[Journal of Modern Applied Statistical Methods](#)

Volume 5 | Issue 2 — Article 22

11-1-2005

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Recommended Citation

Aggarwal, M. L.; Chowdhury, S. Roy; Bansal, Anita; and Mital, Neena (2005) "Interaction Graphs for 4^r2^{n-p} Fractional Factorial Designs," *Journal of Modern Applied Statistical Methods*: Vol. 5 : Iss. 2 , Article 22.
DOI: 10.22237/jmasm/1162354860

Journal of Modern Applied Statistical Methods
November, 2006, Vol. 5, No. 2, 475-489

1538 – 9472/06/$95.00

Interaction Graphs for 4^r2^{n-p} Fractional Factorial Designs

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Interaction graphs have been developed for two-level and three-level fractional factorial designs under different design criteria. A catalogue is presented of all possible non-isomorphic interaction graphs for 4^r2^{n-p} (r=1; n=2,…, 10; p=1,…,8 and r=2; n=1,…, 7; p=1,…,7) fractional factorial designs, and non-isomorphic interaction graphs for asymmetric fractional factorial designs under the concept of combined array.

Key words: Minimum aberration designs, asymmetric orthogonal arrays, combined array, control and noise factors, pattern and extended pattern.

Introduction

Taguchi (1959, 1987) introduced the concept of linear graphs associated with various orthogonal arrays. Linear graphs are the graphical representation of allocation of main effects and two-factor interactions among various columns of orthogonal array. Ankenman and Dean (2003) have given an excellent review on Taguchi's methodology. Joglekar and Kacker (1989) and Kacker and Tsui (1990) discussed the concept of linear graphs for planning industrial experiments. Barton (1999) and Wu and Hamada (2000) discussed the concept of linear graphs. Li *et al.* (1991), Wu and Chen (1992), Chen, Sun and Wu (1993) and Sun and Wu (1994) developed interaction graphs (linear graphs) for two-level and three-level fractional factorial designs under different design criteria. These designs enable one to estimate all main effects and required two-factor interactions Aggarwal, Gupta, and Chowdhury (2001) developed interaction graphs which enable one to estimate three factor interactions along with all main effects and required two-factor interactions.

In the literature, interaction graphs are available for either two level or three level fractional factorial designs. Dey and Mukerjee (1999) discussed the concept of asymmetric orthogonal arrays which has been extensively used in industrial experiments for quality improvement. Further, Dey, Suen and Das (2005) have developed asymmetric fractional factorials plans which are universally optimal using the concept of finite projection geometry. Wu and Hamada (2000) extended the minimum aberration criterion of 2^{n-p} designs to 4^r2^{n-p} asymmetric designs for r = 1and 2. Xu and Wu (2001) generalized minimum aberration criterion for asymmetrical fractional factorial

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designs. In this article below, an algorithm has been developed to generate all possible non-isomorphic interaction graphs for 4^r2^{n-p} (r=1; n=2,…, 10; p=1,…,8 and r=2; n=1,…, 7; p=1,…,7) fractional factorial designs.. Some of these designs satisfy minimum aberration criterion. Interaction graphs were developed for only minimum aberration designs for r = 2. These designs can estimate all main effects and required two-factor interactions.

Many times, in industry, there are uncontrollable (noise) factors which induce variations in the system. Taguchi (1959) introduced robust design methodology to develop experiment that allows us to identify the settings of the control factors that make the product or process insensitive to the effects of the noise factors. Taguchi suggested the use of crossing of two OA's, inner array involving control factors and outer array involving noise factors and named it Crossed array. Welch *et al.* (1990), Shoemaker *et al*. (1991) and Montgomery (1991) independently proposed the concept of combined array where they combine control factors and noise factors in a single design matrix. This approach postulated a single response model of the type:

$$Y = f(X, Z) \quad (1)$$

where X and Z represent the settings in the control and noise variables respectively. Aggarwal *et al.* (2002) have developed a catalogue of all possible non-isomorphic interaction graphs for two level combined arrays. In this article, non-isomorphic interaction graphs were developed for asymmetric designs where one of the four levels factor may be treated as noise or control factor along with other two level control and noise factors. These designs enable one to estimate:

(a) Control and Noise main effects
(b) Control-by-Noise interactions (CxN)
(c) Control-by-Control interactions (CxC)

Also below, an algorithm is developed to generate non-isomorphic interaction graphs for 4^12^{n+m-p} combined array fractional factorial designs, where 'n' and 'm' are the number of control factors and noise factors respectively. A catalogue has been developed giving the number of non-isomorphic interaction graphs for n=2,…, 6; m=1,…,3; p=1,…,3 when a four level factor can be treated as a control or a noise factor.

Algorithm for developing interaction graphs for 4^r2^{n-p} fractional factorial designs.

In case of interaction graphs for two level symmetric fractional factorial designs, there exists only one edge representing two-factor interaction corresponding to linear x linear component. Whereas, in case of asymmetric designs with one factor at four-level its interactions with other two-level factors will generate at the most triple edges representing linear x linear, quadratic x linear and cubic x linear components. Similarly, when two factors are at four levels then the interaction between two four level factors will generate at the most nine edges. Therefore, the interaction graphs differ from each other on the basis of number of edges depending on type of interaction effects between two factors.

The method for developing non-isomorphic interaction graphs for 4^r2^{n-p} fractional factorial designs is an extension of the algorithm for symmetric designs based on the technique given by Li *et al.* (1991) and Wu and Chen (1992).

