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On a Test of Independence via Quantiles that is Sensitive to Curvature

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Let $(Y_i, \mathbf{X}_i)$, $i = 1,...,n$, be a random sample from some p+1 variate distribution where $\mathbf{X}_i$ is a vector having length p. Many methods for testing the hypothesis that Y is independent of $\mathbf{X}$ are relatively insensitive to a broad class of departures from independence. Power improvements focus on the median of Y or some other quantile and test the hypothesis that the regression surface is a horizontal plane versus some unknown form. A wild bootstrap method (Stute et al. 1998) can be used based on quantiles, but with small or moderate sample sizes, control over the probability of a Type I error can be unsatisfactory when sampling from asymmetric distributions. He and Zhu (2003) is readily adapted to testing the hypothesis that the conditional γ quantile of Y does not depend on $\mathbf{X}$ where critical values are determined via simulations. A modification is suggested that avoids the need for simulations to obtain critical values, and perform wells in terms of Type I errors even when sampling from asymmetric distributions.

Keywords: Curvature, quantile regression, robust methods.

Introduction

Let $(Y_i, \mathbf{X}_i)$, $i = 1,...,n$, be a random sample from some p+1 variate distribution where $\mathbf{X}_i$ is a vector of length p. Certainly one of the most common methods for attempting to detect an association between Y and $\mathbf{X}$ is to test the hypothesis that the corresponding (Pearson) correlations are zero using Student's t test. One well-known limitation of this approach is that true associations can be missed due to curvature.

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Another concern is that classic methods can be highly unsatisfactory in terms of controlling the probability of a Type I error.

For example, a general method for testing the hypothesis of independence among sets of variables, which assumes multivariate normality, is available (e.g., Muirhead, 1982, chapter 11). As a special case, the method can be used to test

$$H_0 : \rho_{y1} = ... = \rho_{yp} = 0$$

where ρ_{yj} is Pearson's correlation between Y and X_j, j=1,…,p. But it is known that the level of this test cannot be controlled in an adequate fashion (e.g., Reddon, Jackson, & Schopflocher, 1985; Wilcox, 1997). One could use a method based on Fisher's r-to-z transformation, but this can be unsatisfactory when sampling from nonnormal distributions because under general conditions, Fisher's transformation is not even asymptotically correct; it results in using the wrong standard error (Duncan and Layard, 1973). Geiser and Randle (1997) derived a nonparametric method, but it assumes that Y has a symmetric distribution and that the distribution of $\mathbf{X}$ is elliptically symmetric, a restriction that is avoided here. There are many robust correlation coefficients as well as robust regression methods that might be used (e.g., Wilcox, 2005), as well as improved methods when focusing on Pearson's correlation (e.g., Boik & Haaland, 2006), but it is evident that these methods can miss true associations that are detected when focusing on quantiles via the method considered here.

Let $m(\mathbf{X}) = E(Y \mid \mathbf{X})$ and let $\mu_y = E(Y)$. A general and relatively flexible approach to detecting dependence is to test

$$H_0 : m(\mathbf{X}) = \mu_y \qquad (1)$$

versus the alternative hypothesis that $m(\mathbf{X})$ depends in some (unspecified) manner on $\mathbf{X}$, possibly in a nonlinear fashion. A test of (1) can be performed using a method that stems from general theoretical results reported by Stute, Gonzalez Manteiga and Presedo Quindimil (1998) who were concerned with testing the hypothesis that a regression surface belongs to a specified family of functions. Unlike conventional methods, by design the method is not sensitive to heteroscedasticity. That is, if we model the data with $Y = m(\mathbf{X}) + \lambda(\mathbf{X})\varepsilon$, where the error term ε independent of $\mathbf{X}$, and $\lambda(\mathbf{X})$ is some unknown function. The assumption $\lambda(\mathbf{X}) \equiv 1$ (homoscedasticity) is not made nor required when testing (1). In principle, the method can be extended by replacing the conditional mean of Y with the median or some other robust estimator. When ε has a symmetric distribution, control over the probability of a Type I error has been found to be satisfactory in simulations, but when ε has an asymmetric distribution, this is no longer the case (Wilcox, 2007).

