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Beta-Weibull Distribution: Some Properties and Applications to Censored Data

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Some properties of a four-parameter beta-Weibull distribution are discussed. The beta-Weibull distribution is shown to have bathtub, unimodal, increasing, and decreasing hazard functions. The distribution is applied to censored data sets on bus-motor failures and a censored data set on head-and-neck-cancer clinical trial. A simulation is conducted to compare the beta-Weibull distribution with the exponentiated Weibull distribution.

Key words: Bathtub, unimodal, censored data, bootstrap.

Introduction

Let $F(x)$ be the cumulative distribution function of a Weibull random variable *X*. Famoye, Lee, and Olumolade (2005) defined the cumulative distribution function for beta-Weibull random variable as

$$G(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{F(x)} t^{\alpha-1}\left(1-t\right)^{\beta-1}dt,\ \ 0<\alpha,\ \beta<\infty. \tag{1}$$

From (1), the corresponding probability density function for the beta-Weibull distribution is given by

$$g(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{c}{\gamma}\left(\frac{x}{\gamma}\right)^{c-1}\left[1-e^{-(x/\gamma)^c}\right]^{\alpha-1}e^{-\beta(x/\gamma)^c} \tag{2}$$

for $x > 0$, $\alpha, \beta, c, \gamma > 0$. One may introduce a location parameter ξ in the density in (2) by replacing x with $x-\xi$ where $-\infty<\xi<\infty$. In the rest of this article, take ξ to be zero.

The Weibull distribution has wide applications in many disciplines. See, e.g., Hallinan (1993), Johnson, Kotz, and Balakrishnan (1994). Various extensions have appeared in the literature. For instance, Zacks (1984) introduced the Weibull-exponential distribution. Mudholkar and Kollia (1994) defined a generalized Weibull distribution by introducing an additional shape parameter. Mudholkar, Srivastava, and Kollia (1996) applied the generalized Weibull distribution to model survival data. They showed that the distribution has increasing, decreasing, bathtub, and unimodal hazard functions.

Mudholkar, Srivastava, and Freimer (1995), Mudholkar and Hutson (1996) and Nassar and Eissa (2003) studied various properties of the exponentiated Weibull distribution. Mudholkar et al. (1995) applied exponentiated Weibull distribution to model failure data. Mudholkar and Hutson (1996) applied exponentiated Weibull distribution to extreme value data. They showed that exponentiated Weibull distribution has increasing, decreasing, bathtub, and unimodal hazard rates. The exponentiated exponential distribution proposed by Gupta and Kundu (1999, 2001) is a special case of the exponentiated Weibull family.

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Recently, Famoye et al. (2005) introduced a four-parameter beta-Weibull distribution. They showed that the beta-Weibull distribution is unimodal and obtained some results on the non-central moments. The maximum likelihood technique was used for parameter estimation and a likelihood ratio test was derived for the beta-Weibull distribution. The exponentiated Weibull distribution, Rayleigh distribution (Johnson et al., 1994, p. 686), the Type 2 extreme value distribution (Johnson, Kotz and Balakrishnan, 1995, p. 3), Burr Type (X) distribution (Johnson et al., 1994, p. 54), and the distribution of the order statistic from a Weibull population are special cases of the beta-Weibull distribution (Famoye et al., 2005).

In this article, the hazard function and entropy of the beta-Weibull distribution is examined. It is applied to several failure rate data and survival data. Some properties of the beta-Weibull model are discussed and the shapes of the hazard function are provided. Application of the beta-Weibull distribution to censored data sets is presented. Finally, the results of a simulation study are presented. The simulation study compares the beta-Weibull distribution with the exponentiated Weibull distribution.

Some Properties of Beta-Weibull Distribution

The survival function is given by $S(x) = 1 - G(x)$. The hazard function (or failure rate) of beta-Weibull distribution is given by

$$h(x) = \frac{g(x)}{1-G(x)} = \frac{g(x)}{S(x)}, \tag{3}$$

where $G(x)$ and $g(x)$ are given by (1) and (2) respectively and $S(x)$ is the survival function.

Theorem 1: The limit of beta-Weibull hazard function as $x \to 0$ is

$$\lim_{x\to 0} h(x) = \begin{cases} \infty, & \text{when } \alpha c < 1 \\ \dfrac{c\Gamma(\alpha+\beta)}{\gamma\,\Gamma(\alpha)\Gamma(\beta)}, & \text{when } \alpha c = 1 \\ 0, & \text{when } \alpha c > 1. \end{cases} \tag{4}$$

and the limit of beta-Weibull hazard function as $x \to \infty$ is given by

$$\lim_{x\to \infty} h(x) = \begin{cases} \infty, & \text{when } c > 1 \\ \dfrac{\beta}{\gamma}, & \text{when } c = 1 \\ 0, & \text{when } c < 1. \end{cases} \tag{5}$$

Proof: When $x \to 0$, the limit of $h(x)$ is the same as the limit of $g(x)$. Famoye et al. (2005) obtained the limit in (4) for $g(x)$. When $x \to \infty$, the beta-Weibull hazard function in (3) is indeterminate as both numerator and denominator become 0. By using L'Hôpital's rule, the limit of $h(x)$ as $x \to \infty$ is given by (5). This completes the proof.