Non-isomorphic Interaction graphs for 4^12^{n-p} fractional factorial designs

Consider defining relation with one four levels and 'n' two level factors. First allocate the linear, quadratic and cubic effects viz. A_1, A_2 and A_3 in column 1, 3 and 2 respectively of an orthogonal array. The advantage of the (A_1, A_3, A_2) system is that the three vectors are of the form (α, β, $\alpha\beta$). This relationship makes it easier to relate and trace each A_i to a factorial effect in the original two-level design from which 4^12^{n-p} design has been generated. Then two level factors are allocated to remaining columns of orthogonal array depending on the defining relation. Develop alias structure neglecting three and higher factor interactions. Select one two-factor interaction from each of the aliased two-factor interactions along with all clear two-factor interactions. Construct interaction matrix for each design with column and row headed alphabetically (representing both four level and two level factors) and $(ij)^{th}$ entry value 1 if one of the component is present between i^{th} row and j^{th} column, value 2 if two components are present, value 3 if all the three components are present and 0 otherwise. Corresponding to the interaction matrix, calculate pattern P_4 and P_2 which are the column total of the interaction matrix corresponding to four level factor and two level factors respectively. In other words P_4 is just the number of edges from four level factors to two level factors and vice-versa. Similarly, P_2 is the number of edges between two level factors only.

Next calculate the extended pattern in two parts viz. EP_4 defined as $D_i = \Sigma d_{ij}$, where d_{ij} are the P_4 patterns of j^{th} factor adjacent to i^{th} factor and EP_2 defined as $D_k = \Sigma d_{kl}$, where d_{kl} are the P_2 patterns of l^{th} factor adjacent to k^{th} factor. To obtain non-isomorphic interaction graphs sort the patterns (P_2 and P_4) and extended pattern (EP_2 and EP_4) in ascending order separately. Repeat the procedure for all combinations. The combinations are non-isomorphic if the patterns are distinct or if patterns are same but the corresponding extended patterns are distinct. Corresponding to each distinct combination develop an interaction graph. The following example gives the procedure to develop non-isomorphic interaction graphs for $4^1 2^{n-p}$ fractional factorial designs.

Example1

Consider a $4^1 2^{3-1}$ minimum aberration design with 16 runs where A is a four-level factor and B, C and D are two-level factors with the defining relation: $I = A_1BCD$. The alias structure is shown in Table 1.

Here, three sets of aliased two-factor interactions have been provided. This gives 8 combinations of nine two-factor interactions taking six clear two-factor interactions together. Corresponding to each unique combination there is a distinct interaction graph. Consider one of the combinations:

CD A_3B A_2B BD A_3C A_2C BC A_2D A_3D

There are 3 double edges representing interactions between four level factor and three two-level factors and 3 single edges representing interactions among two-level factors. The interaction matrix with pattern and extended pattern for this combination is shown in table 2.

Repeat the procedure for remaining combinations. This gives four non-isomorphic interaction graphs corresponding to four unique combinations. Table 3 gives pattern and extended pattern of all unique combinations for the design $I = A_1BCD$.

Table 1. Alias Structure

A_1B	A_3B	A_2B	A_1C	A_3C	A_2C	A_1D	A_2D	A_3D
CD			BD			BC		

Table 2. Interaction Matrix

	A	B	C	D
A	0	2	2	2
B	2	0	1	1
C	2	1	0	1
D	2	1	1	0
P_4	6	2	2	2
P_2	0	2	2	2
EP_4	12	16	16	16
EP_2	12	4	4	4

Table 3: Patterns and Extended Patterns for the design I = A_1BCD

S.No.	Design Combinations	P_4	P_2	EP_4	EP_2
1	A_1B A_3B A_2B A_1C A_3C A_2C A_1D A_2D A_3D	9 3 3 3	0 0 0 0	27 27 27 27	0 0 0 0
2	A_1B A_3B A_2B A_1C A_3C A_2C BC A_2D A_3D	8 2 3 3	0 0 1 1	22 16 27 27	6 0 1 1
3	A_1B A_3B A_2B BD A_3C A_2C BC A_2D A_3D	7 2 2 3	0 1 1 2	17 17 17 25	10 2 2 2
4	CD A_3B A_2B BD A_3C A_2C BC A_2D A_3D	6 2 2 2	0 2 2 2	12 16 16 16	12 4 4 4

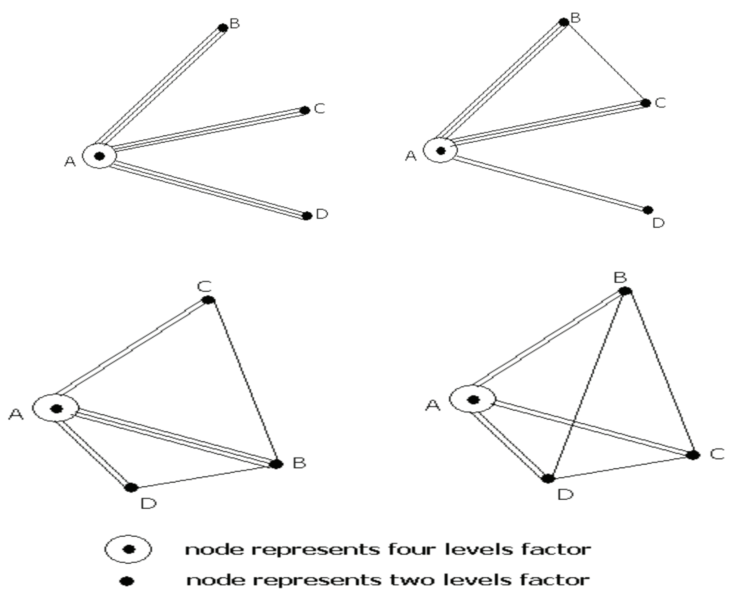

Figure 1: Interaction Graphs for the design I = A_1BCD.

Note: Interaction graphs correspond to the designs outlined in Table 3

Non-isomorphic Interaction graphs for $4^2 2^{n-p}$ fractional factorial designs

Suppose that there are two factors at four levels and n factors at two levels. Optimal minimum aberration designs developed by Wu and Hamada (2000) have been considered, to allocate the two four level factors viz. A in column 1, 2 and 3, B in columns 4, 8 and 12. Depending on the defining relation, n two-level factors are allocated in the remaining columns of orthogonal array. Develop alias structure neglecting three and higher factor interactions. Construct interaction matrix and calculate pattern and extended pattern in the similar manner as discussed above. Due to two factors at four level the entries in the interaction matrix corresponding to interaction between the two four level factors varies from 0 to 9 depending upon the presence of the component combination. The rest of the procedure remains the same as when only one factor is at four levels. The following example gives the procedure to develop non-isomorphic interaction graphs for $4^2 2^{n-p}$ fractional factorial designs.