Let $Y_\gamma(\mathbf{X})$ be the conditional γ quantile of Y given $\mathbf{X}$. A general method derived by He and Zhu (2003) is readily adapted to the problem of testing

$$H_0 : Y_\gamma(\mathbf{X}) = Y_\gamma \qquad (2)$$

where Y_γ is the γ quantile of the marginal distribution of Y. The .5 quantile is perhaps the most obvious choice, but in some situations associations are more pronounced when considering other quantiles, and in some cases other quantiles are intrinsically interesting. The He and Zhu method is based in part on using simulations to estimate the null distribution of their test statistic. Execution time is reasonably low with small sample sizes, but despite the speed of modern computers, execution time can be high. For example, with a sample size of n=100 and p=4, execution time was over 8 minutes on a SUN BLADE 150.

The goal in this paper is to suggest a simple modification of the method derived by He and Zhu (2003) that, for a wide range of situations, can be used to test (2) without resorting to simulation estimates of critical values.

Simulation results reported here find that the actual level of the test is reasonably close to the nominal level, even when sampling from asymmetric distributions and there is a fair degree of heteroscedasticity.

Method

Let $\mathbf{x}$ be the n by (p+1) matrix with the first column containing all ones and the remaining p columns are the columns of $\mathbf{X}$. Following He and Zhu, it is assumed that the design has been normalized so that $n^{-1}\sum \mathbf{x}_j \mathbf{x}_j' - \mathbf{I} = o(1)$.

Let $r_i = Y_i - \hat{Y}_\gamma$, where $\hat{Y}_\gamma$ is some estimate of the γth quantile of Y. Here, the focus is on the quartiles. For the .5 quantile, $\hat{Y}_{.5}$ is taken to be the usual sample median. For the lower and upper quartiles, the so-called ideal fourths are used (e.g., Frigge, Hoaglin, and Iglewicz, 1989), which are computed as follows. Let j=(n/4)+(5/12), rounded down to the nearest integer. Let

$$h = \frac{n}{4} + \frac{5}{12} - j\,.$$

Then the estimate of the lower quartile is given by

$$q_1 = (1-h)Y_{(j)} + hY_{j+1},$$

where $Y_{(1)} \le \ldots \le Y_{(n)}$. Letting k=n-j+1, the estimate of the upper quartile, is

$$q_2 = (1-h)Y_{(k)} + hY_{k-1}$$

There are many other quantile estimators, comparisons of which are reported by Parrish (1990) as well as Dielman, Lowry, and Pfaffenberger (1994). Perhaps they offer some practical advantage for the situation at hand, but this is not pursued here.

Following He and Zhu, for any $\mathbf{x}$, $\mathbf{t} \in R^p$, $\mathbf{x} \le \mathbf{t}$ if and only if each component of $\mathbf{x}$ is less than or equal to each component of $\mathbf{t}$. Let $\psi(r) = \gamma I(r>0) + (\gamma-1)I(r<0)$, and let

$$\mathbf{R}_n\mathbf{t} = n^{-1/2}\sum_{j=1}^{n}\psi(r_j)I(\mathbf{x}_j \le \mathbf{t})$$