Theorem 2: The beta-Weibull distribution has

(a) a constant ($= \beta/\gamma$) failure rate when $\alpha = c = 1$,
(b) a decreasing failure rate when $\alpha c \leq 1$ and $c \leq 1$,
(c) an increasing failure rate when $\alpha c \geq 1$ and $c \geq 1$,
(d) a bathtub failure rate when $\alpha c < 1$ and $c > 1$, and
(e) upside down bathtub (or unimodal) failure rate when $\alpha c > 1$ and $c < 1$.

Proof: It follows from Theorem 1.

Glaser (1980) gave sufficient conditions to characterize a given failure rate distribution as being bathtub shaped (BT), increasing failure rate (IFR), upside-down bathtub (UBT), or decreasing failure rate (DFR). Glaser defined the quantity $\eta(t) = -g'(t)/g(t)$ where $g(t)$ is the probability density function and gave a list of conditions to characterize a given failure rate based on $\eta'(t)$. It is not difficult to show that the beta-Weibull distribution satisfies all the conditions given by Glasser (1980). In Figure 1, the various shapes for the beta-Weibull hazard functions are provided.

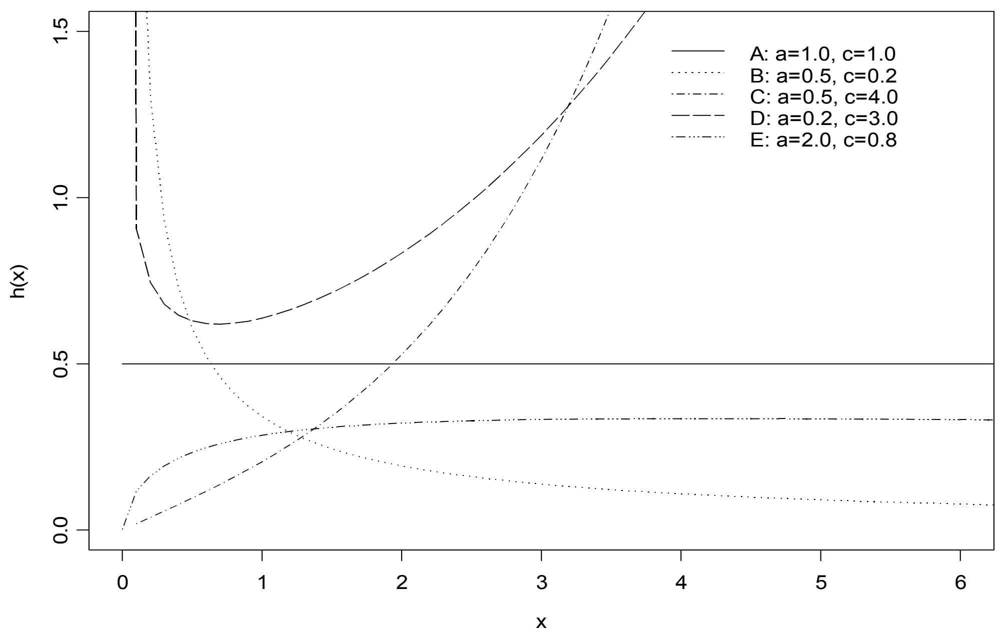

Figure 1: Beta-Weibull hazard functions for β = 2.0, γ = 4.0 and various values of α = a and c

Entropies

Entropy has been used in various situations in science and engineering. Numerous entropy measures have been studied and compared in the literature. See the recent work of Nadarajah and Zografos (2005) and the references therein. Nadarajah and Zografos (2003) derived formulas for Renyi and Shannon entropies for 26 continuous univariate distributions.

The entropy of a random variable X with density $g(x)$ is a measure of variation of the uncertainty. Renyi entropy is defined by

$$I_R(\rho) = \frac{1}{1-\rho} \log\left\{ \int g^{\rho}(x)dx \right\}, \qquad (6)$$

where $\rho > 0$ and $\rho \neq 1$, Renyi (1961). For the beta-Weibull density see equation 7. By using the substitution $t = \left(x/\gamma\right)^c$ and simplifying the resulting quantity, equation 8 is obtained. Hence,

$$\int_0^\infty g^\rho(x)dx = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{c}{\gamma}\right]^\rho \int_0^\infty \left(x/\gamma\right)^{\rho(c-1)} \left[1-\exp[-(x/\gamma)^c]\right]^{\rho(\alpha-1)} \exp[-\rho\beta(x/\gamma)^c]dx \tag{7}$$