Example 2

Consider a $4^2 2^{2-1}$ design with 32 runs where A and B are four level factors and C and D are two level factors with the defining relation: I = A_3B_1CD. The alias structure as shown in Table 4.

Here, there are 16 clear two-factor interactions and three sets of aliased two-factor interactions. This gives all together 8 combinations. For this design, there are 4 unique combinations, which give 4 non-isomorphic interaction graphs. Table 5 gives pattern and extended pattern of all unique combinations for the design I = A_3B_1CD.

Table 4. Alias Structure

A_1B_1	A_3B_1 CD	A_2B_1	A_1B_3	A_3B_3	A_2B_3	A_1B_2	A_3B_2	A_2B_2	A_1C	A_3C B_1D	A_2C
A_2D	A_3D B_1C	A_1D	B_3C	B_2D	B_2C	B_3D					

Table 5. Patterns and Extended Patterns for the design I = A_3B_1CD

S.No.	Design Combinations	P_4	P_2	EP_4	EP_2
1	A_1B_1 A_3B_1 A_2B_1 A_1B_3 A_3B_3 A_2B_3 A_1B_2 A_3B_2 A_2B_2 A_1C A_3C A_2C A_3D A_2D A_1D B_3C B_2D B_2C B_3D	13 15 5 5	0 0 0 0	147 155 71 71	0 0 0 0
2	A_1B_1 CD A_2B_1 A_1B_3 A_3B_3 A_2B_3 A_1B_2 A_3B_2 A_2B_2 A_1C A_3C A_2C A_3D A_2D A_1D B_3C B_2D B_2C B_3D	12 14 5 5	0 0 1 1	126 132 71 71	4 6 1 1
3	A_1B_1 CD A_2B_1 A_1B_3 A_3B_3 A_2B_3 A_1B_2 A_3B_2 A_2B_2 A_1C B_1D A_2C A_3D A_2D A_1D B_3C B_2D B_2C B_3D	13 13 4 6	0 0 1 1	130 130 58 82	5 5 1 1
4	A_1B_1 A_3B_1 A_2B_1 A_1B_3 A_3B_3 A_2B_3 A_1B_2 A_3B_2 A_2B_2 A_1C B_1D A_2C A_3D A_2D A_1D B_3C B_2D B_2C B_3D	14 14 4 6	0 0 0 0	152 152 56 84	0 0 0 0

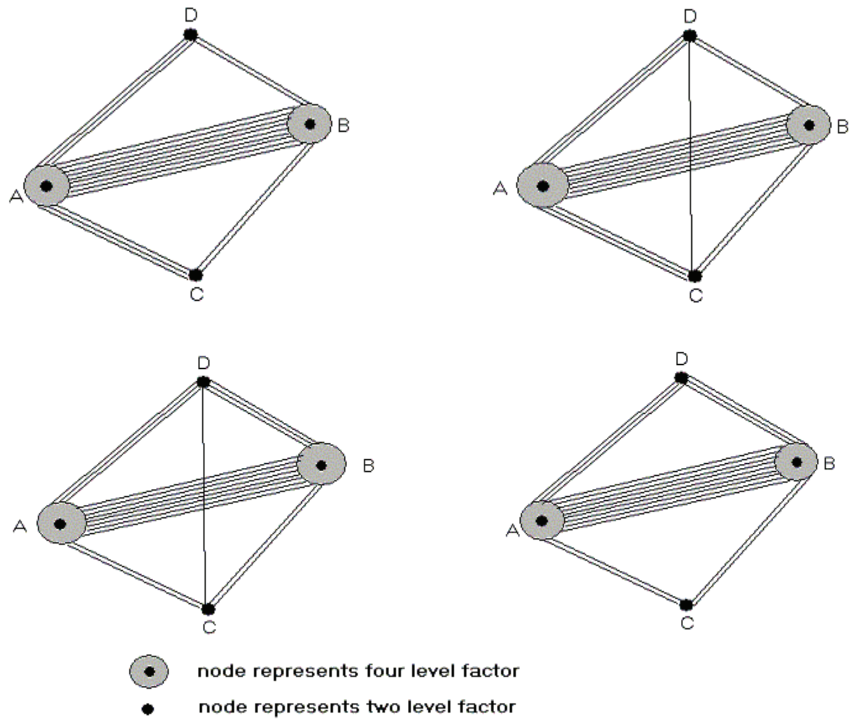

Figure 2. Interaction Graphs for the above I = A_3B_1CD

Note. All the non-isomorphic interaction graphs correspond to the designs outlined in Table 5.

A catalogue has been developed for 4^12^{n-p} and 4^22^{n-p} designs giving the number of non-isomorphic interaction graphs. A part of the catalogue is shown in Appendix – I. A complete catalogue is available with the authors.

Algorithm for developing non-isomorphic interaction graphs for 4^12^{n+m-p} combined array fractional factorial designs.

Consider 4^12^{n+m-p} combined array fractional factorial designs where n and m are the number of control and noise factors respectively. If one of the control factors is at four-level then the model will be of the form:

$$y = b_0 + b_1x_1 + \sum_{i=2}^{n} b_ix_i + \sum_{j=1}^{m} b_jz_j + \sum_{j=1}^{m} b_{1j}x_1z_j + \sum_{j=2}^{n} \gamma_{1j}x_1x_j + \sum_{i=2}^{n}\sum_{j=1}^{m} b_{ij}x_iz_j + \sum_{i=2}^{n}\sum_{j>i}^{n} \gamma_{ij}x_ix_j + \varepsilon \quad (2)$$

where x_1 denotes a four-level control factor, x_i denotes two-level control factors and z_j denotes two-level noise factors.

