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Global Measure of the Deviation of a Wavelet Density Estimator

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A wavelet estimator $f^*(x)$ of an unknown probability density function $f(x) \in \mathcal{L}^2(\mathbf{R})$ is considered. A conditional central limit theorem for martingales is used to show that $\int [f^*(x) - f(x)]^2 dx$ is asymptotically normally distributed. Results obtained can be used in a test of goodness-of-fit.

Key words: Wavelet estimator, martingale difference array, multiresolution analysis, asymptotic distribution.

Introduction

The problem of finding the asymptotic distribution of the quadratic norm of the deviation of the probability density function $f(x)$ from its estimator $f^*(x)$ have been studied by many authors. Bickel and Rosenblatt (1973) obtained the asymptotic distribution of

$$T_n = n\,h(n)\int [f^*(x) - f(x)]^2 a(x)\,dx$$

where $f^*(x)$ is the kernel estimator of $f(x)$, $h(n) \to 0$, $n\,h(n) \to \infty$, and $a(x)$ is a weight function. The basic technique in obtaining the result consists in finding the asymptotic distribution of T_n with $f^*(x)$ replaced by conveniently chosen Gaussian process and showing that two functionals converge to the same law. Viollaz (1980) considered orthogonal series estimators and Lii (1978) considered spline estimators, in both cases the above method is used to establish limit theorems for the quadratic norm of the deviation of the probability density function from its estimator. A method using a conditional central limit theorem for martingales due to Adnan (1981) was used by Ghorai (1980) to find the asymptotic distribution of the quadratic norm of the deviation of the orthogonal series estimator.

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Rosenblatt (1975) used a method involving the Poissonization of the sample size to obtain the asymptotic distribution of the quadratic norm of the deviation of the two-dimensional kernel estimator. Alyass and Sun (1994) considered two-dimensional orthogonal series estimators; they used the method of Poissonization to establish a limit theorem for the properly normalized quadratic norm of the deviation of the estimator.

A wavelet estimator $f^*(x)$ is used here to estimate the probability density function $f(x)$. Then, a martingale central limit theorem is used to show that $\int [f^*(x) - f(x)]^2\ dx$ is asymptotically normally distributed.

A brief review and a statement of a conditional central limit theorem for martingales will now be given. For further details, refer to Adnan (1981). Let $\{V_n, n \geq 1\}$ be a sequence of integrable random variables on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \ldots\ldots$ be an increasing sequence of sub-σ-fields of $\mathcal{F}$. Suppose the sequence $\{(V_n, \mathcal{B}_n), n \geq 1\}$ is a martingale, then the sequence $\{(V_n - V_{n-1}, \mathcal{B}_n), n \geq 1\}$ is called a martingale difference. A double sequence $\{(W_{nj}, \mathcal{B}_{nj}), n \geq 1, j \geq 0\}$ is said to be a martingale difference array if it is a martingale difference for each n.

Suppose that $\{Y_n, n \geq 1\}$ is a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $\{\mathcal{F}_n, n \geq 1\}$ be a sequence of sub-σ-fields of $\mathcal{F}$. $Y_n \mid \mathcal{F}_n$ converges weakly to a random variable Y defined on (Ω, $\mathcal{F}$, P) if and only if

$$E\left[f(Y_n)\,|\,\mathcal{F}_n\right] \to Ef(Y),$$

for every bounded continuous function f. This convergence will be denoted by

$$Y_n \,|\, \mathcal{F}_n \xrightarrow{d} Y .$$

Methodology

The following theorem due to Adnan (1981) will be used in the proof of the main result in this article.

Theorem 1:

Suppose $\{(W_{n\,j}, \mathcal{B}_{n\,j}),\ n \geq 1,\ j \geq 0\}$ is a martingale difference array. Assume that:

(i) $\sup_n \sum_{j=1}^{\infty} EW_{nj}^2 < \infty,$

(ii) $\sum_{j=1}^{\infty} W_{nj}^2 \xrightarrow{p} c^2$ for some positive constant c

(iii) $\sup_j \left|W_{nj}\right| \xrightarrow{p} 0.$

Then, $\sum_{j=1}^{\infty} W_{nj} \,|\, \mathcal{B}_{n0} \xrightarrow{d} N(0, c^2).$

Remark:

Let γ denote the trivial σ-field. If $\gamma \subset \mathcal{B}_{n0}$ then the conditional convergence in the above theorem is equivalent to the usual unconditional convergence in distribution (see Adnan, 1981).