The He and Zhu test statistic, for the situation at hand, is

$$T_n = \max_{\|a\|=1} n^{-1}\sum(\mathbf{a}'\mathbf{R}_n(\mathbf{x}_j))^2 \qquad (3)$$

the largest eigenvalue of $n^{-1}\sum \mathbf{R}_n(\mathbf{x}_i)\mathbf{R}_n'(\mathbf{x}_i)$. A simple strategy for determining an appropriate critical value is to temporarily assume normality, use simulations to approximate the 1-α quantile of the null distribution, say c, and then reject the null hypothesis if $T_n \ge c$ even when sampling from a non-normal distribution. It was found, however, that this strategy performed in an unsatisfactory manner, in simulations, when sampling from heavy-tailed distributions. (The actual Type I error probability can exceed .08 when testing at the .05 level.) However, a simple modification was found to give better results. The modification consists of using a different partial ordering on the design space; otherwise the test statistic is computed in the same manner as T_n. Let

$$\mathbf{R}_n(\mathbf{x}_i) = n^{-1/2}\sum_{k=1}^{n}\psi(r_k)\mathbf{x}_k I(\mathbf{x}_k \le \mathbf{x}_i)$$

For fixed j, let U_{ij} be the ranks of the n values in the jth column of $\mathbf{x}$, j=2,…,q. Let $F_i = \max U_{ij}$, the maximum being taken over j=2,…,q. If $\mathbf{x}_k \le \mathbf{x}_i$, then $F_k \le F_i$. Let

$$\mathbf{W}_i = n^{-1/2}\sum_{k=1}^{n}\psi(r_k)\mathbf{x}_k I(F_k \ge F_i)$$

The test statistic used here is D_n, the largest eigenvalue of

$$\mathbf{Z} = n^{-1}\sum \mathbf{W}_i\mathbf{W}_i'$$

Numerical checks on this test statistic indicate that it is invariant when the design space, $\mathbf{X}$, is shifted in location. This is in contrast to a related method for testing the fit of a quantile regression model, currently under investigation, which Xuming He (personal communication) pointed out does not enjoy this property. (Centering the design space eliminates this problem, but here this does not seem to be necessary.)

Note that a major component of the test statistic D_n is invariant under monotone transformations of the covariates; only the ranks of the marginal distributions of $\mathbf{X}$ are needed. However, the test statistic can be affected by monotone transformations because this can alter the $\psi(r_i)$ values. But it was found among the simulations reported later in this paper that typically the $\psi(r_i)$ values are altered by a relatively small amount suggesting a simple approach toward determining an appropriate critical value: Momentarily assume that both $\mathbf{X}$ and ε have standard normal distributions, then use simulations to determine a critical value given n, p and γ, and use this critical value for the more general case where $\mathbf{X}$ and ε do not have normal distributions. (In essence, this is the same strategy used by Gosset to derive Student's T test.)

Simulations are not needed once a critical value has been estimated. (For p>1, all indications are that it suffices to assume that the correlations among the covariates are zero when determining critical values.)

Some Special Cases

Simulations were used to approximate critical values in the manner just described for p=1,…,8 predictors; n=10, 20, 30, 50, 100, 200 and 400; γ=.5, .25 and .75; and α=.1, .05, .025 and .01. The results are reported in Tables 1-4. Regarding sample sizes not tabled, it was found that there is an approximately linear association between the α level critical value, c_α, and 1/n suggesting that a single regression line might be used to determine c_α given 1/n. However, slightly better control over the probability of a Type I error is obtained by using the critical values in Tables 1-4 and interpolating on 1/n for critical values not tabled.

Results

Simulations were used to the check the small sample properties of the method just described. Included were situations where p=1 and 4, γ=.5, .25 and .75, and where for p=4 there is a common correlation ρ or .5. Here the results for n=20, ρ=.5 and γ=.75 are reported because the largest deviations from the nominal Type I error probability occurred for this special case. In the simulations, observations were generated with the model

$$Y_i = \lambda(X_{i1})\varepsilon,$$

where λ is some function for modeling heteroscedasticity. The distribution of ε was taken to be one of four g-and-h distributions (Hoaglin, 1985), which contains the standard normal distribution as a special case. If Z has a standard normal distribution, then

$$W = \frac{\exp(gZ)-1}{g}\exp(hZ^2/2)$$

if g>0. has a g-and-h distribution where g and h are parameters that determine the first four moments. When g=0, then

$$W = Z\exp(hZ^2/2).$$

The four distributions used here were the standard normal (g=h=0), a symmetric heavy-tailed distribution (h=.2, g=0), an asymmetric distribution with relatively light tails (h=0, g=.2), and an asymmetric distribution with heavy tails (g=h=.2).

