ. Two values are considered for α, 0.05 and 0.10. The following were obtained for each interval:

1- The expected width (IW): the average of widths of the 2000 intervals.
2- Lower error rate (LER): the fraction of intervals that fall entirely above the true parameter.

3- Upper error rate (UER): the fraction of intervals that fall entirely below the true parameter.
4- Total error rate (TER): the fraction of intervals that did not contain the true parameter value.

Results and Conclusions

The results are given in tables 1 – 12. Table 1 has simulation results of the percentile interval of the parameter C using $\alpha = 0.05$. Table 2 has simulation results of the BCa interval of the parameter C using $\alpha = 0.05$. Table 3 has simulation results of the Jacknife interval of the parameter C using $\alpha = 0.05$. Table 4 has simulation results of the percentile interval of the parameter C using $\alpha = 0.1$. Table 5 has simulation results of the BCa interval of the parameter C using $\alpha = 0.1$. Table 6 has simulation results of the Jacknife interval of the parameter C using $\alpha = 0.1$. Table 7 has simulation results of the percentile interval of the parameter σ using $\alpha = 0.05$. Table 8 has simulation results of the BCa interval of the parameter σ using $\alpha = 0.05$. Table 9 has simulation results of the Jacknife interval of the parameter σ using $\alpha = 0.05$. Table 10 has simulation results of the percentile interval of the parameter σ using $\alpha = 0.1$. Table 11 has simulation results of the BCa interval of the parameter σ using $\alpha = 0.1$. Table 12 has simulation results of the Jacknife interval of the parameter σ using $\alpha = 0.1$. From these results the following can be concluded:

For the parameter C, the three intervals have almost the same expected width, and the expected width decreases as the sample sizes increases. In term of attainment of coverage probability and symmetry of lower and upper rates, the three intervals behave in the same way. It is recommended that the Jacknife interval be used because its calculation is simpler than the BCa and the percentile intervals.

For the parameter σ, the expected width for the percentile interval is nearly smaller than the other two intervals. On the other hand, in term of attainment of coverage probability and symmetry of lower and upper rates, the BCa interval behaves the best. It is therefore recommended that the BCa interval be used in this case.

Table 1. Percentile Interval of the parameter C with $\alpha = 0.05$

n_1	n_2	IW	LER	UER	TER
5	5	4.983	0.048	0.054	0.102
5	10	4.341	0.051	0.031	0.081
5	30	3.083	0.061	0.016	0.077
10	5	4.067	0.053	0.029	0.082
10	10	3.654	0.051	0.025	0.075
10	30	2.811	0.047	0.016	0.063
30	5	2.649	0.051	0.013	0.064
30	10	2.540	0.050	0.012	0.062
30	30	2.180	0.047	0.017	0.063

Table 2. BCa Interval of the parameter C with $\alpha = 0.05$

n_1	n_2	IW	LER	UER	TER
5	5	5.007	0.048	0.053	0.101
5	10	4.362	0.056	0.026	0.082
5	30	3.118	0.071	0.011	0.082
10	5	4.085	0.054	0.026	0.079
10	10	3.684	0.057	0.020	0.077
10	30	2.841	0.054	0.008	0.062
30	5	2.688	0.061	0.009	0.069
30	10	2.577	0.058	0.008	0.065
30	30	2.205	0.052	0.010	0.062

Table 3. Jacknife Interval of the parameter C with $\alpha = 0.05$

n_1	n_2	IW	LER	UER	TER
5	5	5.266	0.043	0.047	0.089
5	10	4.495	0.043	0.033	0.076
5	30	3.120	0.045	0.020	0.065
10	5	4.211	0.043	0.030	0.073
10	10	3.762	0.042	0.024	0.066
10	30	2.842	0.037	0.021	0.058
30	5	2.685	0.041	0.016	0.056
30	10	2.570	0.038	0.016	0.053
30	30	2.197	0.038	0.025	0.063

Table 4. Percentile Interval of the parameter C with $\alpha = 0.1$

n_1	n_2	IW	LER	UER	TER
5	5	4.182	0.074	0.081	0.154
5	10	3.641	0.083	0.056	0.139
5	30	2.571	0.087	0.036	0.123
10	5	3.410	0.077	0.065	0.142
10	10	3.062	0.084	0.049	0.133
10	30	2.346	0.072	0.042	0.114
30	5	2.211	0.072	0.034	0.105
30	10	2.119	0.075	0.038	0.112
30	30	1.820	0.070	0.046	0.116