$$\int_0^\infty g^\rho(x)dx = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right]^\rho \left(\frac{c}{\gamma}\right)^{\rho-1} \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma\left(\rho(\alpha-1)+1\right)\Gamma\left(\rho(1-1/c)+1/c\right)}{k!\Gamma\left(\rho(\alpha-1)-k+1\right)\left(k+\beta\rho\right)^{\rho(1-1/c)+1/c}} \tag{8}$$

$$I_R(\rho) = -\log(c/\gamma) + \frac{\rho}{1-\rho}\log\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right) + \frac{1}{1-\rho}\log\left\{\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(\rho(\alpha-1)+1\right)\Gamma\left(\rho(1-1/c)+1/c\right)}{k!\Gamma\left(\rho(\alpha-1)-k+1\right)\left(k+\beta\rho\right)^{\rho(1-1/c)+1/c}}\right\} \tag{9}$$

$$\begin{aligned} \mathrm{E}\left[-\log\left(g(x)\right)\right] &= -\log\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right) - \left[(\alpha-1)\psi(\alpha) + (1-1/c)\Gamma'(1)\right] \\ &+ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k=0}^{\infty}\binom{\alpha-1}{k}\frac{(-1)^k}{k+\beta}\left[(\alpha-1)\psi(\alpha-k) + (1-1/c)\log(k+\beta) + \beta/(k+\beta)\right] \end{aligned} \tag{10}$$

Renyi entropy for the beta-Weibull density is given by equation 9.

The Shannon's (1948) entropy is defined as $\mathrm{E}\left[-\log\left(g(x)\right)\right]$. This is a special case of (6) when $\rho \to 1$. Hence, the Shannon entropy is obtained by taking the limit of (9) as $\rho \to 1$. On taking the limit of (9) as $\rho \to 1$, 0/0 is obtained and hence, the L'Hopital's rule is applied. After using this rule and simplifying, equation 10 is obtained, where $\psi(\cdot)$ is the digamma function and $\Gamma'(\cdot)$ is the derivative of the gamma function.

Applications of beta-Weibull distribution to censored data

In survival analysis, the data may be in grouped form or in ungrouped form and quite often, the data involve censoring. In the case of grouped data, the right censoring is in the form of a last open interval as provided in Tables 1 and 2. Suppose a grouped data consisting of k intervals and the j^{th} interval $\left(I_{j-1}, I_j\right)$ contains n_j observations for j = 1, 2, 3, …, k–1. The boundary I_0 is equal to 0 and the k^{th} interval

$\left(I_{j-1},\ \infty\right)$ has n_k observations. The total number of observations is $n = \sum_{j=1}^{k} n_j$. By using the result in Lawless (1982), the log-likelihood function for the grouped data is

$$\begin{aligned} &\ell(\alpha,\beta,c,\gamma) \\ &= \sum_{j=1}^{k-1} n_j \log\left[S(I_{j-1}) - S(I_j)\right], \\ &+ n_k \log\left[S(I_{k-1})\right] \end{aligned} \tag{11}$$

where $S(.)$ is the beta-Weibull survival function. Estimates of the parameters are obtained by maximizing (11), the logarithm of the censored likelihood function.

The log-likelihood function for the ungrouped data x_j, j = 1, 2, 3, …, n is given by

$$\ell(\alpha,\beta,c,\gamma) = \sum\nolimits_u \log\left[h(x_j)\right] + \sum_{j=1}^{n} \log\left[S(x_j)\right], \tag{12}$$

where $h(.)$ is the beta-Weibull hazard function given by (3) and Σ_u denotes the summation over the uncensored observations. Estimates of the parameters are obtained by maximizing (12), the log-likelihood function. Both the log-likelihood functions in (11) and (12) are maximized directly by using *nlminb*, an SPLUS non-linear optimization routine with bounds. Taking the first and second partial derivatives of (11) and (12) with respect to the model parameters are quite involving. Hence, the Bootstrap method is used, Efron (1981), to estimate the standard errors of the parameter estimates for the beta-Weibull distribution.

Mudholkar et al. (1995) re-analyzed the classical bus-motor-failure data, first considered by Davis (1952), for a fleet of 191 buses. Mudholkar et al. (1995) re-analyzed the first, second, third, fourth, and fifth motor failures. They found that only the exponentiated Weibull provides a good fit to the first two data sets. However, the exponential, the Weibull, and the exponentiated Weibull provide good fits to the last three data sets. In this article, the beta Weibull is applied to all data sets and it provides excellent fits to all. However, the result for the first and the second motor failures are presented in Tables 1 and 2.

The beta-Weibull parameter estimates (standard errors in parentheses) in Table 1 are as follows: $\hat{\alpha}$ = 0.3707(.0610), $\hat{\beta}$ = 0.1256(.0189), $\hat{c}$ = 4.5753(.1853), $\hat{\gamma}$ = 76.2155(1.5219). The beta-Weibull model has an increasing hazard rate for these parameter estimates because $\hat{\alpha}\hat{c} > 1$ and $\hat{c} > 1$.