Whereas if one of the noise factors is at four levels then the model will be of the form:

$$y = b_0 + b_1 z_1 + \sum_{j=2}^{m} b_j z_j + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} b_{i1} x_i z_1 + \sum_{i=1}^{n} \sum_{j=2}^{m} b_{ij} x_i z_j + \sum_{i=1}^{n} \sum_{j>i}^{n} \gamma_{ij} x_i x_j + \varepsilon \qquad (3)$$

where z_1 denotes a four level noise factor.

The method of developing non-isomorphic interaction graphs for $4^1 2^{n+m-p}$ combined array designs is an extension of the algorithm based on the technique given by Aggarwal *et al.* (2002) for symmetric designs. In this case, some of the two-level factors are treated as control and noise factor along with a four-level factor which may be treated as control or noise factor. Capital letters indicate control factors and small letters indicate noise factors. For a given defining relation, an alias structure is first constructed with pre-defined number of control and noise factors neglecting two-factor interactions of type noise x noise and all three and higher order interactions. In order to define non-isomorphic alias structure for a given defining relation, the following is first counted:

(a) Number of clear C_2 xC_2, C_2 x C_4, C_2xN_2, C_2xN_4 and C_4x N_2 interactions.

(b) Number of alias C_2 xC_2 with C_2 xC_2, C_2 xC_2 with C_2xC_4, C_2xC_2 with C_2xN_2, C_2x C_2 with C_2xN_4, C_2x C_2 with C_4x N_2, C_2x N_2 with C_2x N_2, C_2x N_2 with C_2xN_4, C_2x N_2 with C_4x N_2 interaction (any two-factor interaction aliased with N_2xN_2 and N_2x N_4 interaction is assumed to be clear two-factor interaction).

where C_2 and N_2 are control and noise factors at two level respectively and C_4 and N_4 are control and noise factors at four level respectively.

The above counting technique gives all possible non-isomorphic alias structures for different number of control and noise factors at four-level and two-level for a given defining relation.

Calculate P_4, P_2, EP_4 and EP_2 as mentioned earlier. While sorting the patterns and extended patterns in ascending order, divide each P_4, P_2, EP_4 and EP_2 further into two groups, corresponding to the control factors and other corresponding to noise factors. The combinations are non-isomorphic if the patterns are distinct or if patterns are same but the corresponding extended patterns are distinct. Corresponding to each distinct combination develop an interaction graph. For various designs with one factor at four-level and other factors at two-levels a catalogue has been developed highlighting the number of non-isomorphic interaction graphs for each design. A part of the catalogue for $4^1 2^{n+m-p}$ combined array fractional factorial designs corresponding to different number of control and noise factors are shown in Appendix II. A complete catalogue is available with the authors. The concept for developing non-isomorphic interaction graphs for $4^1 2^{n+m-p}$ combined array is explained with the help of following example.

Example 3

Consider a $4^1 2^{5-2}$ design with 32 runs with defining relation I = A_1BCDE = A_2CDF = A_3BEF. Suppose there are 3 control factors and 3 noise factors, in which one control or noise factor is at 4 levels and rest are at two-level. This gives resolution IV design. There are 6 non-isomorphic defining relations of same word length pattern (0, 0, 0, 2, 1, 0) but with different alias structure according to the criteria given above. These defining relations are defined in Table 6.

Now consider the first defining relation i.e., I= $a_1$bcDE = a_2cDF= a_3bEF. The alias structure for the given defining relation is as shown in Table 7.

There are 16 possible combinations of eligible but not clear two-factor interactions along with clear two-factor interactions. For this design there are 9 non-isomorphic interaction graphs corresponding to unique combinations as shown in Figure 3. The pattern and extended pattern of all unique combinations for above design are shown in Table 8.

Table 6. Defining Relations

1. $I= a_1bcDE = a_2cDF$
2. $I= a_1bCDe = a_2CDF$
3. $I= a_1bCDE = a_2CDf$
4. $I= A_1bcdE = A_2cdF$
5. $I= A_1bcDE = A_2cDf$
6. $I= A_1bCDe = A_2CDf$

Table 7. Alias Structure

EF	DF	DE	a_1D	a_3D	a_2D	bD	cE	a_2F	a_3F	a_1F	a_1E	a_2E	a_3E
					cF			cD	bE				bF

TABLE 8. Patterns and extended patterns for the design: $I= a_1bcDE = a_2cDF= a_3bEF$

S.No.	Design Combinations	P_4	P_2	EP_4	EP_2
1.	FE DF DE a_1D a_3D a_2D bD cE a_2F a_3F a_1F a_1E a_2E a_3E	9 0 0 3 3 3	0 1 1 2 3 3	27 3 3 33 33 33	24 3 3 6 6 6
2.	FE DF DE a_1D a_3D a_2D bD cE a_2F a_3F a_1F a_1E a_2E bF	8 0 0 2 3 3	0 1 2 3 3 3	22 2 6 22 29 29	24 3 6 7 8 8
3.	FE DF DE a_1D a_3D a_2D bD cE a_2F bE a_1F a_1E a_2E bF	7 0 0 2 2 3	0 1 3 3 3 4	17 2 7 19 19 25	23 4 10 10 10 10
4.	FE DF DE a_1D a_3D a_2D bD cE a_2F bE a_1F a_1E a_2E a_3E	8 0 0 2 3 3	0 1 2 2 3 4	22 3 6 22 29 29	25 4 7 7 8 8
5.	FE DF DE a_1D a_3D a_2D bD cE cD a_3F a_1F a_1E a_2E bF	7 0 0 2 2 3	0 2 2 3 3 4	17 5 5 19 19 25	24 7 7 9 9 10
6.	FE DF DE a_1D a_3D a_2D bD cE cD bE a_1F a_1E a_2E a_3E	7 0 0 1 3 3	0 2 2 2 4 4	19 6 6 13 25 25	26 8 8 8 10 10
7.	FE DF DE a_1D a_3D cF bD cE a_2F bE a_1F a_1E a_2E bF	6 0 0 2 2 2	0 2 3 3 4 4	12 4 6 16 16 16	22 8 11 11 12 12
8.	FE DF DE a_1D a_3D a_2D bD cE cD bE a_1F a_1E a_2E bF	6 0 0 1 2 3	0 2 3 3 4 4	14 5 6 11 16 21	23 8 11 11 12 12
9.	FE DF DE a_1D a_3D cF bD cE cD bE a_1F a_1E a_2E bF	5 0 0 1 2 2	0 3 3 4 4 4	9 5 5 9 13 13	20 12 12 14 14 14