Table 5 shows the skewness (κ_1)and kurtosis (κ_2) for each distribution considered. When g>0 and h>1/k, $E(W^k)$ is not defined and the corresponding entry in Table 1 is left blank. Additional properties of the g-and-h distribution are summarized by Hoaglin (1985).

The function λ was chosen to reflect three types of variance patterns: $\lambda(\mathbf{X})$=1 (homoscedasticity) $\lambda(\mathbf{X})=X_1^2$, and $\lambda(\mathbf{X})=1+1/(|X_1|+1)$. For convenience, these three λ functions will be called variance patterns VP1, VP2, and VP3.

Each replication in the simulations consisted of generating n vectors for **X**, n values for ε, determining Y according to equation (3), then applying the test of (2). Here, 1,000 replications were used to estimate the actual probability of a type I error. With 1,000 replications, if the actual probability of a type I error is .05, the standard error associated with the proportion of rejections is .007.

Table 6 shows the estimated Type I error probabilities for n=20, p=4, a common correlation ρ=.5, and α=.05. As can be seen, the estimates range between .039 and .071. There are only two situations where the estimate is greater than or equal to .07.

Table 1: Critical values for γ=.5, $1 \le p \le 4$

p=1

n	α=.100	α=.050	α=.025	α=.001
10	0.033939	0.04408	0.050923	0.064173
20	0.015323	0.021007	0.027687	0.032785
30	0.010648	0.014778	0.01825	0.023639
50	0.006619	0.009078	0.011691	0.014543
100	0.003156	0.004375	0.005519	0.007213
200	0.001545	0.002232	0.002748	0.003726
400	0.000772	0.001022	0.001371	0.001818

p=2

n	α=.100	α=.050	α=.025	α=.001
10	0.052848	0.061919	0.071347	0.079163
20	0.021103	0.027198	0.031926	0.035084
30	0.013721	0.018454	0.022177	0.026052
50	0.00839	0.01059	0.012169	0.015346
100	0.004262	0.005514	0.007132	0.008417
200	0.001895	0.002416	0.003085	0.003925
400	0.001045	0.001348	0.001579	0.001864

p=3

n	α=.100	α=.050	α=.025	α=.001
10	0.071556	0.082938	0.089555	0.097538
20	0.031061	0.035799	0.043863	0.053712
30	0.019504	0.023776	0.02718	0.030991
50	0.01103	0.013419	0.015557	0.01798
100	0.005634	0.006805	0.007878	0.008808
200	0.002552	0.003604	0.004276	0.005022
400	0.001251	0.001532	0.001801	0.002038

p=4

n	α=.100	α=.050	α=.025	α=.001
10	0.093268	0.101584	0.108734	0.11834
20	0.038678	0.04552	0.051403	0.060097
30	0.024205	0.02936	0.034267	0.039381
50	0.013739	0.015856	0.018066	0.019956
100	0.006468	0.007781	0.009038	0.010127
200	0.003197	0.003934	0.004657	0.005929
400	0.001653	0.001926	0.002364	0.002657

