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Early Scholars
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A block design set up is considered in presence of a number of controllable covariates. The problem is that of choosing the values of the covariates so that for a given block design, it is optimum in the sense of attaining minimum variance for the estimation of each of the covariate parameters. In case of incomplete block designs, the choice of the values of the covariates depends heavily on the allocation of treatments to the plots of blocks; more specifically on the method of construction of the incomplete block design. In this paper the situation where the block design is a member of the complementary series of balanced incomplete block design (BIBD) with parameters $b = v = s^N + s^{N-1} + \ldots + s + 1$, $r = k = s^N$, $\lambda = s^N - s^{N-1}$ of symmetric balanced incomplete block design (SBIBD) obtained through projective geometry is considered.

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Introduction

The following non stochastic controllable covariates model in a block design set up

$$(Y, \mu 1 + X_1\beta + X_2\tau + Z\gamma, \sigma^2 I) \qquad (1)$$

is considered, where μ is the intercept term, σ^2 is the common variance of the observations, β is the vector of block effects of order $b \times 1$, τ is the vector of treatment effects of order $v \times 1$, γ is the vector of covariate effects of order $c \times 1$ and Y is

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the uncorrelated observation vector of order $n \times 1$; X_1, X_2 are the incidence matrices of block effects, treatment effects respectively and Z is a design matrix of covariate effects.

For the covariates, without loss of generality, the (location-scale)- transformed version: $|z_{ij}| \leq 1$ is assumed. It is evident that for orthogonal estimation of treatment and block effect contrasts on one hand and covariate effects on the other, the following condition must be satisfied.

$$Z'X_1 = 0, \qquad Z'X_2 = 0 \qquad (2)$$

For most efficient estimation of each of the regression parameters the following condition must hold (Pukelsheim, 1993)

$$Z'Z = nI_c. \qquad (3)$$

This means that all the elements in each column of Z must be ± 1, and the columns must be mutually orthogonal.

In the block design set up, the optimum

properties of randomized block design (RBD) and BIBD with respect to a class of optimality criteria for the estimation of treatment effects are well known (see e.g., Shah & Sinha, 1989). The choice of covariates in a design set up was earlier considered by Troya (1982a, 1982b), Liski et al (2002), Das et al. (2003), Dutta (2004), Rao et al. (2006) and others. Troya (1982a, 1982b) first considered the problem of choice of the levels of the covariates, i.e., Z matrix in a completely randomized design (CRD) model. Das et al. (2003) extended it to the set up of RBD and some series of BIBDs. As mentioned earlier, the choice of covariate values depends heavily on the block design set up as is evidenced from (2). In the case of incomplete block designs, the allocation of treatments to the plots of the blocks depends on the method of construction of designs. Das et al. (2003) considered symmetric balanced incomplete block design (SBIBDs) with parameters b=v, r=k, λ constructed through Bose's difference method and some BIBDs with repeated blocks. Dutta (2004) also considered some series of BIBDs obtained through Bose's difference technique together with some arbitrary BIBDs. However, as is well known, there are different methods of construction leading to different series of BIBDs and the choice of the Z matrices also varies from series to series. Here, the problem of choice of Z for the series of complements of SBIBDs, which are obtained through Projective Geometry, is considered. It may be mentioned in this connection that in the series considered in the previous works (Das et al., 2003, Dutta, 2004), the layouts have cyclical pattern which simplified the choice of Z. However, the series of SBIBDs considered here do not have the above cyclical property. Following Das et al. (2003), each column of the Z matrix is transformed to a W-matrix where the element in the ith row and j^{th} column of $W^{(s)}$ is $z_{ij}^{(s)}$; $z_{ij}^{(s)}$ being the element of Z corresponding to j^{th} treatment in i^{th} block of the design for the sth covariate. Corresponding to the block and treatment classification, conditions (2) and (3) in terms of W-matrices reduce to:

(C₁)   Each W-matrix has all column-sums equal to zero;

(C₂)   Each W-matrix has all row-sums equal to zero;

(C₃)   The grand total of all the entries in the Hadamard product (vide Rao, 1973) of any two distinct W-matrices reduces to zero.