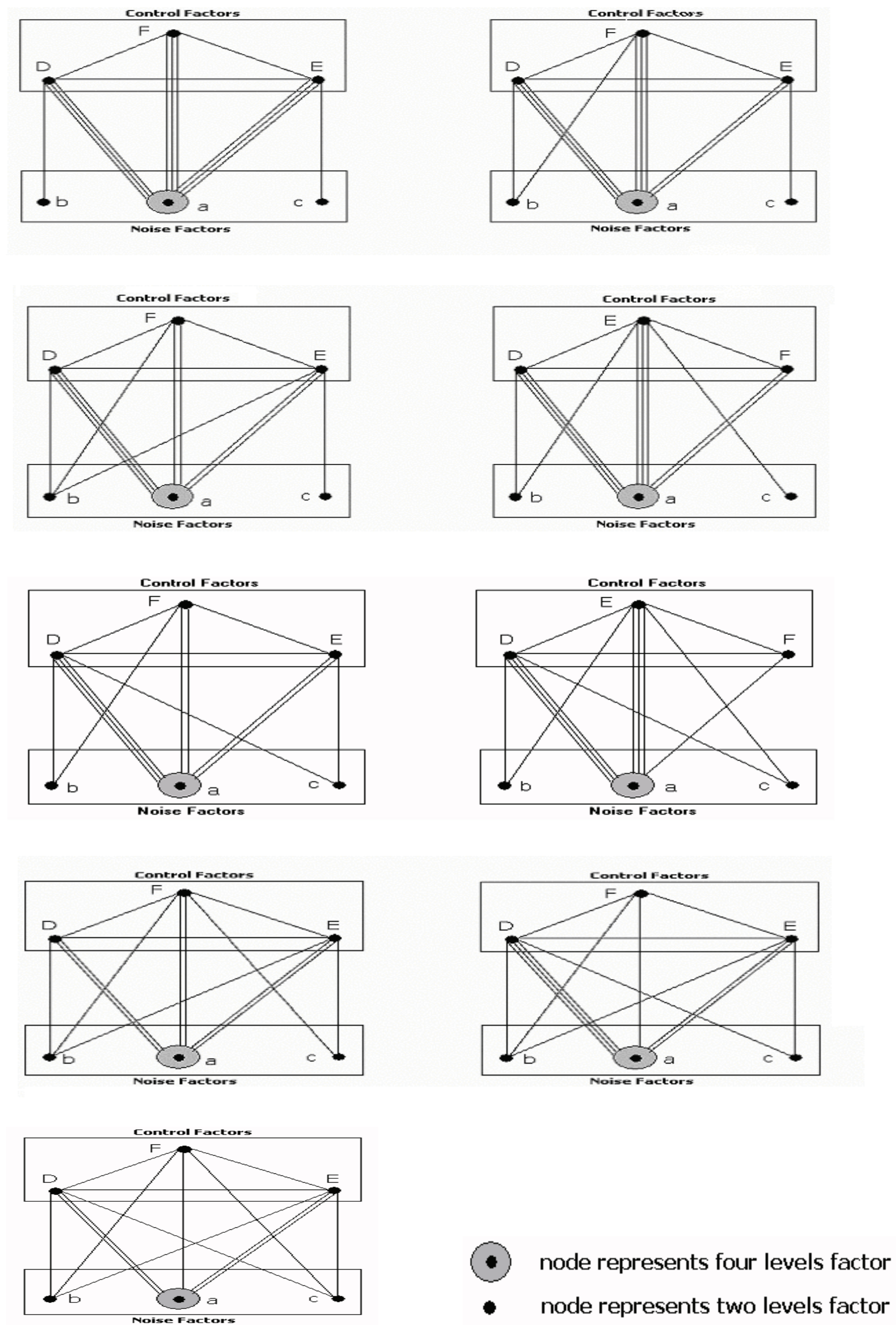

Figure 3. Interaction Graphs for the Design I= $a_1bcDE = a_2cDF$

Conclusion

The non-isomorphic interaction graphs for 4^r2^{n-p} (r=1; n=2,…, 10; p=1,…, 8 and r=2; n=1,…, 7; p=1,…,7) fractional factorial designs developed in this article will enable the engineers to work on experiments when mixed level factors are present say one or two factors are at four level and rest are at two level. These designs will allow the estimation of all the main effects and required two-factor interactions for certain type of asymmetric designs.

All possible non-isomorphic interaction graphs for $4^1 2^{n+m-p}$ (n=2,…, 6; m=1,…, 3; p=1,…, 3) combined array fractional factorial designs are also presented which will allow the estimation of all the main effects and required two-factor interactions when one of the control or noise factor is at four level and rest of the control and noise factors are at two level.