The beta-Weibull maximum likelihood estimates (standard errors in parentheses) in Table 2 are as follows: $\hat{\alpha}$ = 0.1479(0.0634), $\hat{\beta}$ = 0.1757(0.0821), $\hat{c}$ = 5.5104(1.3385), $\hat{\gamma}$ = 81.4003(5.6775). The beta-Weibull model has a bathtub hazard rate for these parameter estimates because $\hat{\alpha}\hat{c} < 1$ and $\hat{c} > 1$.

The exponentiated Weibull and beta-Weibull distributions provided adequate fits to the two data sets, but the fit from beta-Weibull distribution is better by using the chi-square goodness of fit measure. Also, the expected frequencies from the beta-Weibull model are much closer to the observed frequencies than the corresponding results from exponentiated Weibull model. In particular, it is noticed that only Beta-Weibull identifies that the failure rate has a bathtub shape, which logically fits the failure rate of motors well as shown in the above data. The last class (120,000 miles and up) had lower occurrence because the data is right-censored.

Table 1. Re-analysis of the First Bus-Motor Failure

Class interval (1,000 miles)	Observed frequency	Expected frequency Weibull	Expected frequency Exponentiated Weibull	Expected frequency Beta Weibull
0 – 20	6	1.4066	3.8965	5.2925
20 – 40	11	8.9031	11.7722	11.8987
40 – 60	16	21.2228	19.6848	17.4895
60 – 80	25	33.5374	27.4955	24.2573
80 – 100	34	39.8566	34.5251	34.1451
100 – 120	46	36.7799	38.3690	42.5039
120 – 140	33	26.3822	33.8352	35.5682
140 – 160	16	14.5357	18.0184	16.2516
160 – up	4	8.3757	3.4034	3.5932
Total	191	191.0	191.0	191.0
Pearson χ^2		26.218	3.979	0.836
df		6	5	4
p-value		0.0002	0.5524	0.9336
Log-likelihood		–389.936	–381.811	–380.335

Table 2. Re-analysis of the Second Bus-Motor Failure

Class interval (1,000 miles)	Observed frequency	Expected frequency Weibull	Expected frequency Exponentiated Weibull	Expected frequency Beta Weibull
0 – 20	19	13.3474	16.7866	18.6316
20 – 40	13	19.4117	15.8037	14.1624
40 – 60	13	18.7796	15.4234	13.0820
60 – 80	15	15.7765	15.1924	13.4357
80 – 100	15	12.1399	15.0160	16.0268
100 – 120	18	8.7520	14.6341	17.5898
120 – up	11	15.7929	11.1438	11.0717
Total	104	104.0	104.0	104.0
Pearson χ^2		18.2291	1.9485	0.3611
df		4	3	2
p-value		0.0011	0.5832	0.8348
Log-likelihood		–208.872	–201.707	–200.918

Mudholkar et al. (1995) applied the exponentiated Weibull distribution to model Efron's (1988) Arm A data on the survival times of 51 head-and-neck cancer patients given in Table 3. The beta-Weibull model was applied to fit the data in Table 3 and the result were grouped into 13 classes as in Table 12 of Mudholkar et al. (1995). For more details about the data, see Mudholkar et al. (1995). The results of the analysis and that of Mudholkar et al. (1995) are presented in Table 4. The fits from both exponentiated Weibull and beta-Weibull distributions are very similar for the data. It appears the exponentiated Weibull distribution is slightly better because it has only three parameters compared to the beta-Weibull distribution with four parameters. A likelihood ratio test can be applied to test the adequacy of beta-Weibull distribution against a reduced special case (Famoye et al. 2005).

Table 3. Survival Times (in days) for the Patients in Arm A of the Head-and-Neck-Cancer Trial

7, 34, 42, 63, 64, 74+, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 173, 176, 185+, 218, 225, 241, 248, 273, 277, 279+, 297, 319+, 405, 417, 420, 440, 523, 523+, 583, 594, 1101, 1116+, 1146, 1226+, 1349+, 1412+, 1417.

Note. Data is from Efron (1988); + indicates observations lost to follow-up.

Table 4. Re-analysis of Arm A of the Head-and-Neck-Cancer Trial

j^{th} class interval (in months)	N_j	S_j	Expected Deaths (E_j) Weibull	Exponentiated Weibull	Beta Weibull
0 – 1	51	1	4.2739	1.8814	1.8374
1 – 2	50	2	3.8787	4.2669	4.2335
2 – 3	48	5	3.5922	4.6938	4.6845
3 – 4	42	2	3.0697	4.1702	4.1676
4 – 6	72	15	5.1380	6.8828	6.8742
6 – 8	49	3	4.4120	4.3158	4.3023
8 – 11	56	4	3.8190	4.4572	4.4353
11 – 14	45	3	3.0079	3.1773	3.1583
14 – 18	45	2	2.9567	2.8248	2.8091
18 – 24	46	2	2.9666	2.5099	2.5019
24 – 31	49	0	3.0988	2.2784	2.2833
31 – 38	47	2	2.9258	1.9072	1.9241
38 – 47	28	1	1.7189	1.0029	1.0197
$\sum_{j=1}^{13} R_j^2$			27.930	17.490	17.410
Approx. df			11	10	9
p-value			0.0033	0.0642	0.0427

Note. $R_j = \sqrt{2}\,\text{sign}\left(S_j - E_j\right)\left[S_j \log(S_j / E_j) + (N_j - S_j)\log\left(\dfrac{N_j - S_j}{N_j - E_j}\right)\right]^{1/2}$.