In a BIBD set up with parameters v,b,r,k and λ, W-matrix of order b×v can be constructed from the incidence matrix of the BIBD by placing judiciously ±1's in the non-zero k-positions in every row and in the non-zero r positions in every column such that each W-matrix satisfies conditions C_1, C_2 and C_3 mentioned above. The paper is organized as follows: In Section 2 an outline of the construction of BIBDs through PG(N,s) and a method of partitioning of the blocks into different sets useful for the choice of W-matrices are given and in Section 3 methods of constructing optimum W-matrices by using sets described in Section 2 have been considered

BIBDs through Projective Geometry: Partitioning of blocks

With the help of the Galosis field GF(s), a finite projective geometry of N dimensions, to be written as PG(N,s), where $s=p^n$, p is a prime number and n is any positive integer, can be constructed. Any ordered set of (N+1) elements $(x_0,x_1,\ldots,x_N)$ where the x_i's belong to GF(s) and are not simultaneously zero, is called a point of the projective geometry PG(N,s). It is known that the number of points in PG(N,s) is equal to

$$\phi(N,0,s) = \frac{s^{N+1}-1}{s-1}$$ and the number of m-flats is equal to φ(N,m,s) where

$$\phi(N,m,s) = \frac{(s^{N+1}-1)(s^N-1)...(s^{N-m+1}-1)}{(s^{m+1}-1)(s^m-1)...(s-1)}$$

By making correspondence between points and m-flats of PG(N,s) with varieties and blocks respectively, a BIBD with parameters v = φ(N,0,s), b = φ(N,m,s), r = φ(N-1, m-1, s), k = φ(m,0,s), λ = φ(N-2, m-2, s) can be obtained (cf. Bose, 1939). The following series of SBIBDs with m=N-1 has parameters

b=v=$s^N+s^{N-1}+\ldots+s+1$, r = k = $s^{N-1}+\ldots+s+1$,

$\lambda = s^{N-2}+s^{N-3}+\dots+s+1.$ (4)

The complementary SBIBD of (4) has the parameters

$$b = v = s^N+s^{N-1}+\dots+s+1,$$ (5)
$$r = k = s^N, \lambda = s^N-s^{N-1}.$$

It is mentioned above that the choice of the levels of the covariates in BIBD set up depends on the method of its construction and the maximum number of covariates satisfying (2)-(3) varies from series to series.

The blocks of the SBIBD are partitioned into $(s^{N-1} + s^{N-3}+\dots+s^2+1)$ (=t, say) disjoint sets; each set containing (s+1) blocks such that the portion of the incidence matrix of the complementary design corresponding to each set conforms to that of the incidence matrix of an RBD with suitable parameters. This fact has been used for the choice of the Z matrix.

It is to be noted that the number of (N-1)-flats passing through a particular (N-2)-flat is the number of (N-1)-flats on which a particular (N-2)-flat lies. This number is given by $\phi(1,0,s) = s+1$. Such (s+1), (N-1)-flats passing through a particular (N-2)-flat can be obtained as follows:

Consider an (N-2)-flat of PG(N,s) given by

$$a'x=0, \quad b'x=0$$ (6)

where, a′ and b′ are two row vectors of a matrix A of order 2×(N+1) with elements from GF(s) such that rank (A)=2.

The (s+1), (N-1)-flats containing the same (N-2)-flat in (6), are given by $(\lambda_1 a'+\lambda_2 b')x=0$; $(\lambda_1,\lambda_2) \neq (0,0)$ and $(\lambda_1,\lambda_2) \equiv \rho(\lambda_1,\lambda_2)$ where, ρ is a non-zero element of GF(s). If N is odd, then the full set of ϕ(N, N-1, s), (N-1)-flats can be partitioned into

$$\frac{\phi(N,N-1,s)}{s+1} = \frac{s^{N+1}-1}{(s+1)(s-1)}$$
$$= (s^{N-1} + s^{N-3} +\dots+ s^2 +1)$$

