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On the Product of Maxwell and Rice Random Variables

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The distributions of the product of independent random variables arise in many applied problems. These have been extensively studied by many researchers. In this paper, the exact distributions of the product $|XY|$ have been derived when X and Y are Maxwell and Rice random variables respectively, and are distributed independently of each other. The associated cdfs, pdfs, and kth moments have been given.

Key words: Maxwelll distribution, products, Rice distribution.

Introduction

The distributions of the product $|X\,Y|$, when X and Y are independent random variables, arise in many applied problems of biology, economics, engineering, genetics, hydrology, medicine, number theory, order statistics, physics, psychology, etc, (see, for example, Cigizoglu & Bayazit (2000), Galambos & Simonelli (2005), Grubel (1968), Ladekarl, et al. (1997), and Rokeach & Kliejunas (1972), among others, and references therein). The distributions of the product $|X\,Y|$, when X and Y are independent random variables and come from the same family, have been extensively studied by many researchers, (see, for example, Bhargava & Khatri (1981), Malik & Trudel (1986), Rathie & Rohrer (1987), Springer & Thompson (1970), Stuart (1962), and Wallgren (1980), among others,). In recent years, there has been a great interest in the study of the above kind when X and Y belong to different families, (see, for example, Nadarajah (2005), and Nadarajah & Kotz (2005), among others). In this paper, the distributions of the product $|X\,Y|$, when X and Y are independent random variables having Maxwell and Rice distributions respectively, have been investigated.

The derivation of the cdf, pdf, and kth moment of $Z = |X\,Y|$ involve some special functions, which are defined as follows, (see, for example, Abramowitz & Stegun, 1970, Gradshteyn & Ryzhik, 2000, and Prudnikov, et al., 1986, among others, for details). The series

$$
{}_pF_q\left(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; z\right) = \sum_{k=0}^{\infty}\left\{\frac{(\alpha_1)_k\,(\alpha_2)_k\cdots(\alpha_p)_k}{(\beta_1)_k\,(\beta_2)_k\cdots(\beta_q)_k}\,\frac{z^k}{k!}\right\},
$$

is called a generalized hypergeometric series of order (p, q), where $(\alpha)_k$ and $(\beta)_k$ represent Pochhammer symbols. For $p = 1$ and $q = 2$, we have generalized hypergeometric function ${}_1F_2$ of order $(1, 2)$, given by

$$
{}_1F_2(\alpha_1; \beta_1, \beta_2; z) = \sum_{k=0}^{\infty}\left\{\frac{(\alpha_1)_k}{(\beta_1)_k\,(\beta_2)_k}\,\frac{z^k}{k!}\right\}.
$$

The integral

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$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt\ ,\ \alpha > 0\,,$$

is defined as a (complete) gamma function, whereas the integrals

$$\gamma(\alpha,\, x) = \int_0^x t^{\alpha-1} e^{-t}\, dt,\, \alpha > 0\,,$$

and

$$\Gamma(\alpha,\, x) = \int_x^\infty t^{\alpha-1} e^{-t}\, dt,\, \alpha > 0\,,$$

are respectively known as incomplete gamma and complementary incomplete gamma functions. For negative values, gamma function can be defined as

$$\Gamma\left(-n+\frac{1}{2}\right) = \frac{(-1)^n\, 2^n \sqrt{\pi}}{1.3.5.\ldots(2n-1)}\text{, where } n \geq 0$$

is an integer (e.g., Andrews, et al., 1999, and Bohr & Mollerup, 1922).The error function is defined by

$$erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du\ ,$$

whereas the complementary error, $erfc(x)$, is defined as

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du = 1 - erf\,(x)\,.$$

The modified Bessel function of first kind, $I_\nu(x)$, for a real number ν, is defined by

$$I_\nu(x) = \left(\frac{1}{2}x\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{1}{4}\,x^2\right)^k}{(k!)\;\Gamma(\nu+k+1)}\,,$$

where $\Gamma(.)$ denotes gamma function. Also,

$$I_\nu(x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu e^{-x}\ {}_1F_1\left(\frac{1}{2}+\nu\,,1+2\nu\,;2\,x\right),$$

where ${}_1F_1$ denotes the confluent hypergeometric function. When $\nu = 0$, modified Bessel function of first kind, $I_0(x)$, of order 0 is obtained as follows:

$$I_0(x) = \sum_{k=0}^\infty \frac{\left(\frac{1}{4}\,x^2\right)^k}{(k!)^2}\ \ldots \qquad (1)$$