The parameter estimates (standard errors in parentheses) from beta-Weibull model are as follows: $\hat{\alpha}$ = 11.2139(3.3705), $\hat{\beta}$ = 0.5874(0.1791), $\hat{c}$ = 0.3859(0.0622), $\hat{\gamma}$ = 0.2947(0.1451). The estimates show that the model has a unimodal hazard function because $\hat{\alpha}\hat{c} > 1$ and $\hat{c} < 1$.

Comparison between beta-Weibull and exponentiated Weibull distributions

In the previous section, the fits from both beta-Weibull and exponentiated Weibull distributions are very close. In this section, a simulation is conducted to compare these two distributions. The parameters are estimated by the method of maximum likelihood. Samples of sizes n = 250, 500, and 1000 were generated from beta-Weibull and exponentiated Weibull distributions. The parameter sets for which the beta-Weibull hazard function is bathtub (Table 5), unimodal (Table 6), increasing (Table 7), and decreasing (Table 8) are simulated. For each simulated sample, the likelihood ratio test proposed by Famoye et al. (2005) is applied to compare the beta-Weibull and exponentiated Weibull distributions. In each case, there is no significance difference between the two models. The biases were examined (actual parameter value minus the estimated value) and the standard errors of the maximum likelihood estimates. These biases and the standard errors tell a different story.

For each sample size, 100 different samples were generated in order to obtain 100 parameter estimates which are used to compute the biases and the standard errors. The biases and the standard errors of the maximum likelihood estimates (mle) are reported in Tables 5 through 8. When the parameter β = 1, the simulated data is considered to be from the exponentiated Weibull distribution. The following are some observations from the simulation study.

a. For the parameter set of a bathtub hazard function (Table 5):

a.1 When $\beta < 1$, the biases of the mle from beta-Weibull estimates are smaller than the corresponding biases from the exponentiated Weibull distribution. The standard errors of the mle of α and c for the two distributions are comparable, while the standard errors of the mle of γ are larger for beta-Weibull distribution.

a.2 When β = 1, the biases and standard errors of the mle of α and c for the two distributions are comparable. When comparing the mle of γ, the beta-Weibull distribution seems to have larger bias and standard error.

a.3 When $\beta > 1$, the biases and standard errors of the mle for beta-Weibull distribution seem to be larger than the biases and standard errors of the mle for exponentiated Weibull distribution.

b. For the parameter set of a unimodal hazard function (Table 6):

b.1 When $\beta < 1$, similar results as in (a.1) are observed.

b.2 When β = 1, similar results as in (a.2) are observed.

b.3 When $\beta > 1$, the biases of the mle of α and c are larger for beta-Weibull, while the standard errors of the mle of α and c for the two distributions are comparable. The mle of γ have comparable biases for the two distributions. The mle of γ have larger standard errors for the beta-Weibull distribution.

c. For the parameter set of an increasing hazard function (Table 7):

c.1 When $\beta < 1$, similar results as in (a.1) are observed.

c.2 When β = 1, the biases and standard errors of the mle of α for beta-Weibull are smaller than the biases and standard errors of the mle from exponentiated Weibull. The biases of the mle of c are larger for beta-Weibull but the standard errors are comparable for the two distributions. Both biases and standard errors of the mle of γ are larger for beta-Weibull.

c.3 When $\beta > 1$, the biases and standard errors of the mle of α for the two distributions are comparable. The biases of the mle of c are slightly larger for beta-Weibull but the standard errors are comparable for the two distributions. The estimates of γ have larger biases and standard errors for the beta-Weibull.

d. For the parameter set of a decreasing hazard function (Table 8):

d.1 When $\beta < 1$, both biases and standard errors of the mle of α, c and γ are smaller for beta-Weibull.

d.2 When $\beta = 1$, similar results as in (c.2) are observed.

d.3 When $\beta > 1$, the biases and standard errors of the mle of α and c for the two distributions are comparable. The estimates of γ have comparable biases with larger standard errors for the beta-Weibull.

Conclusion

The biases of the mle from beta-Weibull distribution are smaller than the biases of the mle from exponentiated Weibull model with comparable standard errors when $\beta < 1$. The biases and standard errors are, in general, smaller for the exponentiated Weibull distribution when $\beta \geq 1$. In all the three examples in previous section, the estimates for parameter β are less than 1.0 and thus, this simulation study supports the use of the beta-Weibull distribution for describing the data sets. In addition, another implication of the simulation results is that one can take the advantage of the Beta-Weibull distribution and the exponentiated Weibull distribution by using the Beta-Weibull distribution and setting up the upper bound of parameter estimate of β to be one.

