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An alternative Wald type test called the quel test is developed for two linear restrictions by finding the critical region based on the quel utilizing the repeated values of estimated parameters of interest under the null. Simulation shows evidence that the full quel test performs best in that it holds nominal level well and shows monotonic increasing power properties.

Key words: Bootstrap technique, nonlinear models, percentile confidence contour, power, quel, size.

Introduction

Considerable interest exists in testing linear restrictions in the nonlinear models such as logit, Tobit and exponential models. For testing such hypotheses in the context of nonlinear models, an asymptotic test such as the Wald test or the likelihood ratio test is usually employed. The Wald test has an advantage over the likelihood ratio test since the Wald test requires the maximum likelihood estimates of the parameters only under the alternate hypothesis.

Unfortunately, for small samples the Wald test does not perform well in terms of size and power property. In some situations, the power of the Wald test first increases then eventually starts decreasing when alternative hypothesis parameters increase in distance from the null hypothesis. The Wald test behaves this way because, for certain parameter values, the estimated covariance matrix of the maximum likelihood estimator increases faster than the square of the distance between the parameter estimate and null value (see, for example, Hauck & Donner, 1977; Vaeth, 1985; Mantel, 1987; Nelson & Savin, 1988, 1990). Moreover, the biased estimates of the parameters being tested can cause the power of the Wald test to drop below its size at local alternatives (for example see, Goh, 1998). These two types of behavior discussed above are usually known as non-monotonicity in the power function and local biasedness respectively.

Other important situations exist in which the estimated covariance matrix cannot be assessed and may not have an explicit form. For example, testing for the presence of first-order moving average disturbances in a linear regression model the information matrix is not well defined if the parameter of the moving average process is 1 or -1 (see Goh, 1998).

This article introduces to construct the alternative Wald type tests that do not depend on the estimated covariance matrix, and use nonparametric ideas and computer simulation to judge whether the estimates observed are likely to have come from a null hypothesis data generating process. Applying the above concepts, we construct the bivariate generalizations of the boxplot based on a generalized quel introduced by Goldberg and Iglewicz (1992) which is defined as four separate quarter ellipses matched on their major and minor axes so that the quel is continuous and smooth. Previously, many authors including Turkey (1947), Scott (1985) and Becketti and Gould (1987) attempted to estimate the confidence contours of a bivariate density, but those approaches had serious shortcomings.

The primary aim of this article is to construct new tests that solve the problem of non-monotonicity in the power function, but do not face the limitations discussed above in

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practice. These new tests only require simulated estimates of the parameters of interest under the null hypothesis but do not involve estimating the covariance matrix. Moreover, these new tests require defining the rejection region based on the contour points of the percentile confidence limits of quels (half and full) of the values of estimated parameters of interest under the null hypothesis. Furthermore, the null hypothesis is rejected if the sample data estimates fall outside the percentile confidence limit of a quel (half or full).

Methodology

The Wald and LR Tests

Let $y_1, \cdots, y_n$ be n independent observations distributed with density function $f(y_t \mid x_t, \theta)$, t=1,…,n, where x_t is a vector of covariates and θ is an unknown $k \times 1$ parameter vector. Let $\theta = (\beta', \eta')'$, where $\beta = (\beta_{1,} \beta_2)'$ are two parameters of interest and η is a $(k-2) \times 1$ vector of nuisance parameters. The log-likelihood function is given by $l(\theta) = \sum_{t=1}^{n} \ln f(y_t \mid x_t, \theta)$. The interest lies in testing the composite hypotheses

$$H_0 : \beta = \beta_0 \quad \text{against} \quad H_1 : \beta \neq \beta_0, \qquad (1)$$

where β_0 is a 2×1 vector of known constants. Let $\hat{\theta} = (\hat{\beta}', \hat{\eta}')'$ be the maximum likelihood estimators of θ under the alternative hypotheses. Then the Wald test statistic is

$$W = (\hat{\beta} - \beta_0)(R\hat{V}(\hat{\theta})R')^{-1}(\hat{\beta} - \beta_0), \quad (2)$$

where $R = (I_2 : 0)$, I_2 is the 2×2 identity matrix, $\hat{V}(\hat{\theta})$ is a constant estimator of $V(\hat{\theta})$ with replace θ by $\hat{\theta}$ and

$$V(\hat{\theta}) = \left(E\left[-\frac{\partial^2 l(\theta)}{\partial \theta_i \partial \theta_j} \right] \right)$$

is the covariance matrix of $\hat{\theta}$. Under the standard regularity conditions (see, for example, Godfrey, 1988), W asymptotically follows a χ^2 distribution with 2 degrees of freedom under the null hypothesis. The null hypothesis is rejected for large values of W (for details see Goh, 1998).