Table 2: Critical values for $\gamma=.5,\ 5\leq p\leq 8$

p=5

n	α=.100	α=.050	α=.025	α=.001
10	0.117217	0.124714	0.129459	0.136456
20	0.048839	0.055609	0.06058	0.067944
30	0.030595	0.035004	0.040434	0.047649
50	0.01694	0.019527	0.022047	0.025313
100	0.008053	0.009779	0.01149	0.013384
200	0.003761	0.004376	0.005098	0.005866
400	0.001895	0.002254	0.002612	0.002939

p=6

n	α=.100	α=.050	α=.025	α=.001
10	0.136962	0.14412	0.149004	0.152667
20	0.055909	0.062627	0.069978	0.08119
30	0.034635	0.040741	0.044161	0.047722
50	0.020165	0.023075	0.025881	0.02848
100	0.009436	0.011247	0.013221	0.015101
200	0.004645	0.005334	0.006041	0.007237
400	0.002278	0.002636	0.002997	0.003669

p=7

n	α=.100	α=.050	α=.025	α=.001
10	0.15618	0.16322	0.17175	0.17714
20	0.07011	0.07705	0.08272	0.09041
30	0.04177	0.04737	0.0531	0.05767
50	0.02338	0.02601	0.0296	0.03261
100	0.01085	0.01256	0.01374	0.01625
200	0.00516	0.00613	0.00686	0.00835
400	0.00253	0.00304	0.00362	0.00397

p=8

n	α=.100	α=.050	α=.025	α=.001
10	0.17839	0.18	0.19379	0.19958
20	0.07803	0.08562	0.09151	0.10249
30	0.04599	0.05218	0.05736	0.06263
50	0.02589	0.02973	0.03374	0.03787
100	0.01219	0.01365	0.01548	0.01771
200	0.00589	0.00687	0.00789	0.00852
400	0.00283	0.00324	0.00373	0.00412

Table 3: Critical values for γ=.25 or .75, $1 \le p \le 4$

p=1

n	α=.100	α=.050	α=.025	α=.001
10	0.029933	0.039598	0.054088	0.062961
20	0.011122	0.014989	0.018154	0.022685
30	0.009207	0.011302	0.014872	0.019931
50	0.004824	0.00704	0.010357	0.013177
100	0.00237	0.003315	0.004428	0.005123
200	0.001106	0.001611	0.001984	0.00265
400	0.000517	0.00068	0.000869	0.001202

p=2

n	α=.100	α=.050	α=.025	α=.001
10	0.044842	0.06026	0.066001	0.087041
20	0.017341	0.022471	0.027371	0.033436
30	0.012121	0.015041	0.018939	0.022644
50	0.006489	0.008461	0.0107	0.013232
100	0.002973	0.004064	0.004911	0.005769
200	0.001515	0.002058	0.002583	0.003114
400	0.000798	0.000993	0.001183	0.001399

p=3

n	α=.100	α=.050	α=.025	α=.001
10	0.063653	0.072975	0.083841	0.097222
20	0.021659	0.027437	0.031875	0.03683
30	0.01529	0.018964	0.021729	0.02896
50	0.008357	0.010072	0.012713	0.015255
100	0.003903	0.004764	0.005577	0.00666
200	0.001914	0.002343	0.002834	0.003465
400	0.00096	0.001147	0.001356	0.001548

p=4

n	α=.100	α=.050	α=.025	α=.001
10	0.085071	0.095948	0.104197	0.11845
20	0.029503	0.034199	0.039543	0.045044
30	0.019203	0.022769	0.026887	0.033482
50	0.01144	0.013555	0.016139	0.018298
100	0.004863	0.005756	0.007385	0.009115
200	0.002635	0.003111	0.003769	0.004216
400	0.001189	0.001435	0.001728	0.001956