Table 5: Bias (standard error) of parameter estimate for $\alpha = 0.5$, $c = 1.5$ and various values of β and γ (bathtub hazard function)

Actual values			Exponentiated Weibull distribution		
β	γ	n	$\hat{\alpha}$	$\hat{c}$	$\hat{\gamma}$
0.5	2	250	.006 (.097)	-.093 (.224)	-1.457 (0.397)
		500	.036 (.067)	-.151 (.164)	-1.577 (0.262)
		1000	.036 (.048)	-.143 (.118)	-1.597 (0.217)
0.5	4	250	.005 (.097)	-.093 (.224)	-2.909 (0.793)
		500	.037 (.067)	-.155 (.168)	-3.158 (0.528)
		1000	.036 (.047)	-.144 (.119)	-3.192 (0.431)
1.0	2	250	-.030 (.147)	-.036 (.317)	.021 (0.364)
		500	-.009 (.105)	-.035 (.229)	-.003 (0.270)
		1000	.000 (.058)	-.023 (.133)	-.010 (0.157)
1.0	4	250	-.031 (.147)	-.038 (.318)	.046 (0.722)
		500	-.011 (.105)	-.030 (.228)	.004 (0.537)
		1000	.000 (.058)	-.023 (.133)	-.021 (0.314)
2.0	2	250	-.017 (.137)	-.024 (.289)	.821 (0.200)
		500	-.012 (.101)	.000 (.200)	.815 (0.149)
		1000	-.023 (.070)	.042 (.133)	.839 (0.102)
2.0	4	250	-.020 (.134)	-.016 (.278)	1.658 (0.378)
		500	-.012 (.100)	.001 (.198)	1.631 (0.294)
		1000	-.023 (.070)	.042 (.133)	1.677 (0.204)

Actual values			Beta-Weibull distribution			
β	γ	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{\gamma}$
0.5	2	250	-.030 (.094)	-.104 (.253)	.032(.210)	-.171 (0.897)
		500	-.005 (.068)	-.066 (.237)	-.002 (.163)	-.143 (0.853)
		1000	-.013 (.051)	-.018 (.232)	.026 (.123)	.024 (0.825)
0.5	4	250	-.030 (.094)	-.106 (.250)	.030 (.210)	-.360 (1.778)
		500	-.004 (.069)	-.069 (.239)	-.007 (.170)	.311 (1.728)
		1000	-.013 (.052)	-.018 (.231)	.025 (.123)	.041 (1.641)
1.0	2	250	-.049 (.145)	.185 (.395)	.033 (.291)	.392 (0.731)
		500	-.035 (.107)	.261 (.385)	.053 (.227)	.508 (0.749)
		1000	-.027 (.065)	.245 (.378)	.061 (.157)	.489 (0.710)
1.0	4	250	-.049 (.143)	.163 (.409)	.028 (.292)	.717 (1.481)
		500	-.037 (.106)	.253 (.405)	.058 (.224)	1.003 (1.539)
		1000	-.027 (.065)	.245 (.378)	.061 (.157)	.977 (1.421)
2.0	2	250	-.036 (.132)	1.126 (.497)	.047 (.255)	.999 (0.496)
		500	-.040 (.100)	1.261 (.384)	.086 (.198)	1.127 (0.429)
		1000	-.051 (.075)	1.203 (.509)	.119 (.154)	1.098 (0.516)
2.0	4	250	-.038 (.133)	1.124 (.490)	.047 (.259)	1.994 (0.992)
		500	-.039 (.099)	1.240 (.415)	.085 (.197)	2.217 (0.902)
		1000	-.052 (.073)	1.224 (.486)	.123 (.148)	2.237 (0.990)

Table 6: Bias (standard error) of parameter estimate for $\alpha = 1.5$, $c = 0.75$ and various values of β and γ (unimodal hazard function)

Actual values			Exponentiated Weibull distribution		
β	γ	n	$\hat{\alpha}$	$\hat{c}$	$\hat{\gamma}$
0.5	2	250	-.562 (.449)	.119 (.056)	-1.080 (0.399)
		500	-.450 (.303)	.103 (.043)	-1.277 (0.555)
		1000	-.343 (.197)	.090 (.029)	-1.496 (0.402)
0.5	4	250	-.594 (.511)	.121 (.059)	-2.090 (1.457)
		500	-.446 (.303)	.103 (.043)	-2.572 (1.110)
		1000	-.337 (.198)	.089 (.029)	-3.016 (0.800)
1.0	2	250	-.052 (.458)	-.022 (.120)	-.077 (0.680)
		500	-.028 (.367)	-.010 (.088)	-.046 (0.509)
		1000	-.045 (.266)	.002 (.066)	.019 (0.388)
1.0	4	250	-.018 (.431)	-.028 (.116)	-.243 (1.310)
		500	-.022 (.363)	-.012 (.087)	-.108 (1.006)
		1000	-.022 (.250)	-.003 (.064)	-.018 (0.747)
2.0	2	250	-.058 (.532)	-.035 (.137)	1.085 (0.328)
		500	.004 (.338)	-.032 (.099)	1.090 (0.238)
		1000	.061 (.213)	-.035 (.066)	1.077 (0.159)
2.0	4	250	-.055 (.523)	-.030 (.131)	2.172 (0.633)
		500	.015 (.335)	-.035 (.098)	2.164 (0.472)
		1000	.061 (.204)	-.035 (.065)	2.151 (0.308)