For

$$\text{Re}\left(\nu + \frac{1}{2}\right) > 0,\ \left|\arg\,(z)\right| < \frac{\pi}{2};$$

or

$$\text{Re}\ (z) = 0$$

and

$$\nu = 0\,,$$

the modified Bessel function of second kind of order ν is given by

$$K_\nu(x) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_1^\infty e^{-zt}\,(t^2-1)^{\nu-\frac{1}{2}}\, dt\,.$$

For

$$\left|\arg\,(z)\right| < \frac{\pi}{2},\ \text{Re}(z^2)\,,$$

one would have

$$K_\nu(x) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{e^{-t-\frac{z^2}{4t}}}{(t)^{\nu+1}}\, dt\,.$$

For non-integer ν ,

$$K_\nu(x) = \frac{\pi\,\{I_{-\nu}(x) - I_\nu(x)\}}{2\sin(\nu\pi)}.$$

The following Lemmas will also be needed in our calculations.

LEMMA 1 (Gradshteyn & Ryzhik (2000), Equation (3.381.4), Page 317). For $\mathrm{Re}(\mu) > 0$, and $\mathrm{Re}(\nu) > 0$,

$$\int_0^\infty t^{\nu-1} e^{-\mu t}\, dt = \frac{1}{\mu^\nu}\Gamma(\nu).$$

LEMMA 2 (Prudnikov et al. (1986), Volume 2, Equation (2.8.5.15), Page 106). For $a > 0$,

$$\int_0^\infty t^{\alpha-1} e^{-p/t^2}\operatorname{erf}(c\,t)\,dt$$
$$= \frac{c\,p^{(\alpha+1)/2}}{\sqrt{\pi}}\Gamma\left(-\frac{\alpha+1}{2}\right) {}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \frac{3+\alpha}{2}; c^2 p\right)$$
$$- \frac{1}{\sqrt{\pi}\,\alpha c^\alpha}\Gamma\left(\frac{\alpha+1}{2}\right) {}_1F_2\left(-\frac{\alpha}{2}; 1-\frac{\alpha}{2}, \frac{1-\alpha}{2}; c^2 p\right)$$

LEMMA 3 (Prudnikov et al., 1986, Volume 2, Equations 2.10.3.14, Page 151). For $\mathrm{Re}(\alpha) < 0$, $\mathrm{Re}(p) > 0$, $\mathrm{Re}(\nu) > 0$, and $\mathrm{Re}(c) > 0$,

$$\int_0^\infty x^{\alpha-1} e^{-p/x}\,\gamma(\nu, c\,x)\,dx$$
$$= \frac{c^\nu (p)^{\alpha+\nu}}{\nu}\Gamma(-\alpha-\nu)\, {}_1F_2(\nu; \nu+1, 1+\alpha+\nu; c\,p)$$
$$- \frac{\Gamma(\alpha+\nu)}{\alpha c^\alpha}\, {}_1F_2(-\alpha; 1-\alpha, 1-\alpha-\nu; c\,p)$$

where ${}_1F_2$ denotes generalized hypergeometric function of order $(1, 2)$, (see definition above).

LEMMA 4 (Gradshteyn & Ryzhik (2000), Equation (3.471.9), Page 340). For $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$,

$$\int_0^\infty x^{\nu-1} e^{-\frac{\beta}{x} - \gamma x}\, dx = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{\beta\gamma}\right)$$

where $K_\nu(.)$ denotes modified Bessel function of the second kind, (see definition above).