sets each containing (s+1), (N-1)-flats having a common (N-2)-flat. It is clear that the

$$\frac{\phi(N,N-1,s)}{s+1}, \quad (N-1)\text{-flats passing through}$$

a particular (N-2)-flat are disjoint. As the blocks correspond to (N-1)-flats, through one to one correspondence, partition the blocks into $(s^{N-1}+s^{N-3}+\dots+s^2+1)$ disjoint sets each containing (s+1) blocks. It will be clear from the following two examples covering both the situations where s is prime or prime power.

Example 1: N=3, m=2, s=2. There are 15 blocks which can be partitioned into 5 sets each of size 3 as mentioned above.

$$
\begin{array}{lll}
x_0 = 0 & x_1 = 0 & x_2 = 0 \\
S_1 : x_1 + x_2 = 0 & S_2 : x_0 + x_3 = 0 & S_3 : x_1 + x_3 = 0 \\
x_0 + x_1 + x_2 = 0 & x_0 + x_1 + x_3 = 0 & x_1 + x_2 + x_3 = 0 \\
\end{array}
$$

$$
\begin{array}{ll}
x_3 = 0 & x_0 + x_1 = 0 \\
S_4 : x_0 + x_2 = 0 & S_5 : x_2 + x_3 = 0 \\
x_0 + x_2 + x_3 = 0 & x_0 + x_1 + x_2 + x_3 = 0 \\
\end{array}
$$

It is to be noted that only two equations in each set S_i are independent and these can conveniently be represented as Ax=0. It is clear that the choice of A matrix in S_1 is given by:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

The choice of A matrices for other S's are obvious.

Example 2: $N=3$, $m=2$ and $s=2^2$. There are 85 blocks which can be partitioned into 17 sets each of size 5.

Let the elements of $GF(2^2)$ be $\alpha_0=0$, $\alpha_1=1$, $\alpha_2=x$, $\alpha_3=1+x$; x being a primitive element of $GF(2^2)$. Then the 17 sets are:

$$
\begin{array}{lll}
x_0 = 0 & x_2 = 0 & x_0 + x_2 = 0 \\
x_1 = 0 & x_3 = 0 & x_1 + x_3 = 0 \\
S_1 : x_0 + x_1 = 0 & S_2 : x_2 + x_3 = 0 & S_3 : x_0 + x_1 + x_2 + x_3 = 0 \\
x_0 + \alpha_2 x_1 = 0 & x_2 + \alpha_2 x_3 = 0 & x_0 + \alpha_2 x_1 + x_2 + \alpha_2 x_3 = 0 \\
x_0 + \alpha_3 x_1 = 0 & x_2 + \alpha_3 x_3 = 0 & x_0 + \alpha_3 x_1 + x_2 + \alpha_3 x_3 = 0
\end{array}
$$

$$
\begin{array}{ll}
x_0 + \alpha_2 x_2 = 0 & x_0 + \alpha_3 x_2 = 0 \\
x_1 + \alpha_3 x_3 = 0 & x_1 + \alpha_2 x_3 = 0 \\
S_4 : x_0 + x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 & S_5 : x_0 + x_1 + \alpha_3 x_2 + \alpha_2 x_3 = 0 \\
x_0 + \alpha_2 x_1 + \alpha_2 x_2 + x_3 = 0 & x_0 + \alpha_2 x_1 + \alpha_3 x_2 + \alpha_3 x_3 = 0 \\
x_0 + \alpha_3 x_1 + \alpha_2 x_2 + \alpha_2 x_3 = 0 & x_0 + \alpha_3 x_1 + \alpha_3 x_2 + x_3 = 0
\end{array}
$$