Table 5. BCa Interval of the parameter C with $\alpha = 0.1$

n_1	n_2	IW	LER	UER	TER
5	5	4.194	0.074	0.077	0.151
5	10	3.658	0.090	0.052	0.142
5	30	2.603	0.097	0.030	0.126
10	5	3.423	0.082	0.058	0.139
10	10	3.086	0.093	0.042	0.135
10	30	2.372	0.085	0.033	0.118
30	5	2.243	0.083	0.024	0.107
30	10	2.149	0.088	0.025	0.113
30	30	1.840	0.084	0.036	0.119

Table 6. Jacknife Interval of the parameter C with $\alpha = 0.1$

n_1	n_2	IW	LER	UER	TER
5	5	4.419	0.065	0.073	0.138
5	10	3.772	0.067	0.057	0.124
5	30	2.619	0.076	0.040	0.116
10	5	3.534	0.067	0.061	0.128
10	10	3.157	0.071	0.049	0.119
10	30	2.385	0.063	0.045	0.108
30	5	2.254	0.065	0.039	0.104
30	10	2.157	0.065	0.040	0.105
30	30	1.844	0.061	0.053	0.113

Table 7. Percentile Interval of the parameter σ with $\alpha = 0.05$

n_1	n_2	IW	LER	UER	TER
5	5	0.790	0.000	0.198	0.198
5	10	0.667	0.000	0.145	0.145
5	30	0.456	0.003	0.081	0.084
10	5	0.669	0.001	0.149	0.150
10	10	0.584	0.002	0.128	0.130
10	30	0.427	0.003	0.090	0.093
30	5	0.455	0.004	0.094	0.098
30	10	0.428	0.006	0.069	0.075
30	30	0.350	0.007	0.067	0.074

Table 8. BCa Interval of the parameter σ with $\alpha = 0.05$

n_1	n_2	IW	LER	UER	TER
5	5	0.904	0.017	0.067	0.084
5	10	0.760	0.021	0.041	0.062
5	30	0.499	0.029	0.028	0.057
10	5	0.763	0.021	0.044	0.065
10	10	0.660	0.019	0.031	0.050
10	30	0.463	0.028	0.026	0.054
30	5	0.497	0.029	0.031	0.060
30	10	0.464	0.026	0.021	0.047
30	30	0.371	0.026	0.025	0.051

Table 9. Jacknife Interval of the parameter σ with $\alpha = 0.05$

n_1	n_2	IW	LER	UER	TER
5	5	0.847	0.005	0.165	0.170
5	10	0.702	0.006	0.119	0.125
5	30	0.467	0.008	0.076	0.084
10	5	0.702	0.004	0.124	0.128
10	10	0.608	0.008	0.109	0.117
10	30	0.435	0.009	0.082	0.091
30	5	0.462	0.009	0.087	0.096
30	10	0.432	0.013	0.072	0.085
30	30	0.354	0.009	0.066	0.075

Table 10. Percentile Interval of the parameter σ with $\alpha = 0.1$

n_1	n_2	IW	LER	UER	TER
5	5	0.666	0.000	0.253	0.253
5	10	0.561	0.003	0.206	0.209
5	30	0.383	0.012	0.134	0.146
10	5	0.563	0.002	0.215	0.217
10	10	0.490	0.006	0.199	0.204
10	30	0.358	0.012	0.133	0.144
30	5	0.381	0.012	0.139	0.151
30	10	0.359	0.015	0.116	0.131
30	30	0.294	0.016	0.113	0.129

Table 11. BCa Interval of the parameter σ with $\alpha = 0.1$

n_1	n_2	IW	LER	UER	TER
5	5	0.770	0.042	0.098	0.140
5	10	0.636	0.042	0.065	0.106
5	30	0.414	0.060	0.049	0.109
10	5	0.639	0.047	0.069	0.115
10	10	0.547	0.045	0.058	0.103
10	30	0.385	0.054	0.060	0.114
30	5	0.412	0.059	0.059	0.117
30	10	0.386	0.052	0.046	0.098
30	30	0.310	0.051	0.047	0.098

Table 12. Jacknife Interval of the parameter σ with $\alpha = 0.1$

n_1	n_2	IW	LER	UER	TER
5	5	0.711	0.012	0.202	0.214
5	10	0.590	0.013	0.171	0.184
5	30	0.392	0.026	0.113	0.139
10	5	0.590	0.013	0.160	0.172
10	10	0.511	0.015	0.160	0.175
10	30	0.366	0.023	0.117	0.139
30	5	0.388	0.023	0.128	0.151
30	10	0.363	0.024	0.101	0.125
30	30	0.298	0.026	0.099	0.125

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