A multiresolution analysis $\ldots \subseteq A_{-2} \subseteq A_{-1} \subseteq A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$ of $\mathcal{L}^2(\mathbf{R})$ is an increasing sequence of subspaces $A_j, j \in \mathbf{Z}$, of $\mathcal{L}^2(\mathbf{R})$ satisfying the following conditions:

(M1) $\bigcup_{j \in \mathbf{Z}} A_j$ is dense in $\mathcal{L}^2(\mathbf{R})$,

(M2) $\bigcap_{j \in \mathbf{Z}} A_j = \{0\}$,

(M3) $g(x) \in A_j$ if and only if $g\left(2^{-j}x\right) \in A_0$,

(M4) there exists a function $\varphi(x)$ in A_0 such that $\{\varphi(x-k)\}_{k \in \mathbf{Z}}$ is an orthonormal basis for A_0.

Remarks:

(i) It follows that

$$\left\{ 2^{j/2}\, \varphi\left(2^j x - k\right) \right\}_{k \in \mathbf{Z}}$$

forms an orthonormal basis for A_j.

(ii) Assume that φ is integrable and $\int \varphi(x)dx \neq 0$ because if $\int \varphi(x)dx = 0$ then the same is true for all functions in all A_j, and one would not expect to have condition (M1). In fact one can show that if φ has compact support and $\int \varphi(x)\,dx = 1$ then condition (M1) holds (see Strichartz (1993)).

In order to construct the wavelets, let B_j be the orthogonal complement of A_j in A_{j+1}, thus $A_{j+1} = A_j \oplus B_j$. There exists a function $\psi(x)$ called the wavelet such that the family $\{\varphi(x-k), \psi(x-k)\}_{k \in \mathbf{Z}}$ is an orthonormal basis for A_1. This implies that $\{\psi(x-k)\}_{k \in \mathbf{Z}}$ is an orthonormal basis for B_0. The space $\mathcal{L}^2(\mathbf{R})$ is represented as a direct sum

$$\mathcal{L}^2(\mathbf{R}) = \bigoplus_{j \in Z} B_j .$$

Also

$$\left\{ 2^{j/2}\, \psi\left(2^j x - k\right) \right\}_{k \in \mathbf{Z}}$$

is an orthonormal basis for B_j and that the spaces B_j are all mutually orthogonal. Therefore, it is possible to combine all the orthonormal bases for B_j into one orthonormal basis:

$$\left\{ 2^{j/2} \varphi\left(2^j x - k\right) \right\}_{j\in\mathbf{Z},\, k\in\mathbf{Z}}$$

for $\mathcal{L}^2(\mathbf{R})$. Because the following decomposition of $\mathcal{L}^2(\mathbf{R})$ is also true

$$\mathcal{L}^2(\mathbf{R}) = A_q \oplus \left[\bigoplus_{j=q}^{\infty} B_j \right], \qquad q \in \mathbf{Z},$$

then one can combine the basis

$$\left\{ 2^{q/2} \varphi\left(2^q x - k\right) \right\}_{k\in\mathbf{Z}}$$

for A_q with the bases

$$\left\{ 2^{j/2} \psi\left(2^j x - k\right) \right\}_{k\in\mathbf{Z}}$$

for B_j with $j \geq q$ to obtain an orthonormal basis for $\mathcal{L}^2(\mathbf{R})$. Then, if

$$\varphi_{j,k}(x) = 2^{j/2} \varphi\left(2^j x - k\right)$$

and

$$\psi_{j,k}(x) = 2^{j/2} \psi\left(2^j x - k\right),$$

the family

$$\left\{ \varphi_{q,k}(x), \psi_{j,k}(x) \right\}_{j\geq q,\, k\in\mathbf{Z}}$$

forms an orthonormal basis for $\mathcal{L}^2(\mathbf{R})$. Thus, for any $f(x)\in \mathcal{L}^2(\mathbf{R})$. there is