$$
\begin{array}{ll}
x_0 + x_3 = 0 & x_0 + \alpha_2 x_3 = 0 \\
x_1 + x_2 + x_3 = 0 & x_1 + \alpha_3 x_2 + \alpha_2 x_3 = 0 \\
S_6 : x_0 + x_1 + x_2 = 0 & S_7 : x_0 + x_1 + \alpha_3 x_2 = 0 \\
x_0 + \alpha_2 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 & x_0 + \alpha_2 x_1 + x_2 + x_3 = 0 \\
x_0 + \alpha_3 x_1 + \alpha_3 x_2 + \alpha_2 x_3 = 0 & x_0 + \alpha_3 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0
\end{array}
$$

$$
\begin{array}{ll}
x_0 + \alpha_3 x_3 = 0 & x_0 + x_2 + x_3 = 0 \\
x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0 & x_1 + x_2 = 0 \\
S_8 : x_0 + x_1 + \alpha_2 x_2 = 0 & S_9 : x_0 + x_1 + x_3 = 0 \\
x_0 + \alpha_2 x_1 + \alpha_3 x_2 + \alpha_2 x_3 = 0 & x_0 + \alpha_2 x_1 + \alpha_3 x_2 + x_3 = 0 \\
x_0 + \alpha_3 x_1 + x_2 + x_3 = 0 & x_0 + \alpha_3 x_1 + \alpha_2 x_2 + x_3 = 0
\end{array}
$$

$$
\begin{array}{ll}
x_0 + \alpha_2 x_2 + \alpha_3 x_3 = 0 & x_0 + \alpha_3 x_2 + \alpha_2 x_3 = 0 \\
x_1 + \alpha_2 x_2 = 0 & x_1 + \alpha_3 x_2 = 0 \\
S_{10} : x_0 + x_1 + \alpha_3 x_3 = 0 & S_{11} : x_0 + x_1 + \alpha_2 x_3 = 0 \\
x_0 + \alpha_2 x_1 + x_2 + \alpha_3 x_3 = 0 & x_0 + \alpha_2 x_1 + \alpha_2 x_2 + \alpha_2 x_3 = 0 \\
x_0 + \alpha_3 x_1 + \alpha_3 x_2 + \alpha_3 x_3 = 0 & x_0 + \alpha_3 x_1 + x_2 + \alpha_2 x_3 = 0
\end{array}
$$

.

Example 2 (cont.) N=3, m=2 and s=2^2. There are 85 blocks which can be partitioned into 17 sets each of size 5.

$$x_0 + \alpha_3 x_2 + \alpha_3 x_3 = 0$$
$$x_1 + \alpha_2 x_2 + x_3 = 0$$
$$S_{12} : x_0 + x_1 + x_2 + \alpha_2 x_3 = 0$$
$$x_0 + \alpha_2 x_1 + x_3 = 0$$
$$x_0 + \alpha_3 x_1 + \alpha_2 x_2 = 0$$

$$x_0 + \alpha_2 x_2 + \alpha_2 x_3 = 0$$
$$x_1 + \alpha_3 x_2 + x_3 = 0$$
$$S_{13} : x_0 + x_1 + x_2 + \alpha_3 x_3 = 0$$
$$x_0 + \alpha_2 x_1 + \alpha_3 x_2 = 0$$
$$x_0 + \alpha_3 x_1 + x_3 = 0$$

$$x_0 + x_2 + \alpha_3 x_3 = 0$$
$$x_1 + \alpha_2 x_2 + \alpha_2 x_3 = 0$$
$$S_{14} : x_0 + x_1 + \alpha_3 x_2 + x_3 = 0$$
$$x_0 + \alpha_2 x_1 + \alpha_2 x_2 = 0$$
$$x_0 + \alpha_3 x_1 + \alpha_2 x_3 = 0$$

$$x_0 + x_2 + \alpha_2 x_3 = 0$$
$$x_1 + \alpha_3 x_2 + \alpha_3 x_3 = 0$$
$$S_{15} : x_0 + x_1 + \alpha_2 x_2 + x_3 = 0$$
$$x_0 + \alpha_2 x_1 + \alpha_3 x_3 = 0$$
$$x_0 + \alpha_3 x_1 + \alpha_3 x_2 = 0$$