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Appendix 1
Non-Isomorphic Interaction Graphs for $4^1 2^{n-p}$ Fractional Factorial Designs

S.No	Design	Design generator	No. of 2 level factors (n)	No. of non-isomorphic interaction graphs
1	$4^1 2^{2-1}$	$I=A_3BC$	2	2
2	$4^1 2^{3-2}$	$I=A_1BC=A_2BD$	3	1
3	$4^1 2^{3-1}$	*$I=A_1BCD$	3	4
4	$4^1 2^{4-2}$	*$I=A_3BD=A_1BCE$	4	25
5	$4^1 2^{4-2}$	$I=A_2BD=A_3BE$	4	1
6	$4^1 2^{5-3}$	*$I= A_3BD = A_3CE = A_1BCF$	5	75
7	$4^1 2^{5-3}$	$I= A_1BD = A_2BE = A_3BF$	5	1
8	$4^1 2^{5-3}$	$I= BCD = A_1CE = A_2BF$	5	69
9	$4^1 2^{5-3}$	$I= A_2BD = A_1BE = BCF$	5	14
10	$4^1 2^{5-3}$	$I= A_1BCD = A_2BCE = A_3BCF$	5	35
11	$4^1 2^{6-4}$	*$I=A_3BD = A_3CE = A_1BCF = A_2BCG$	6	92
12	$4^1 2^{6-4}$	$I=A_1BCD = A_2BCE = A_3BCF = BCG$	6	13
13	$4^1 2^{6-4}$	$I=A_1BCD = A_2CE = A_3BF = BCG$	6	49
14	$4^1 2^{7-5}$	*$I=A_1BD=A_1CE=A_3BF=A_3CG=A_2CH$	7	12
15	$4^1 2^{7-5}$	$I=A_1BCD=A_2BCE=A_3BCF=BCG=A_2CH$	7	42
16	$4^1 2^{7-5}$	$I=A_1BD=A_1CE=A_3BF= A_3CG=A_2BH$	7	12
17	$4^1 2^{8-6}$	*$I=A_1BD=A_1CE=A_3BF=A_3CG=A_2BH=A_2CJ$	8	6
18	$4^1 2^{8-6}$	$I=A_1BCD=A_2BCE=A_3BCF=A_1BG=A_2BH=A_3BJ$	8	26
19	$4^1 2^{8-6}$	$I=A_1BCD=A_2BCE=A_3BCF=BCG=A_2CH=A_3BJ$	8	26
20	$4^1 2^{9-7}$	*$I=A_1BD=A_3BE=BCF=A_2BG=A_1BCH=A_3BCJ=A_2BCK$	9	8
21	$4^1 2^{9-7}$	$I=A_1BCD=A_2BCE=A_1BF=A_2BG=A_1CH=A_2CJ=BCK$	9	10
22	$4^1 2^{10-8}$	*$I=A_1BD=A_1CE=A_3BF=BCG=A_2BH=A_1BCJ=A_3BCK= A_2BCL$	10	8
23	$4^1 2^{10-8}$	$I=A_1BCD=A_2BCE=A_3BCF=A_1BG=A_2BH=A_3BJ=A_1CK=A_2CL$	10	5
24	$4^1 2^{4-1}$	*$I=A_1BCDE$	4	1
25	$4^1 2^{4-1}$	$I=BCDE$	4	2
26	$4^1 2^{5-2}$	*$I=A_2CDE = A_1BCDF$	5	19
27	$4^1 2^{5-2}$	$I=A_1BCDE = A_3BCDF$	5	2
28	$4^1 2^{6-3}$	*$I=A_2BDE = A_2CDF=A_1BCDG$	6	392
29	$4^1 2^{6-3}$	$I=A_1BCDE = A_2BCDF=A_3BCDG$	6	1
30	$4^1 2^{6-3}$	$I=A_1BCDE = A_2BCDF=BCDG$	6	7
31	$4^1 2^{7-4}$	$I=A_1BCDE = A_2BCDF = A_3BCDG = BCDH$	7	4
32	$4^1 2^{7-4}$	$I=A_1BCDE = A_2BCDF = A_3BCDG = A_3CDH$	7	124
33	$4^1 2^{5-1}$	*$I=A_2BCDEF$	5	1
34	$4^1 2^{5-1}$	$I=CDEF$	5	2
35	$4^1 2^{5-1}$	$I=A_3BCF$	5	4
36	$4^1 2^{6-2}$	*$I=A_1BCDF = A_3CDEG$	6	1
37	$4^1 2^{6-2}$	$I=A_1BCDEF = A_2BCDEG$	6	2
38	$4^1 2^{6-2}$	$I=A_1BCDEF = A_2BCDG$	6	4
39	$4^1 2^{6-2}$	$I=A_1BCDF = A_2BCG$	6	19
40	$4^1 2^{6-2}$	$I=BCDEF = CDEG$	6	4
41	$4^1 2^{6-2}$	$I=BCDF = CDEG$	6	7
42	$4^1 2^{7-3}$	*$I= A_1BCDF= A_3BCEG= A_2BDEH$	7	1
43	$4^1 2^{7-3}$	$I= A_1BCDEF= A_2BCDEG= A_3BCDEH$	7	2

44	4^12^{7-3}	I= A_1BCDF= A_2BCEG= BCDH	7	11
45	4^12^{7-3}	I= BCDEF= BCDG= CDEH	7	150
46	4^12^{8-4}	I= BDEF=A_1BCDG= A_3CDEH= A_2BCEJ	8	3
47	4^12^{8-4}	I= A_1BCDEF=A_2BCDEG= A_3BCDEH= BCDEJ	8	1
48	4^12^{8-4}	I= A_1BCDEF=A_2BCDEG= A_1BCDH= A_2BCEJ	8	15
49	4^12^{8-4}	I= A_1BCDEF=A_1BCDG= A_1CDEH= A_1BDEJ	8	6
50	4^12^{8-4}	I= A_2BCEF=A_1BCDG= A_2CDEH= A_3BDEJ	8	7
51	4^12^{8-4}	I= A_1BCDF=A_2BCDG= A_3BCDH= BCDJ	8	4
52	4^12^{8-4}	I= A_1BCDF=A_2BCDG= A_3BCDH= A_3DEJ	8	4
53	4^12^{8-4}	I= A_1BCDF=BCEG= BDEH= CDEJ	8	17
54	4^12^{8-4}	I= BCDF= BCEG = BDEH= CDEJ	8	26
55	4^12^{9-5}	I=BDEF=A_2BDG=A_1BCDH=A_3CDEJ=A_2BCEK	9	660