Actual values			Beta-Weibull distribution			
β	γ	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{\gamma}$
0.5	2	250	-.412 (.399)	-.107 (.221)	.072 (.065)	.153 (0.765)
		500	-.305 (.269)	-.071 (.225)	-.052 (.052)	.147 (0.791)
		1000	-.209 (.177)	-.047 (.222)	.039 (.042)	.116 (0.782)
0.5	4	250	-.439 (.436)	-.115 (.216)	.075 (.066)	.291 (1.519)
		500	-.306 (.268)	-.086 (.221)	.055 (.051)	.191 (1.562)
		1000	-.207 (.180)	-.056 (.222)	.040 (.043)	.168 (1.565)
1.0	2	250	-.019 (.439)	.147 (.395)	-.038 (.116)	.227 (0.940)
		500	-.001 (.373)	.156 (.388)	-.026 (.088)	.281 (0.877)
		1000	-.006 (.247)	.163 (.409)	-.018 (.066)	.320 (0.887)
1.0	4	250	.005 (.426)	.139 (.393)	-.041 (.115)	.392 (1.833)
		500	.004 (.373)	.162 (.394)	-.027 (.089)	.580 (1.769)
		1000	.009 (.244)	.148 (.402)	-.020 (.067)	.561 (1.744)
2.0	2	250	-.041 (.558)	1.013 (.553)	-.047 (.135)	1.074 (0.644)
		500	.024 (.342)	.975 (.678)	-.045 (.098)	1.027 (0.749)
		1000	.071 (.224)	.995 (.531)	-.043 (.070)	1.055 (0.522)
2.0	4	250	-.048 (.575)	1.011 (.623)	-.041 (.133)	2.179 (1.321)
		500	.032 (.343)	.974 (.678)	-.047 (.099)	2.046 (1.495)
		1000	.076 (.214)	1.025 (.538)	-.046 (.068)	2.159 (1.066)

Table 7: Bias (standard error) of parameter estimate for $\alpha = 1.5$, $c = 1.5$ and various values of β and γ (increasing hazard function)

Actual values			Exponentiated Weibull distribution		
β	γ	n	$\hat{\alpha}$	$\hat{c}$	$\hat{\gamma}$
0.5	2	250	-.149 (.451)	.042 (.219)	-.926 (0.494)
		500	-.146 (.449)	.067 (.190)	-.908 (0.444)
		1000	-.120 (.274)	.076 (.120)	-.897 (0.293)
0.5	4	250	-.160 (.507)	.039 (.225)	-1.861 (1.037)
		500	-.087 (.421)	.038 (.185)	-1.942 (0.849)
		1000	-.083 (.261)	.063 (.114)	-1.861 (0.562)
1.0	2	250	-.054 (.621)	-.064 (.284)	-.034 (0.386)
		500	-.046 (.401)	-.017 (.187)	.003 (0.274)
		1000	-.034 (.274)	-.005 (.137)	.009 (0.206)
1.0	4	250	-.047 (.620)	-.068 (.284)	-.081 (0.770)
		500	-.007 (.375)	-.037 (.181)	-.047 (0.521)
		1000	-.006 (.261)	-.017 (.134)	-.024 (0.396)
2.0	2	250	-.054 (.530)	-.065 (.266)	.667 (0.243)
		500	.011 (.330)	-.066 (.194)	.659 (0.170)
		1000	.058 (.206)	-.068 (.130)	.646 (0.112)
2.0	4	250	-.038 (.526)	-.074 (.271)	1.321 (0.488)
		500	.026 (.330)	-.074 (.194)	1.303 (0.343)
		1000	.068 (.208)	-.075 (.130)	1.281 (0.226)