Neyman and Pearson (1928) first proposed the likelihood ratio (LR) test for testing a composite hypothesis. Note that the Wald and LR tests have the same first-order asymptotic properties and they are asymptotically equivalent (see Rao, 1973). Several authors have studied the asymptotic relationship between these two tests (see, for example, Gourieroux and Monfort 1995, Chapter 17; Hendry 1995, Chapter 13). Let $\hat{\theta}_0$ be the maximum likelihood estimators of θ under the null hypotheses. Then the LR test for the hypothesis in (1) involves rejecting the null hypothesis for large values of

$$LR = 2\left[l(\hat{\theta}) - l(\hat{\theta}_0) \right], \qquad (3)$$

which, under the standard regularity conditions, follows a χ^2 distribution with 2 degrees of freedom asymptotically under the null hypothesis.

The Quel (Full or Half) Test

As observed for some nonlinear models, the estimated covariance matrix is not always available. Thus, some new test procedures namely, full quel and half quel tests, for two linear restrictions, are outlined which do not require an expression of this matrix. As only the quel for a two-dimensional case can be constructed (see Goldberg & Iglewicz, 1992), attention is limited to testing problems involving only two restrictions.

The Percentile Confidence Contour Points of a Quel (Full or Half)

Let (u_i, v_i), $i = 1, 2, \cdots, N$, be a set of simulated maximum likelihood estimates of (β_1, β_2) for the i^{th} sample under the null hypothesis. Specifically, $(u_1, v_1), (u_2, v_2), \ldots,$

(u_N, v_N), are bivariate observations of size N for (U,V). Following the method of an asymmetric plot provided by Goldberg and Iglewicz (1992), the two-dimensional confidence contour based on the standardized errors of each point of (U,V) can be found. To obtain these errors, requires first finding the location, scale, and correlation estimators for (U,V) as well as the two additional parameters represented by the proportions of the total standard deviation due to residuals in the positive direction of the major and minor axes of the asymmetric plot. Goldberg and Iglewicz (1992) introduced the estimation of those parameters by an extended biweight bivariate estimator (BIWT) and one-step biweight estimator (BIWT-1) as being efficient. Based on the extended BIWT and BIWT-1 method provided by Goldberg and Iglewicz (1992), the location, scale, and correlation estimators for (U,V) can be easily obtained, as well as the two additional parameters.

Let μ_{ub}^{*q}, μ_{vb}^{*q}, σ_{ub}^{q}, σ_{vb}^{q} and r_{uvb}^{q} be the location, scale, and correlation estimates of U and V respectively, and let γ_1 and γ_2 be the estimates of the two additional parameters. In order to find the boundary points of the confidence contour of a quel, regardless of size, define $G_1 = U_s^q + V_s^q$ and $G_2 = U_s^q - V_s^q$, as the major and minor axes in this order, where U_s^q and V_s^q are the standardized values of U and V based on the location and scale estimators for a quel as $u_{si}^q = (u_i - \mu_{ub}^{*q})/\sigma_{ub}^q$ and $v_{si}^q = (v_i - \mu_{ub}^{*q})/\sigma_{ub}^q$, respectively. Note that $sign(G_1 - G_2) = sign(r_{uvb}^q)$. Therefore, the major and minor axes must be redefined with respect to the correlation estimator r_{uvb}^q as $G_{1i}^* = (u_{si}^q + v_{si}^q)/r_{1uvb}^{*q}$ and $G_{2i}^* = (u_{si}^q - v_{si}^q)/r_{2uvb}^{*q}$, where $r_{1uvb}^{*q} = \sqrt{2(1 + r_{uvb}^q)}$ and $r_{2uvb}^{*q} = \sqrt{2(1 - r_{uvb}^q)}$, respectively. In addition, the standardized errors for the construction of a quel whose percentile point approximately determines the percentile confidence contour of the rejection region for the quel (see Figure 1) must be computed. Compute the standardized errors, ξ_i^q, based on G_{1i}^* and G_{2i}^* using the additional parameter estimates γ_1 and γ_2 as

$$\xi_i^q = \sqrt{\nabla_{1i}^2 + \nabla_{2i}^2}, \quad \text{for } i = 1,2,\cdots,N, \quad (4)$$

where for l = 1, 2,

$$\nabla_{li} = \begin{cases} G_{li}^*/2\gamma_l & \text{if } G_{li}^* > 0 \\ G_{li}^*/2(1-\gamma_l) & \text{otherwise.} \end{cases}$$

Note that these errors assess the distances of each point obtained from the observations of U and V to the center (μ_{ub}^{*q}, μ_{vb}^{*q}). Let $\xi_{percentile}^q$ be the percentile of the standardized errors ξ_i^q (i = 1, 2, …, N) in equation (4).