$$x_0 + \alpha_2 x_2 + x_3 = 0$$
$$x_1 + x_2 + \alpha_2 x_3 = 0$$
$$S_{16} : x_0 + x_1 + \alpha_3 x_2 + \alpha_3 x_3 = 0$$
$$x_0 + \alpha_2 x_1 + \alpha_2 x_3 = 0$$
$$x_0 + \alpha_3 x_1 + x_2 = 0$$

$$x_0 + \alpha_3 x_2 + x_3 = 0$$
$$x_1 + x_2 + \alpha_3 x_3 = 0$$
$$S_{17} : x_0 + x_1 + \alpha_2 x_2 + \alpha_2 x_3 = 0$$
$$x_0 + \alpha_2 x_1 + x_2 = 0$$
$$x_0 + \alpha_3 x_1 + \alpha_3 x_3 = 0$$

where (x_0, x_1, x_2, x_3) is a point of PG(3,2^2). As an illustration, the choice of A matrix corresponding to S_1 and S_4 are given respectively by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & \alpha_2 & 0 \\ 0 & 1 & 0 & \alpha_3 \end{pmatrix}.$$

Similarly A matrices for other S_i's can be written.

Choice of Covariates

From (4), it is seen that any block of the design contains $k = (s^{N-1}+\lambda)$ treatments and any two blocks have exactly λ treatments in common. As any two blocks of the set S_i ($i=1(1)t$; $t=(s^{N-1}+s^{N-3}+\ldots+s^2+1)$), have the same λ treatments common, without loss of any generality, the portion N_i of the incidence matrix corresponding to the blocks in S_i ($i=1(1)t$) can be written in the following form (with some rearrangement of blocks if necessary):

$$
N_i = \begin{pmatrix} 1'_{s^{N-1}} & 0' & . & . & . & . & 0' & 1'_\lambda \\ 0' & 1'_{s^{N-1}} & . & . & . & . & 0' & 1'_\lambda \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0' & 0' & . & & & & 1'_{s^{N-1}} & 1'_\lambda \end{pmatrix}^{s+1 \times v} \tag{7}
$$

The part of the incidence matrix of the design with parameters in (5) corresponding to the part N_i of the design with parameters in (4) is obtained by replacing one's by zero's and zero's by one's in (7) and is given by :

$$
N_i^c = \begin{pmatrix} 0'_{s^{N-1}} & 1'_{s^{N-1}} & . & . & . & . & 1'_{s^{N-1}} & 0'_\lambda \\ 1'_{s^{N-1}} & 0'_\lambda & . & . & . & . & 1'_{s^{N-1}} & 0'_\lambda \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1'_{s^{N-1}} & 1'_{s^{N-1}} & . & & & & 0'_{s^{N-1}} & 0'_\lambda \end{pmatrix}^{s+1 \times v} \tag{8}
$$

Using the structure (8) above, a method for choosing the values of the covariates optimally for the complementary design with parameters in (5) is developed.

Theorem 1:

If $s=2^p$ where p be any positive integer, $(s^{N-1}-1)(s-1)+(s-1)$, W-matrices for the design with parameters in (5), where N is an odd integer can be constructed.

Proof

Because s is a power of 2, Hadamard matrices of orders s^{N-1} and s exist and can be written as follows:

$$
H_{s^{N-1}} = \left(h_1, \cdots, h_{s^{N-1}-1}, 1 \right)
$$
$$
H_s = \left(h_1^*, \cdots, h_{s-1}^*, 1 \right) \tag{9}
$$

where $1'$ is a row vector with all elements equal to one. Again the matrix (8) can be written as

$$
N_i^c = \left(A_{1i}, A_{2i}, \cdots, A_{ji}, \cdots, A_{s+1i}, 0_i \right)
$$