Actual values			Beta-Weibull distribution			
β	γ	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{\gamma}$
0.5	2	250	-.056 (.385)	-.066 (.220)	-.031(.197)	-.125 (0.542)
		500	-.045 (.378)	-.030 (.214)	-.013 (.165)	-.032 (0.506)
		1000	-.029 (.229)	-.081 (.224)	-.002 (.103)	-.145 (0.507)
0.5	4	250	-.055 (.413)	-.045 (.222)	-.038 (.196)	-.178 (1.129)
		500	-.002 (.368)	-.031 (.215)	-.035 (.168)	-.135 (0.983)
		1000	-.001 (.228)	-.072 (.235)	-.013 (.107)	-.265 (1.028)
1.0	2	250	-.013 (.514)	.113 (.415)	-.090 (.266)	.127 (0.577)
		500	-.012 (.386)	.141 (.408)	-.049 (.179)	.197 (0.518)
		1000	.003 (.251)	.126 (.437)	-.041 (.135)	.184 (0.564)
1.0	4	250	-.009 (.515)	.114 (.416)	-.092 (.267)	.250 (1.150)
		500	.013 (.381)	.108 (.387)	-.059 (.181)	.272 (0.956)
		1000	.013 (.253)	.056 (.415)	-.037 (.139)	.170 (1.038)
2.0	2	250	-.036 (.558)	1.012 (.588)	-.091 (.262)	.714 (0.473)
		500	.030 (.337)	.983 (.683)	-.092 (.194)	.695 (0.482)
		1000	.078 (.207)	1.053 (.531)	-.094 (.132)	.721 (0.398)
2.0	4	250	-.037 (.581)	.962 (.624)	-.091 (.275)	1.356 (0.921)
		500	.038 (.342)	.948 (.671)	-.093 (.197)	1.321 (0.940)
		1000	.075 (.221)	.977 (.523)	-.089 (.139)	1.317 (0.739)

Table 8: Bias (standard error) of parameter estimate for $\alpha = 1.5$, $c = 0.5$ and various values of β and γ (decreasing hazard function)

Actual values			Exponentiated Weibull distribution		
β	γ	n	$\hat{\alpha}$	$\hat{c}$	$\hat{\gamma}$
0.5	2	250	-1.018 (.627)	.111 (.035)	-.644 (0.914)
		500	-.742 (.270)	.097 (.022)	-1.145 (0.634)
		1000	-.633 (.180)	.090 (.016)	-1.433 (0.469)
0.5	4	250	-1.017 (.627)	.112 (.035)	-1.286 (1.829)
		500	-.746 (.272)	.097 (.022)	-2.280 (1.269)
		1000	-.633 (.180)	.090 (.016)	-2.867 (0.937)
1.0	2	250	-.114 (.478)	-.002 (.071)	-.028 (0.896)
		500	-.043 (.363)	-.004 (.056)	-.067 (0.723)
		1000	.034 (.248)	.001 (.042)	-.004 (0.540)
1.0	4	250	-.095 (.467)	-.005 (.072)	-.130 (1.794)
		500	-.043 (.364)	-.004 (.056)	-.136 (1.446)
		1000	-.029 (.246)	-.000 (.042)	-.028 (1.076)
2.0	2	250	-.065 (.524)	-.020 (.089)	1.361 (0.333)
		500	.005 (.326)	-.020 (.062)	1.374 (0.232)
		1000	.063 (.206)	-.024 (.043)	1.366 (0.161)
2.0	4	250	-.054 (.522)	-.022 (.090)	2.711 (0.667)
		500	.019 (.318)	-.022 (.060)	2.737 (0.455)
		1000	.061 (.204)	-.023 (.043)	2.729 (0.320)

Actual values			Beta-Weibull distribution			
β	γ	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{\gamma}$
0.5	2	250	-.777 (.482)	-.125 (.226)	.076(.039)	.564 (0.883)
		500	-.549 (.247)	-.093 (.203)	-.061 (.030)	.473 (0.795)
		1000	-.472 (.165)	-.122 (.207)	.059 (.024)	.226 (0.879)
0.5	4	250	-.773 (.481)	-.120 (.227)	.076 (.039)	1.154 (1.774)
		500	-.555 (.251)	-.097 (.205)	.062 (.031)	.924 (1.598)
		1000	-.472 (.165)	-.122 (.207)	.059 (.024)	.452 (1.758)
1.0	2	250	-.086 (.483)	.164 (.370)	-.012 (.072)	.420 (1.088)
		500	-.008 (.368)	.205 (.357)	-.017 (.057)	.467 (1.055)
		1000	-.000 (.241)	.166 (.400)	-.012 (.044)	.345 (1.142)
1.0	4	250	-.071 (.478)	-.159 (.368)	-.015 (.073)	.772 (2.178)
		500	-.009 (.369)	.205 (.356)	-.017 (.057)	.933 (2.108)
		1000	.001 (.243)	.155 (.402)	-.011 (.045)	.633 (2.287)
2.0	2	250	-.047 (.552)	1.061 (.529)	-.029 (.088)	1.366 (0.607)
		500	.029 (.330)	1.059 (.563)	-.030 (.062)	1.342 (0.636)
		1000	.078 (.215)	1.028 (.541)	-.031 (.046)	1.302 (0.595)
2.0	4	250	-.051 (.576)	1.023 (.578)	-.028 (.091)	2.682 (1.236)
		500	.037 (.329)	1.040 (.566)	-.030 (.061)	2.631 (1.313)
		1000	.075 (.214)	1.023 (.536)	-.030 (.045)	2.597 (1.185)

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