Distribution of the Product |XY|

Let X and Y be Maxwell and Rice random variables respectively, distributed independently of each other and defined as follows.

Maxwell Distribution:

A continuous random variable X is said to have a Maxwell distribution if its pdf $f_X(x)$ and cdf $F_X(x)$ are, respectively, given by

$$f_X(y) = \sqrt{\frac{2}{\pi}}\, a^{3/2} x^2 e^{-a x^2/2}, \; x > 0,\; a > 0 \;\ldots \quad (2)$$

and

$$F_X(x) = \frac{2\gamma\left(\frac{3}{2}, \frac{1}{2} a x^2\right)}{\sqrt{\pi}} \quad ,\ldots (3)$$
$$= erf\left(\sqrt{\frac{a}{2}}\, x\right) - \sqrt{\frac{2a}{\pi}}\, x\, e^{-a x^2/2}$$

where $\gamma(a, x)$ and $erf(x)$ denote incomplete gamma and error functions respectively, (see definition above).

Rice Distribution: A continuous random variable Y is said to have a Rice distribution if its pdf $f_Y(y)$ is given by

$$f_Y(y) = \frac{y}{\sigma^2} e^{-(y^2+\nu^2)/2\sigma^2} I_0\left(\frac{y\nu}{\sigma^2}\right), \quad y>0, \sigma>0, \nu\geq 0 \;\ldots \quad (4)$$

where $I_0(y)$ denotes the modified Bessel function of the first kind, (see definition above).

For $|v|=0$, the expression (4) reduces to a Rayleigh distribution. In what follows, we consider the derivation of the distribution of the product $|X\,Y|$, when X and Y are Maxwell and Rice random variables respectively, distributed independently of each other and defined as above. An explicit expression for the cdf of $|X\,Y|$ in terms of hypergeometric function has been derived in Theorem 1. In Theorem 2, another explicit expression for the cdf of $|X\,Y|$ in terms of hypergeometric function and modified Bessel function of the second kind $K_\nu(x)$ has been derived.

Theorem 1

Suppose X is a Maxwell random variable with pdf $f_X(x)$ as given in (2) and cdf $F_X(x)=P(X\le x)$ given by (3) in terms of the incomplete gamma function. Also, suppose Y is a Rice random variable with pdf $f_Y(y)$ given by (4) in terms of the modified Bessel function of the first kind $I_0(y)$. Then the cdf of $Z=|X\,Y|$ can be expressed as

$$F(z)=\left[\frac{e^{-v^2/2\sigma^2}}{\sqrt{\pi}\sigma^2}\right]\sum_{k=0}^{\infty}\frac{v^{2k}}{2^{2k}\sigma^{4k}(k!)^2}\left[\frac{\Gamma\left(k-\frac{1}{2}\right)2^{k-1}a^{\frac{3}{2}}\sigma^{2k-1}z^3}{3}\,{}_1F_2\left(\frac{3}{2};\frac{5}{2},\frac{3}{2}-k;\frac{a z^2}{4\sigma^2}\right)+\frac{\Gamma\left(\frac{1}{2}-k\right)a^{k+1}z^{2(k+1)}}{2^{k+1}(k+1)}\,{}_1F_2\left(k+1;k+2,k+\frac{1}{2};\frac{a z^2}{4\sigma^2}\right)\right]$$

where ${}_1F_2(.)$ denotes hypergeometric function of order $(1,2)$, (see definition above).