Note. (* Designs are Minimum Aberration Designs (MAD))

Non-Isomorphic Interaction Graphs for 4^22^{n-p} Minimum Aberration Designs

S.No.	Design	Design generator	No. of 2 level factors (n)	No. of non-isomorphic interaction graphs
1	4^22^{1-1}	I=A_1B_3C	1	6
2	4^22^{2-2}	I=A_1B_3C=A_3B_1D	2	42
3	4^22^{3-3}	I= A_1B_3C = A_3B_1D = A_3B_2E	3	110
4	4^22^{4-4}	I= A_1B_3C = A_3B_1D = A_2B_3E=A_3B_2F	4	117
5	4^22^{5-5}	I= A_1B_3C = A_3B_1D = A_3B_3E= A_2B_3F= A_3B_2G	5	37
6	4^22^{6-6}	I= A_1B_1C = A_1B_3D = A_3B_1E=A_3B_3F=A_1B_2G=A_3B_2H	6	17
7	4^22^{7-7}	I=A_1B_1C=A_1B_3D=A_3B_1E=A_3B_3F=A_2B_3G=A_1B_2H = A_3B_2J	7	10
8	4^22^{2-1}	I= A_3B_1CD	2	4
9	4^22^{3-2}	I=A_3B_1CD=A_2B_3CE	3	58
10	4^22^{4-3}	I=A_3B_1CD=A_2B_3CE=A_1B_2CF	4	1730
11	4^22^{3-1}	I=A_2B_2CDE	3	1
12	4^22^{4-2}	I=A_1B_1CDE=A_3B_3CDF	4	4
13	4^22^{5-3}	I=A_1B_1CDE=A_3B_3CDF=A_3B_2DG	5	223

Appendix 2

Non-Isomorphic Interaction Graphs for $4^1 2^{n+m-p}$ Combined Array Fractional Factorial Designs

S. N	Design	Defining Relation	No. of Control Factors (n)	No. of Noise Factors (m)	No. of Non- isomorphic Interaction graphs
1	$4^1 2^{2-1}$	I= A_3bC	2	1	3
2	$4^1 2^{3-2}$	I= $A_1bC = A_2bD$	3	1	2
3	$4^1 2^{3-2}$	I= $a_1BC = a_2BD$	3	1	1
4	$4^1 2^{3-2}$	I= $A_1bc = A_2bD$	2	2	2
5	$4^1 2^{3-2}$	I= $a_1bC = a_2bD$	2	2	1
6	$4^1 2^{4-2}$	I= $A_2bD = A_3bE$	4	1	2
7	$4^1 2^{4-2}$	I= $a_2BD = a_3BE$	4	1	1
8	$4^1 2^{4-2}$	I= $A_1cE = BcD$	4	1	9
9	$4^1 2^{4-2}$	I= $A_1Ce = BCD$	4	1	9
10	$4^1 2^{4-2}$	I= $a_1CE = BCD$	4	1	8
11	$4^1 2^{4-2}$	I= $a_2BD = a_2CE$	4	1	6
12	$4^1 2^{4-2}$	I= $a_2bD = a_3bE$	3	2	1
13	$4^1 2^{4-2}$	I= $a_2BD = a_3BE$	3	2	1
14	$4^1 2^{4-2}$	I= $A_2bD = A_3bE$	3	2	2
15	$4^1 2^{4-2}$	I= $A_2bd = A_3bE$	3	2	2
16	$4^1 2^{4-2}$	I= $a_1BCE = a_1CD$	4	1	8
17	$4^1 2^{4-2}$	I= $A_1BCE = A_1Cd$	4	1	9
18	$4^1 2^{4-2}$	I= $a_1bCE = a_1CD$	3	2	6
19	$4^1 2^{4-2}$	I= $a_1BcE = a_1cD$	3	2	3
20	$4^1 2^{4-2}$	I= $a_1BCE = a_1Cd$	3	2	3
21	$4^1 2^{5-3}$	I= $a_1BD = a_2BE = a_3BF$	5	1	1
22	$4^1 2^{5-3}$	I= $A_1bD = A_2bE = A_3bF$	5	1	1
23	$4^1 2^{5-3}$	I= $a_2BD= a_1BE = BCF$	5	1	14
24	$4^1 2^{5-3}$	I= $A_2bD= A_1bE = bCF$	5	1	17
25	$4^1 2^{5-3}$	I= $a_1bD = a_2bE = a_3bF$	4	2	1
26	$4^1 2^{5-3}$	I= $A_1bd = A_2bE = A_3bF$	4	2	1
27	$4^1 2^{5-3}$	I= $A_1Bd = A_2Be = A_3BF$	4	2	1
28	$4^1 2^{5-3}$	I= $a_1bCD= a_2bCE = a_3bCF$	4	2	10
29	$4^1 2^{5-3}$	I= $a_2bD= a_1bE = bcF$	3	3	4
30	$4^1 2^{5-3}$	I= $a_2bd= a_1bE = bCF$	3	3	10
31	$4^1 2^{5-3}$	I= $a_2Bd= a_1Be = BCF$	3	3	7
32	$4^1 2^{5-3}$	I= $A_2Bd= A_1BE = Bcf$	3	3	8
33	$4^1 2^{5-3}$	I= $a_1bcD= a_2bcE = a_3bcF$	3	3	1
34	$4^1 2^{4-1}$	I= a_1BCDE	4	1	1
35	$4^1 2^{4-1}$	I= A_1bCDE	4	1	1
36	$4^1 2^{4-1}$	I= A_1BcDE	4	1	1
37	$4^1 2^{4-1}$	I= a_1bCDE	3	2	1
38	$4^1 2^{4-1}$	I= A_1bcDE	3	2	1
39	$4^1 2^{4-1}$	I= a_1BCE	4	1	4