Table 4: Critical values for γ=.25 and .75, $5 \le p \le 8$

p=5

n	α=.100	α=.050	α=.025	α=.001
10	0.102894	0.114259	0.122545	0.130222
20	0.036733	0.042505	0.048664	0.055457
30	0.024193	0.028806	0.032924	0.03821
50	0.012663	0.014635	0.017276	0.019736
100	0.006106	0.007311	0.00896	0.009745
200	0.003067	0.003615	0.003998	0.004812
400	0.001441	0.001733	0.002079	0.002308

p=6

n	α=.100	α=.050	α=.025	α=.001
10	0.117643	0.126566	0.133107	0.14228
20	0.044309	0.049732	0.053913	0.060513
30	0.028607	0.033826	0.038616	0.043547
50	0.015445	0.017557	0.020041	0.022748
100	0.007335	0.008406	0.009392	0.01092
200	0.003352	0.003815	0.004381	0.005252
400	0.001704	0.002002	0.002339	0.002773

p=7

n	α=.100	α=.050	α=.025	α=.001
10	0.106573	0.113059	0.117388	0.121287
20	0.05217	0.058363	0.064734	0.069749
30	0.030697	0.035507	0.039266	0.044438
50	0.016737	0.019606	0.021254	0.022923
100	0.007767	0.009232	0.010341	0.011471
200	0.003998	0.00459	0.005507	0.006217
400	0.001903	0.002175	0.002519	0.002859

p=8

n	α=.100	α=.050	α=.025	α=.001
10	0.119571	0.126977	0.130121	0.133258
20	0.0595	0.067185	0.071283	0.079431
30	0.034311	0.039827	0.044452	0.048512
50	0.0186	0.021094	0.023273	0.027471
100	0.009136	0.010902	0.012289	0.01373
200	0.004382	0.005192	0.005598	0.006484
400	0.002197	0.002526	0.002819	0.003242

Table 5: Some properties of the g-and-h distribution

g	h	κ_1	κ_2
0.0	0.0	0.0	3.00
0.0	0.2	0.0	21.46
0.2	0.0	0.61	3.68
0.2	0.2	2.81	155.98

Table 6: Estimated Probability, $\hat{\alpha}$, of a Type I error, n=20, p=4, γ=.75, ρ=.5

x		ε		$\hat{\alpha}$		
g	h	g	h	VP1	VP2	VP3
0.0	0.0	0.0	0.0	0.058	0.057	0.059
0.0	0.0	0.0	0.2	0.052	0.061	0.053
0.0	0.0	0.2	0.0	0.066	0.065	0.048
0.0	0.0	0.2	0.2	0.059	0.063	0.049
0.0	0.2	0.0	0.0	0.041	0.046	0.039
0.0	0.2	0.0	0.2	0.043	0.050	0.045
0.0	0.2	0.2	0.0	0.052	0.065	0.046
0.0	0.2	0.2	0.2	0.046	0.066	0.055
0.2	0.0	0.0	0.0	0.061	0.059	0.058
0.2	0.0	0.0	0.2	0.049	0.051	0.070
0.2	0.0	0.2	0.0	0.047	0.071	0.062
0.2	0.0	0.2	0.2	0.054	0.064	0.054
0.2	0.2	0.0	0.0	0.050	0.056	0.053
0.2	0.2	0.0	0.2	0.042	0.045	0.047
0.2	0.2	0.2	0.0	0.045	0.059	0.049
0.2	0.2	0.2	0.2	0.045	0.060	0.049

Conclusion

One of the main points is that when dealing with the quartiles, the method considered here continues to perform well in simulations, in terms of Type I errors, when sampling from skewed distributions, in contrast to the wild bootstrap method in Stute et al. (1998). Given the ease and flexibility of the method, all indications are that it has practical value. For situations where interpolation is not possible based on the results in Tables 1-4, simulations are still needed to determine critical values, but the results reported here indicate that this needs to be done only once. That is, given n, p and γ, critical values can be determined via simulations, stored in a table, and then used in future studies where these values for n, p and γ occur. An R and S-plus function for applying the method (called medind) is available from the author upon request.

Finally, the modification considered here can be extended to the situation where the goal is to test the fit of a linear quantile regression model. Preliminary results indicate that alternative critical values are now needed and that now critical values have an approximately linear association with $n^{-1.5}$ rather than $1/n$, as was the case here.

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