$$f(x) = \sum_{k=-\infty}^{\infty} \alpha_{q,k}\, \varphi_{q,k} + \sum_{j=q}^{\infty} \sum_{k=-\infty}^{\infty} \beta_{j,k}\, \psi_{j,k}(x), \qquad (1)$$

where

$$\alpha_{q,k} = \int f(x)\varphi_{q,k}(x)dx,$$
$$\beta_{j,k} = \int f(x)\psi_{j,k}(x)dx, \qquad j \geq q.$$

For more detailed account of the subject of multi-resolution analysis and wavelets see Meyer (1990) and Daubechies (1992).

Suppose $X_1, X_2, \ldots, X_n$ are independent, identically distributed, real-valued random variables with common, but unknown, continuous probability density function $f(x)\in \mathcal{L}^2(\mathbf{R})$. Estimate $f(x)$ by

$$f^*(x) = \sum_{k=-\infty}^{\infty} \hat{\alpha}_{q(n),k}\varphi_{q(n),k}(x),$$

where (2)

$$\hat{\alpha}_{q(n),k} = \frac{1}{n}\sum_{i=1}^{n}\varphi_{q(n),k}(x_i).$$

Some of the properties of this estimator may be found in Doukhan and Leon (1990) and Kerkyacharian and Picard (1992). Throughout the remainder of this article, assume the function φ is compactly supported in the interval $[s, t]$. This will ensure that, in (2), only finite random number of coefficients $\hat{\alpha}_{q(n),k}$ are non-zero. To simplify the calculations, assume that $s, t\in \mathbf{Z}$. Under these assumptions,

$$f^*(x) = \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1-t}^{-s} \varphi_{q(n),k}(x-[x_i])\, \varphi_{q(n),k}(x_i-[x_i]), \qquad (3)$$

where $[x]$ denotes the largest integer that is less than or equal to x.

Let

$$Y_i = X_i - [x_i],\ i = 1, 2, \ldots\ldots, n,$$
$$\eta_{q(n),k} = E\varphi_{q(n),k}(Y)$$

and

$$\theta_{q(n),k}(y) = \varphi_{q(n),k}(y) - \eta_{q(n),k}.$$

Using (1) and (3) the result is

$$\int [f^*(x) - f(x)]^2 dx = \frac{2}{n^2}\sum_{j=2}^{n}\sum_{i=1}^{j-1} U_{ij}(n) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1-t}^{-s} \theta^2_{q(n),k}(y_i) \qquad (4)$$

$$+\frac{1}{n^2}\sum_{i\neq j} Z_{ij}(n)+\sum_{j=q(n)}^{\infty}\sum_{k=-\infty}^{\infty}\beta_{j,k}^2,$$

where

$$Z_{ij}(n)= \qquad (5)$$

$$\sum_{k=1-t}^{-s}\Big[\sum_{l=1-t}^{-s}\varphi_{q(n),k}(Y_i)\,\varphi_{q(n),l}(Y_j)$$

$$\delta_{k+2^{q(n)}[x_i]\,,\,l+2^{q(n)}[x_j]}-\varphi_{q(n),k}(Y_i)\,\varphi_{q(n),k}(Y_j)\Big],$$

and

$$U_{ij}(n)=\sum_{k=1-t}^{-s}\theta_{q(n),k}(Y_i)\,\theta_{q(n),k}(Y_j)$$

Now put

$$c_{kl}=\operatorname{cov}\big(\varphi_{q(n),k}(Y),\quad \varphi_{q(n),l}(Y)\big), \quad (6)$$

$$\mu_n=\frac{1}{n}\sum_{k=1-t}^{-s}c_{kk},\ \sigma_n^2=\sum_{k=1-t}^{-s}\sum_{l=1-t}^{-s}c_{kl}^2,$$

$$W_{nj}=\begin{cases}\dfrac{\sqrt{2}}{n\sigma_n}\displaystyle\sum_{i=1}^{j-1}U_{ij}(n), & j=2,3,\ldots,n,\\[2ex] 0\quad, & j=0,1 \quad\text{and}\quad j>n,\end{cases}$$