where A_{ji} is the jth partitioned matrix in the jth column block of N_i^c, $j = 1(1)(s+1)$. Let the kth non-null row of A_{ji} be replaced by the kth row of $h_m^* h_n'$; $k = 1(1)s$ and the resultant matrix be denoted by A_{ji}^*. The procedure for each A_{ji} is repeated with the same $h_m^* h_n'$. This leads to a matrix $W_{i,m,n}^*$ with elements ± 1 satisfying the properties C_1 and C_2. Using the same h_m^* and h_n different $W_{i,m,n}^*$'s corresponding to different N_i^c's are obtained. Now for fixed h_m^* and h_n

$$
W_{m,n}^* = \begin{pmatrix} W_{1,m,n}^* \\ W_{2,m,n}^* \\ . \\ . \\ . \\ W_{t,m,n}^* \end{pmatrix}
$$

satisfies the properties C_1 and C_2. By varying h_m and h_n^*, $(s^{N-1}-1)(s-1)$, $W_{m,n}^*$ − matrices can be constructed. The transformation required to apply on (8) to get back the corresponding portion of the incidence matrix of the design may also be applied on the elements of the above W*- matrix to get the original W-matrix. It is clear that such W-matrices also satisfy all the properties C_1, C_2 and C_3.

Again, note that the number of unit vectors in the rows of N_i^c is s which is the same as that of the elements of h_m^*. Let the qth vector $1'_{s^{N-1}}$ be replaced in the first column

block matrix of N_i^c by $+1'_{s_{N-1}}$ or by $-1'_{s_{N-1}}$ according as the qth element of h_m^* is +1 or -1 respectively to get A_1^{**}. Now the rows of A_1^{**} are permuted cyclically to get $A_2^{**}, A_3^{**}, ..., A_{s+1}^{**}$ and hence a new W-matrix viz. W_m^{**} can be constructed. It is easy to show that these W_m^{**} matrices together with $W_{m,n}^{**}$ satisfy all the conditions C_1, C_2 and C_3. In all, $(s^{N-1}-1)(s-1)+(s-1)$, W-matrices exist. The procedure is illustrated through the following example.

Example: 3

The SBIBD whose blocks are 2-flats of PG (3,2) is considered so that the parameters of the SBIBD are v=b=15, r=k=7, λ=3. Now the complement of this design has parameters v'=b'=15, r'=k'=8, λ'=4.

The sets of blocks of the complementary design of Example 1 where the treatment corresponding to the point (x_0, x_1, x_2, x_3) is indexed by $2^3x_0+2^2x_1+2x_2+x_3$ are:

S_1 = [(8,9,10,11,12,13,14,15), (2,3,4,5,10,11,12,13), (2,3,4,5,8,9,14,15)]
S_2 = [(4,5,6,7,12,13,14,15), (1,3,5,7,8,10,12,14), (1,3,4,6,8,10,13,15)]
S_3 = [(2,3,6,7,10,11, 14,15), (1,3,4,6,9,11,12,14), (1,2,4,7,9,10,12,15)]
S_4 = [(1,3,5,7,9,11, 13,15), (2,3,6,7,8,9,12,13), (1,2,5,6,8,11,12,15)]
S_5 = [(4,5,6,7,8,9,10,11), (1,2,5,6,9,10,13,14), (1,2,4,7,8,11,13,14)].

The Hadamard matrices of orders 2 and 4 exist and are written as:

$$H_2 = \begin{pmatrix} +1 & +1 \\ -1 & +1 \end{pmatrix} = [h_1, 1]$$

and $H_4 = \begin{pmatrix} +1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 \\ -1 & +1 & -1 & +1 \end{pmatrix} = [h_1^*, h_2^*, h_3^*, 1]$