Proof

Using the expressions (3) for cdf of Maxwell random variable X and the expression (4) for pdf of Rice random variable Y, the cdf $F(z)=\Pr(|X\,Y|\le z)$ can be expressed as

$$\begin{aligned}F(z)&=\Pr\left(|X|\le\frac{z}{|Y|}\right)\\&=\int_0^\infty F_X\left(\frac{z}{y}\right)f_Y(y)\,dy \qquad \ldots\\&=\left[\frac{2\,e^{-v^2/2\sigma^2}}{\sqrt{\pi}\,\sigma^2}\right]\int_0^\infty y e^{-y^2/2\sigma^2}\,\gamma\left(\frac{3}{2},\frac{a z^2}{2y^2}\right)I_0\left(\frac{v y}{\sigma^2}\right)dy\end{aligned} \tag{5}$$

where $y>0,\ z>0,\ a>0,\ \sigma>0,\ v\ge 0$. The proof of Theorem 1 easily follows by using definition (1) of modified Bessel function of first kind, $I_0(x)$, of order 0, and Lemma 3 in the integral (5) above.

Theorem 2

Suppose X is a Maxwell random variable with pdf $f_X(x)$ as given in (2) and cdf $F_X(x)=P(X\le x)$ given by (3) in terms of the error function. Also, suppose Y is a Rice random variable with pdf $f_Y(y)$ given by (4) in terms of the modified Bessel function of the first kind $I_0(y)$. Then the cdf of $Z=|X\,Y|$ can be expressed as

$$F(z)=\left[\frac{\sqrt{a}\,e^{-v^2/2\sigma^2}}{\sqrt{\pi}\,\sigma^2}\right]\sum_{k=0}^{\infty}\frac{v^{2k}}{2^{2k}\sigma^{4k}(k!)^2}\left[\Gamma\left(k+\frac{1}{2}\right)2^k\sigma^{2k+1}z\,{}_1F_2\left(\frac{1}{2};\frac{3}{2},\frac{1}{2}-k;\frac{a z^2}{4\sigma^2}\right)+\frac{\Gamma\left(-k-\frac{1}{2}\right)a^{k+\frac{1}{2}}z^{2(k+1)}}{2^{k+2}(k+1)}\,{}_1F_2\left(k+1;k+2,k+\frac{3}{2};\frac{a z^2}{4\sigma^2}\right)-\sqrt{2}\left(\sqrt{a}\,\sigma\right)^{k+\frac{1}{2}}z^{k+\frac{3}{2}}K_{k+\frac{1}{2}}\left(\frac{\sqrt{a}\,z}{\sigma}\right)\right]$$

where ${}_1F_2(.)$ denotes hypergeometric function of order $(1,2)$, and $K_k(.)$ denotes the modified Bessel functions of the second kind of order k, (see definition above).

Proof

Using the expressions (3) for cdf of Maxwell random variable X and the expression (4) for pdf of Rice random variable Y, the cdf $F(z)=\Pr(|X\,Y|\le z)$ can be expressed as

$$\begin{aligned}
&\mathrm{F}(z)\\
&=\Pr\left(|X|\le\frac{z}{|Y|}\right)\\
&=\int_0^\infty F_X\left(\frac{z}{y}\right)f_Y(y)\,dy\\
&=\left[\frac{e^{-v^2/2\sigma^2}}{\sigma^2}\right]\int_0^\infty y e^{-y^2/2\sigma^2}\left\{\begin{array}{l}\mathrm{erf}\left(\sqrt{\dfrac{a}{2}}\dfrac{z}{y}\right)\\ -\sqrt{\dfrac{2a}{\pi}}\dfrac{z}{y}e^{-\frac{a z^2}{2y^2}}\end{array}\right\} I_0\left(\frac{vy}{\sigma^2}\right)dy\ldots
\end{aligned} \tag{6}$$

where $y>0,\ z>0,\ a>0,\ \sigma>0,\ v\ge 0$. The proof of Theorem 2 easily follows by using the definition (1) of modified Bessel function of the first kind, $I_0(x)$, of order 0, substituting $y=\frac{1}{t}$ in the first term and $y^2=u$ in the second term of the integral (6) above, and then using Lemmas 2 and 4 respectively.