$$V_{nj}=\sum_{i=1}^{j}W_{ni}\quad\text{for all } n \text{ and } j,$$

and let $\mathfrak{B}_{nj}$ be the σ-field generated by $Y_1, Y_2, \ldots, Y_j$, and $\mathfrak{B}_{n0}$ be the trivial σ-field. The sequence $\{(V_{nj}, \mathfrak{B}_{nj}), j \geq 0\}$ is a martingale for each $n \geq 1$. Because $W_{nj} = V_{nj} - V_{n,j-1}$, $\{(W_{nj}, \mathfrak{B}_{nj}), n > 0, j \geq 0\}$ is a martingale difference array.

Results

Now, to state and prove the main theorem:

Theorem 2: Assume that

$$\frac{n}{\sigma_n}\sum_{k=1-t}^{-s}\Big[E\big|\varphi_{q(n),k}(Y)\big|\Big]^2\to 0 \qquad (7)$$

as $n\to\infty$

$$\frac{n}{\sigma_n}\sum_{j=q(n)}^{\infty}\sum_{k=-\infty}^{\infty}\beta_{j,k}^2\to 0 \ \text{ as } \ n\to\infty \quad (8)$$

$$\sigma_n^2\to\infty \qquad\text{as}\qquad n\to\infty, \qquad (9)$$

$$\sup_k E\varphi_{q(n),k}^4(Y)\leq M. \qquad (10)$$

for some constant M

It follows that if $t - s < 1+2^{q(n)}$, then

$$\frac{n}{\sqrt{2}\,\sigma_n}\Big(\int\big[f^*(x)-f(x)\big]^2dx-\mu_n\Big)\xrightarrow{d} N(0,1).$$

Proof: Using (4) and (6) gives

$$\frac{n}{\sqrt{2}\sigma_n}\Big(\int\big[f^*(x)-f(x)\big]^2dx-\mu_n\Big)=$$

$$\sum_{j=1}^{\infty}W_{nj}+\frac{1}{\sqrt{2}n\sigma_n}\sum_{i=1}^{n}\sum_{k=1-t}^{-s}\Big[\theta_{q(n),k}^2(Y_i)-c_{kk}\Big]+$$

$$\frac{1}{\sqrt{2}n\sigma_n}\sum_{i\neq j}Z_{ij}$$

$$+\sum_{j=q(n)}^{\infty}\sum_{k=-\infty}^{\infty}\beta_{j,k}^2=H_1+H_2+H_3+H_4\,.$$

By assumption (8), $H_4 \to 0$ as $n \to \infty$. Let

$$A=\left\{\begin{matrix}(k,l):\ k-l=2^{q(n)}\,([X_j]-\\ [x_i])\,,\ k,l=1-t,\ldots\ldots,-s\end{matrix}\right\}$$

and

$$\mathrm{B}=\left\{\begin{matrix}(\mathrm{i,j}):[X_j]-[X_i]>\dfrac{(t-s)-1}{2^{q(n)}},\\ i\neq j,\quad i,j=1,2,\ldots\ldots,n\end{matrix}\right\}$$

From (5) it follows that

$$H_3 = \frac{1}{\sqrt{2}n\sigma_n}$$

$$\left\{ \sum_B \left[\sum_A \varphi_{q(n),k}(Y_i)\varphi_{q(n),1}(Y_j) - \sum_{k=1-t}^{-s} \varphi_{q(n),k}(Y_i)\varphi_{q(n),k}(Y_j) \right] \right\}$$
$$+\sum_{B^C} \left[\sum_A \varphi_{q(n),k}(Y_i)\varphi_{q(n),1}(Y_j) - \sum_{k=1-t}^{-s} \varphi_{q(n),k}(Y_i)\varphi_{q(n),k}(Y_j) \right]$$