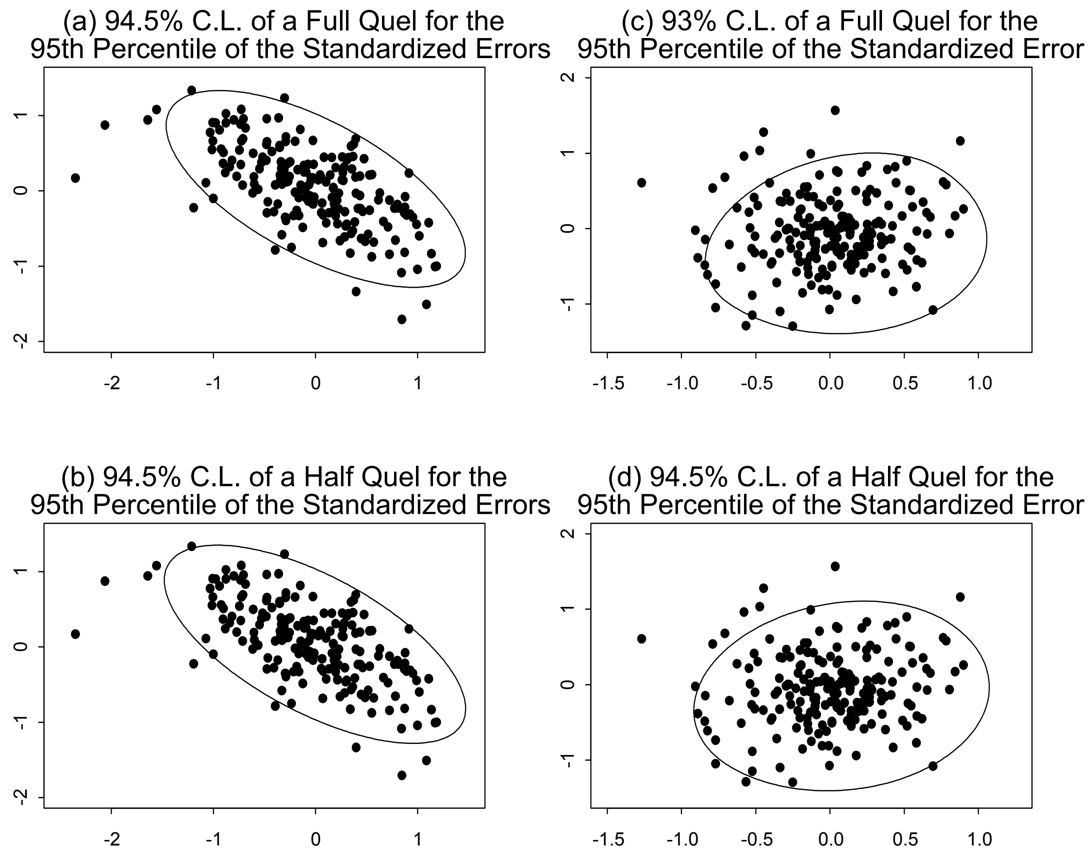
The construction of a quel depends on two things, the percentile of the errors $\xi_{percentile}^q$ and the estimators of two additional parameters γ_1 and γ_2. As a result, options of different values of $\xi_{percentile}^q$, γ_1 and γ_2 create different kinds of quel. In this case, these values are chosen from two different options, which assess two different quels called full and half quels. These two options for constructing full and half quels are discussed in the Appendix.

Upon acquiring the percentile of the standardized errors as well as the two additional estimators from either options for a full or half quel as shown in the Appendix, it is easy to find the boundary points of the percentile confidence limit for the full or half quel. In doing so, based on $\xi_{percentile}^q$ as well as γ_l^{fq} for l = 1, 2 from option-I in the Appendix, the lengths of the vertices in all four quadrants from the origin for the full quel are:

$$\phi_1^{(-1)} = \xi_{percentile}^{fq}\left[2(1-\gamma_1^{fq})\right]\sqrt{\left(1 + r_{uvb}^q\right)/2}$$

$$\phi_1^{(+1)} = \xi_{percentile}^{fq}\left[2(\gamma_1^{fq}\right]\sqrt{\left(1 + r_{uvb}^q\right)/2}$$

Figure 1: Exact Percentile Confidence Contour of a Full or Half Quel for the 95th Percentile of the Standardized Errors ξ_i^q (i = 1, 2, …, N) for a Full or Half Quel when N = 200.*



*The scatter points represent the simulated ML estimates of (β_1, β_2) of size N. The underlying distribution was the two-regressor binary logit model of size n = 30 using design matrix X_3 [(a), (b)] and the three-regressor binary logit model of size n = 30 using design matrix X_2 [(c), (d)].

$$\phi_1^{(+1)} = \xi_{percentile}^{fq}\left[2(\gamma_1^{fq}\right]\sqrt{\left(1 + r_{uvb}^q\right)/2}$$

$$\phi_2^{(+1)} = \xi_{percentile}^{fq}\left[2\gamma_2^{fq}\right]\sqrt{\left(1 - r_{uvb}^q\right)/2}.$$