Using h_1 and h_i^* (i = 1(1)3) and proceeding as in Theorem 1, three W-matrices can be constructed. The construction of a W-matrix viz. W_{11}^* is illustrated using h_1 and h_1^*:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\
0 & +1 & -1 & +1 & -1 & 0 & 0 & 0 & 0 & -1 & +1 & -1 & +1 & 0 & 0 \\
0 & -1 & +1 & -1 & +1 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & -1 & +1 \\
0 & 0 & 0 & +1 & +1 & -1 & -1 & 0 & 0 & 0 & 0 & +1 & +1 & -1 & -1 \\
+1 & 0 & -1 & 0 & -1 & 0 & +1 & +1 & 0 & -1 & 0 & -1 & 0 & +1 & 0 \\
-1 & 0 & +1 & -1 & 0 & +1 & 0 & -1 & 0 & +1 & 0 & 0 & -1 & 0 & +1 \\
0 & +1 & +1 & 0 & 0 & -1 & -1 & 0 & 0 & +1 & +1 & 0 & 0 & -1 & -1 \\
+1 & 0 & -1 & -1 & 0 & +1 & 0 & 0 & +1 & 0 & -1 & -1 & 0 & +1 & 0 \\
-1 & -1 & 0 & +1 & 0 & 0 & +1 & 0 & -1 & -1 & 0 & +1 & 0 & 0 & +1 \\
+1 & 0 & +1 & 0 & -1 & 0 & -1 & 0 & +1 & 0 & +1 & 0 & -1 & 0 & -1 \\
0 & +1 & -1 & 0 & 0 & -1 & +1 & +1 & -1 & 0 & 0 & -1 & +1 & 0 & 0 \\
-1 & -1 & 0 & 0 & +1 & +1 & 0 & -1 & 0 & 0 & -1 & +1 & 0 & 0 & +1 \\
0 & 0 & 0 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & 0 & 0 & 0 & 0 \\
+1 & -1 & 0 & 0 & -1 & +1 & 0 & 0 & -1 & +1 & 0 & 0 & +1 & -1 & 0 \\
-1 & +1 & 0 & -1 & 0 & 0 & +1 & -1 & 0 & 0 & +1 & 0 & -1 & +1 & 0
\end{pmatrix}$$

Similarly, by taking the combinations (h_1, h_2^*), (h_1, h_3^*) W_{12}^* and W_{13}^* can be constructed. Another matrix W_1^{**} using h_1 is given below:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & +1 & +1 & +1 & +1 & -1 & -1 \\
0 & +1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 \\
0 & -1 & -1 & -1 & -1 & 0 & 0 & +1 & +1 & 0 & 0 & 0 & 0 & +1 & +1 \\
0 & 0 & 0 & -1 & +1 & -1 & +1 & 0 & 0 & 0 & 0 & +1 & -1 & +1 & -1 \\
+1 & 0 & +1 & 0 & -1 & 0 & -1 & +1 & 0 & +1 & 0 & -1 & 0 & -1 & 0 \\
-1 & 0 & -1 & +1 & 0 & +1 & 0 & -1 & 0 & -1 & 0 & 0 & +1 & 0 & +1 \\
0 & -1 & +1 & 0 & 0 & +1 & -1 & 0 & 0 & -1 & +1 & 0 & 0 & +1 & -1 \\
+1 & 0 & -1 & +1 & 0 & -1 & 0 & 0 & +1 & 0 & -1 & +1 & 0 & -1 & 0 \\
-1 & +1 & 0 & -1 & 0 & 0 & +1 & 0 & -1 & +1 & 0 & -1 & 0 & 0 & +1 \\
-1 & 0 & +1 & 0 & -1 & 0 & +1 & 0 & +1 & 0 & -1 & 0 & +1 & 0 & -1 \\
0 & +1 & -1 & 0 & 0 & +1 & -1 & +1 & -1 & 0 & 0 & +1 & -1 & 0 & 0 \\
+1 & -1 & 0 & 0 & +1 & -1 & 0 & -1 & 0 & 0 & +1 & -1 & 0 & 0 & +1 \\
0 & 0 & 0 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & 0 & 0 & 0 & 0 \\
+1 & +1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & +1 & +1 & 0 \\
-1 & -1 & 0 & +1 & 0 & 0 & +1 & +1 & 0 & 0 & +1 & 0 & -1 & -1 & 0
\end{pmatrix}$$

Thus four W–matrices W_{11}^*, W_{12}^*, W_{13}^* and W_1^{**} are constructed satisfying conditions C_1-C_3.

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