The second term in the above formula is equal to zero because $\frac{(t-s)-1}{2^{q(n)}} < 1$ forces $[X_j] = [X_i]$. Also for $(i,j) \in B$ $A = \varnothing$. Therefore

$$H_3 = -\frac{1}{\sqrt{2}n\sigma_n} \sum_B \sum_{k=1-t}^{-s} \varphi_{q(n),k}(Y_i)\varphi_{q(n),k}(Y_j),$$

$$\operatorname{var}(H_3) \le \frac{1}{2n^2\sigma_n^2}$$

$$\sum_{i\ne j} \sum_{i'\ne j'} \sum_{k=1-t}^{-s} \sum_{l=1-t}^{-s} E\left|\varphi_{q(n),k}(Y_i)\varphi_{q(n),k}(Y_j)\varphi_{q(n),l}(Y_i')\varphi_{q(n),l}(Y_j')\right|$$
$$= \frac{n(n-1)}{2n^2\sigma_n^2} \sum_{k=1-t}^{-s} \sum_{l=1-t}^{-s} E\left|\varphi_{q(n),k}(Y)\varphi_{q(n),l}(Y)\right| \left[E\left|\varphi_{q(n),k}(Y)\varphi_{q(n),l}(Y)\right| \right.$$
$$\left. + 2E\left|\varphi_{q(n),k}(Y)\right| E\left|\varphi_{q(n),l}(Y)\right| \right] + \frac{n(n-1)(n^2-n-3)}{2n^2\sigma_n^2}$$
$$\left[\sum_{k=1-t}^{-s} \left\{ E\left|\varphi_{q(n),k}(Y)\right| \right\}^2 \right]^2.$$

Hence, if (7), (9) and (10) hold, then $\operatorname{var}(H_3) \to 0$ as $n \to \infty$. Next, observe under assumption (10)

$$\operatorname{var}(H_2) = \frac{1}{2n\sigma_n^2} E\left[\sum_{k=1-t}^{-s} \left(\theta^2_{q(n),k}(Y) - c_{kk} \right) \right]^2$$
$$= O\left(\frac{1}{n\sigma_n^2} \right).$$

Consequently, $\operatorname{var}(H_2) \to 0$ as $n \to \infty$. if assumption (9) holds.

Therefore, to complete the proof of the theorem, it is sufficient to show that $H_1 \xrightarrow{d} N(0,1)$. To prove this, observe:

$$EW_{nj}^2 = \frac{2}{n^2\sigma_n^2} E\left(\sum_{i=1}^{j-1} U_{ij} \right)^2 = \frac{2(j-1)}{n^2}$$

Therefore,

$$\sum_{j=1}^{\infty} EW_{nj}^2 = \frac{2}{n^2\sigma_n^2} E\left(\sum_{i=1}^{j-1} U_{ij} \right)^2 \qquad (11)$$
$$= \frac{2(j-1)}{n^2} \qquad \text{for all } n.$$

Next to be shown is

$$\sum_{j=1}^{\infty} W_{nj}^2 \xrightarrow{p} 1. \qquad (12)$$

In order to establish (12), it is enough, in view of Chebychev's inequality and (11), to show that

$$E\left(\sum_{j=1}^{n} W_{nj}^2 \right)^2 = \qquad (13)$$
$$\sum_{j=1}^{n} EW_{nj}^4 + 2\sum_{j<j'}^{n} EW_{nj}^2 W_{nj'}^2 \longrightarrow 1$$
$$\text{as } n \longrightarrow \infty.$$

Using Holder's inequality,

$$W_{nj}^4 \le \frac{4(t-s)^3}{n^4\sigma_n^4}$$
$$E\left[\sum_{k=1-t}^{-s} \left(\sum_{i=1}^{j-1} \theta_{q(n),k}(Y_i) \right)^4 \theta^4_{q(n),k}(Y_j) \right]$$
$$= \frac{4(t-s)^3}{n^4\sigma_n^4}$$
$$\sum_{k=1-t}^{-s} E\theta^4_{q(n),k}(Y) \left[(j-1) E\theta^4_{q(n),k}(Y) + 3(j-1)(j-2)c_{kk}^2 \right]$$

By summing over j,

$$\sum_{j=1}^{n} EW_{nj}^4 \le \frac{2(n-1)(t-s)^3}{n^3\sigma_n^4}$$

$$\sum_{k=1-t}^{-s} E\theta^4_{q(n),k}(Y)\left[E\theta^4_{q(n),k}(Y)+2(n-s)c^2_{kk}\right]$$