PDF of the Product $Z=|X\,Y|$, and kth Moment of RV $Z=|X\,Y|$

In what follows, without loss of generality, for simplicity of computations, this section discusses the derivation of the pdf of the product $Z=|X\,Y|$, when X and Y are Rice and Maxwell random variables distributed according to (4) and (2), respectively, and independently of each other. An explicit expression for the pdf of the product $Z=|X\,Y|$ in terms of the modified Bessel function of the second kind $K_\nu(x)$ has been derived in Theorem 3. The expression for the kth moment of RV $Z=|X\,Y|$ in terms of gamma functions has been derived in Theorem 4.

Theorem 3

Suppose X and Y are Rice and Maxwell random variables having pdf given by (4) and (2), respectively. Then the pdf of $Z=|X\,Y|$ can be expressed as

$$\begin{aligned}
f_Z(z)=&\left(\sqrt{\frac{2}{\pi}}\,e^{-v^2/2\sigma^2}\right)\\
&\sum_{n=0}^{\infty}\frac{v^{2n}\,a^{n+2}}{\sigma^{2n+1}(n!)^2}K_{n+\frac{1}{2}}\left(\frac{\sqrt{a}\,z}{\sigma}\right)\cdots
\end{aligned} \tag{7}$$

where $K_{n+\frac{1}{2}}(.)$ denotes the modified Bessel functions of the second kind of order $n+\frac{1}{2}$, (see definition above).

Proof

The pdf of $Z=|X\,Y|$ can be expressed as

$$\begin{aligned}
&f_Z(z)\\
&=\int_0^\infty\frac{1}{y}f_X\left(\frac{z}{y}\right)f_Y(y)\,dy\\
&=\left(\sqrt{\frac{2}{\pi}}\frac{a^{\frac{3}{2}}}{\sigma^2}e^{-v^2/2\sigma^2}\,z\right)\\
&\int_0^\infty e^{-\frac{z^2}{2\sigma^2y^2}-\frac{a y^2}{2}}\ I_0\left(\frac{v z}{\sigma^2 y}\right)dy,\ldots
\end{aligned} \tag{8}$$

where $y>0,\ z>0,\ a>0,\ \sigma>0,\ v\ge 0$. The proof of Theorem 3 easily follows by using the definition (1) of modified Bessel function of the first kind, $I_0(x)$, of order 0, substituting $y^2 = \frac{1}{t}$, and then using Lemma 4 in the integral (8) above.

Theorem 4

If Z is a random variable with pdf given by (7), then its kth moment can be expressed as

$$E\left(Z^k\right) = \left(\frac{2^{k-\frac{1}{2}} e^{-\nu^2/2\sigma^2}}{\sqrt{\pi}}\right) \sum_{n=0}^{\infty} \frac{\nu^{2n} a^{n-\frac{k}{2}+\frac{3}{2}}}{\sigma^{2n-k} (n!)^2} \Gamma\left(\frac{2k-2n+1}{4}\right) \Gamma\left(\frac{2k+2n+3}{4}\right)$$

Proof

$$E\left(Z^k\right) = \left(\sqrt{\frac{2}{\pi}}\, e^{-\nu^2/2\sigma^2}\right) \sum_{n=0}^{\infty} \frac{\nu^{2n} a^{n+2}}{\sigma^{2n+1} (n!)^2} \int_0^{\infty} z^k K_{n+\frac{1}{2}}\left(\frac{\sqrt{a}\, z}{\sigma}\right) dz \ldots \tag{9}$$

By using the equation (6.621.3 / page 712) from Gradshteyn and Ryzhik (2000), in the integral (9) above, the result of Theorem 4 easily follows.

Conclusion

This article has derived the exact distributions of the product of two independent random variables X and Y, where X and Y have Maxwell and Rice distributions respectively. The pdf and kth moment of the product of two variables are also given. The distribution is obtained as a function of hypergeometric of order $(1, 2)$, where as the pdf has been obtained as a function of Bessel of the second kind. We hope the findings of the article will be useful for the practitioners which are indicated in the introduction of the article.

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