40	4^12^{4-1}	I= A_1bCE	4	1	6
41	4^12^{4-1}	I= a_1bCE	3	2	3
42	4^12^{4-1}	I= A_1bcE	3	2	3
43	4^12^{4-1}	I= BCDE	4	1	2
44	4^12^{4-1}	I= bCDE	4	1	4
45	4^12^{4-1}	I= bCDE	3	2	4
46	4^12^{4-1}	I= bcDE	3	2	2
47	4^12^{5-2}	I= $a_1BCDE = a_2CDF$	5	1	19
48	4^12^{5-2}	I= $A_1bCDE = A_2CDF$	5	1	44
49	4^12^{5-2}	I= $A_1BCDE = A_2CDf$	5	1	21
50	4^12^{5-2}	I= $a_1BCDE = a_2CDf$	4	2	6
51	4^12^{5-2}	I= $A_1bCDe = A_2CDF$	4	2	16
52	4^12^{5-2}	I= $A_1bCDE = A_2CDf$	4	2	23
53	4^12^{5-2}	I= $a_1bCDe = a_2CDF$	3	3	4
54	4^12^{5-2}	I= $a_1bCDE = a_2CDf$	3	3	3
55	4^12^{5-2}	I= $A_1bcDE = A_2cDf$	3	3	9
56	4^12^{5-2}	I= $A_1bCDe = A_2CDf$	3	3	6
57	4^12^{5-2}	I= $a_1BCDE = a_3BCDF$	5	1	2
58	4^12^{5-2}	I= $A_1bCDE = A_3bCDF$	5	1	2
59	4^12^{5-2}	I= $A_1BCDe = A_3BCDF$	5	1	3
60	4^12^{5-2}	I= $a_1BCDe= a_3BCDF$	4	2	1
61	4^12^{5-2}	I= $A_1bCDe = A_3bCDF$	4	2	3
62	4^12^{5-2}	I= $A_1BCDe = A_3BCDf$	4	2	2
63	4^12^{5-2}	I= $a_1bcDE = a_3bcDF$	3	3	2
64	4^12^{5-2}	I= $a_1bCDe= a_3bCDF$	3	3	1
65	4^12^{5-2}	I= $A_1bcdE = A_3bcdF$	3	3	2
66	4^12^{5-2}	I= $A_1bCDe = A_3bCDf$	3	3	2
67	4^12^{5-2}	I= $a_2BCE = BCDF$	5	1	30
68	4^12^{5-2}	I= $A_2BCe = BCDF$	5	1	36
69	4^12^{5-2}	I= $a_2bCE = bCDF$	4	2	49
70	4^12^{5-2}	I= $a_2BCe = BCDF$	4	2	24
71	4^12^{5-2}	I= $A_2bCe = bCDF$	4	2	53
72	4^12^{5-2}	I= $a_2bCE = bCdF$	3	3	10
73	4^12^{5-2}	I= $A_2bcE = bcdF$	3	3	22
74	4^12^{5-2}	I= $A_2bCe = bCdF$	3	3	10
75	4^12^{6-3}	I= $a_1BCDE = a_2BCDF =a_3BCDG$	6	1	1
76	4^12^{6-3}	I= $A_1bCDE = A_2bCDF =A_3bCDG$	6	1	1
77	4^12^{6-3}	I= $A_1BCDe = A_2BCDF =A_3BCDG$	6	1	2
78	4^12^{6-3}	I= $a_1bCDE = a_2bCDF =a_3bCDG$	5	2	1
79	4^12^{6-3}	I= $A_1bCDe = A_2bCDF =A_3bCDG$	5	2	2

80	$4^1 2^{6-3}$	I= $a_1$bcDE = $a_2$bcDF =$a_3$bcDG	4	3	1
81	$4^1 2^{6-3}$	I= a_1bCDe = a_2bCDF =a_3bCDG	4	3	1
82	$4^1 2^{6-3}$	I= A_1bcdE = A_2bcdF =A_3bcdG	4	3	1
83	$4^1 2^{6-3}$	I= A_1bcDe= A_2bcDF =A_3bcDG	4	3	2
84	$4^1 2^{6-3}$	I= a_1BCDE = a_2BCDF =BCDG	6	1	7
85	$4^1 2^{6-3}$	I= A_1BCDE = A_2BCDF =BCDg	6	1	8
86	$4^1 2^{6-3}$	I= a_1BCDe = a_2BCDF =BCDG	5	2	7
87	$4^1 2^{6-3}$	I= A_1BcdE = A_2BcdF =BcdG	5	2	5
88	$4^1 2^{6-3}$	I= $a_1$bcDE = $a_2$bcDF =bcDG	4	3	5
89	$4^1 2^{6-3}$	I= A_1BCDe = A_2BCDf =BCDg	4	3	7
90	$4^1 2^{6-3}$	I= a_2BCE = a_3CDF =a_1BDG	6	1	92
91	$4^1 2^{6-3}$	I= A_2BCe = A_3CDF =A_1BDG	6	1	251
92	$4^1 2^{6-3}$	I= a_2bCE = a_3CDF =a_1bDG	5	2	69
93	$4^1 2^{6-3}$	I= A_2bcE = A_3cDF =A_1bDG	5	2	136
94	$4^1 2^{6-3}$	I= $a_2$bcE = a_3cDF =a_1bDG	4	3	10
95	$4^1 2^{6-3}$	I= A_2bcE = A_3cdF =A_1bdG	4	3	16
96	$4^1 2^{6-3}$	I= a_2BCE = a_3CDF=BCDG	6	1	392
97	$4^1 2^{6-3}$	I= a_2bCE = a_3CDF=bCDG	5	2	438
98	$4^1 2^{6-3}$	I= A_2bCE = A_3CdF =bCdG	5	2	446
99	$4^1 2^{6-3}$	I= $a_2$bcE = a_3cDF=bcDG	4	3	86
100	$4^1 2^{6-3}$	I= A_2bCe= A_3CdF =bCdG	4	3	264