Therefore, under assumptions (9) and (10)

$$\sum_{j=1}^{n} EW^4_{nj} \to 0 \text{ as } n\to\infty. \qquad (14)$$

Now,

$$2\sum_{j<j'} EW^2_{nj}W^2_{nj'} = \frac{8}{n^4\sigma^4_n}\sum_{j<j'} E\left[\left(\sum_{i=1}^{j-1}U^2_{ij}\right)\left(\sum_{r=1}^{j'-1}U^2_{rj'}\right)+\left(\sum_{i=1}^{j-1}U^2_{ij}\right)\left(\sum_{\substack{r=1\\ r\neq r'}}^{j'-1}\sum_{r'=1}^{j'-1}U_{rj'}U_{r'j'}\right)\right.$$

$$+\left(\sum_{r=1}^{j'-1}U^2_{rj'}\right)\left(\sum_{\substack{i=1\\ i\neq i'}}^{j-1}\sum_{i'=1}^{j-1}U_{ij}U_{i'j}\right)$$

$$\left.+\left(\sum_{\substack{i=1\\ i\neq i'}}^{j-1}\sum_{i'=1}^{j-1}U_{ij}U_{i'j}\right)\left(\sum_{\substack{r=1\\ r\neq r'}}^{j'-1}\sum_{r'=1}^{j'-1}U_{rj'}U_{r'j'}\right)\right]$$

$$= G_1+G_2+G_3+G_4.$$

Note that

$$G_1 = \frac{8}{n^4\sigma^4_n}\sum_{j<j'}\left[2(j-1)\,EU^2_{12}\,U^2_{23}+(j-1)(j'-3)\left\{EU^2_{12}\right\}^2\right]$$

$$= \frac{8(n-1)(n-2)}{3n^3\sigma^4_n}\sum_k\sum_l\sum_{k'}\sum_{l'} c_{kl}c_{k'l'}E\theta_{q(n),k}\,\theta_{q(n),l}\,\theta_{q(n),k'}\,\theta_{q(n),l'} + \frac{(n-1)(n-2)(n-3)}{n^3}.$$

Thus,

$$G_1 \to 1 \quad \text{as} \quad n\to\infty. \qquad (16)$$

If assumptions (9) and (10) hold,

$$G_3 = 0 \quad \text{and } G_2 \to 0 \quad \text{as} \quad n\to\infty. \quad (17)$$

Also, computations (see Ghorai, 1980) show

$$G_4 \to 0 \text{ as } n\to\infty. \qquad (18)$$

Therefore (15) together with (16),(17) and (18) gives

$$2\sum_{j<j'} EW^2_{nj}\,W^2_{nj'} \to 1 \text{ as } n\to\infty. \quad (19)$$

Relation (12) now follows by combining (13), (14) and (19). Finally,

$$P\left(\sup_j\left|W_{nj}\right| > \in\right) \le \sum_{j=1}^{n} P\left(W^2_{nj} > \in^2\right) \le \frac{1}{\in^4}\sum_{j=1}^{n} EW^4_{nj}.$$

By using (14)

$$\sup_j\left|W_{nj}\right| \xrightarrow{p} 0. \qquad (20)$$

may be deduced. The theorem now follows by combining Theorem 1 with (11), (12) and (20).

Conclusion

Tests of goodness-of-fit can be obtained as a direct application to Theorem 2. In fact, a test may be constructed for the hypothesis H: $f(x) = f_0(x)$ at a given level α, where $f_0(x)$ is a given function. To do this, the statistic

$$R_n = \int\left[f^*(x)-f(x)\right]^2 dx$$

is to be computed for $f(x) = f_0(x)$ and the hypothesis is to be rejected if $R_n \ge d_n(\alpha)$ where by Theorem 2

$$d_n(\alpha) = \mu_n + \frac{\sqrt{2}\sigma_n}{n}\Phi^{-1}(1-\alpha),$$

where

$$\Phi(z) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{z} e^{-t^2/2} dt.$$

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