Next, based on the parametric equations of an ellipse in terms of angle θ^{fq} with range 0 to 360 degrees using $\phi_1^{(-1)}$, $\phi_1^{(+1)}$, $\phi_2^{(-1)}$, and $\phi_2^{(+1)}$ as

$$\Phi_1 = \phi_1^{sign(\cos\theta^{fq})}\cos\theta^{fq}$$

and

$$\Phi_2 = \phi_2^{sign(\sin\theta^{fq})}\sin\theta^{fq}, \qquad (5)$$

the boundary points of the percentile confidence contour for the full quel are given by

$$X^{fq} = \mu_{ub}^q + (\Phi_1 + \Phi_2)\sigma_{ub}^q$$

and

$$Y^{fq} = \mu_{vb}^q + (\Phi_1 - \Phi_2)\sigma_{vb}^q. \qquad (6)$$

In a manner similar to the case of a full quel discussed above, the boundary points of the percentile confidence limit for a half quel can also be obtained as in equation (6) by using equation (5) with $\xi^{fq}_{percentile}$ and γ^{fq}_{l} for l = 1, 2 replaced by $\xi^{hq}_{percentile}$ and γ^{hq}_{l} for l = 1, 2 from option-II in the Appendix.

A Point on the Percentile Confidence Limit of a Quel for a Fixed Angle

Consider the case of using a full quel to find the boundary point P_{H_0}. In doing so, it is necessary to find a solution of an angle θ^{fq}, based on the angle A_{H_1} so that the x and y coordinates of a point P_{H_0} using equations (5) and(6) for a particular solution of this θ^{fq} can easily be obtained. To solve this θ^{fq} based on the angle A_{H_1}, define

$$X^{fq} = D_{H_0} \cos A_{H_1}$$
$$Y^{fq} = D_{H_0} \sin A_{H_1} \tag{7}$$

Using equation (5), results in an updated equation for the solution of an angle θ^{fq} from equations (6) and (7) as given by

$$\mu^q_{vb} + \left[\varphi_1^{sign(\cos\theta^{fq})}\cos\theta^{fq} - \varphi_2^{sign(\sin\theta^{fq})}\sin\theta^{fq}\right]\sigma^q_{vb}$$
$$= \left\{\mu^q_{ub} + \begin{bmatrix}\varphi_1^{sign(\cos\theta^{fq})}\cos\theta^{fq} + \\ \varphi_2^{sign(\sin\theta^{fq})}\sin\theta^{fq}\end{bmatrix}\sigma^q_{ub}\right\}\tan A_{H_1} \tag{8}$$

To obtain the solution of θ^{fq} from (8), start with the combination of ϕ_1 and ϕ_2 from the previous section, which depends on the sign of $\cos\theta^{fq}$ and $\sin\theta^{fq}$. Based on the values of $\cos\theta^{fq}$ and $\sin\theta^{fq}$, the solution for an angle θ^{fq} in four different cases is obtained as follows.

Case I: Values of $\cos\theta^{fq}$ and $\sin\theta^{fq}$ are both positive

$$\varphi_1^{sign(\cos\theta^{fq})} = \varphi_1^{(+1)}$$
$$\varphi_2^{sign(\sin\theta^{fq})} = \varphi_2^{(+1)}. \tag{9}$$

Using (9), solve for the angle θ^{fq} from equation (8), which is

$$\theta^{fq} = \arcsin\frac{2b_1 \pm \sqrt{4a_1^2(a_1^2 - 1 + b_1^2)}}{2(a_1^2 + b_1^2)}, \tag{10}$$

where

$$a_1 = \frac{\phi_1^{(+1)}(\sigma^q_{vb} - \sigma^q_{ub}\tan A_{H_1})}{\mu^q_{ub}\tan A_{H_1} - \mu^q_{vb}}$$

and

$$b_1 = \frac{\varphi_2^{(+1)}(\sigma^q_{vb} + \sigma^q_{ub}\tan A_{H_1})}{\mu^q_{ub}\tan A_{H_1} - \mu^q_{vb}}.$$

Case II: Values of $\cos\theta^{fq}$ and $\sin\theta^{fq}$ are positive and negative respectively

$$\varphi_1^{sign(\cos\theta^{fq})} = \varphi_1^{(+1)}$$
$$\varphi_2^{sign(\sin\theta^{fq})} = \varphi_2^{(-1)} \tag{11}$$

Similar to Case-I, find the solution of an angle θ^{fq} from (8) using equation (11) as

$$\theta^{fq} = \arcsin\frac{2b_2 \pm \sqrt{4a_1^2(a_1^2 - 1 + b_1^2)}}{2(a_1^2 + b_1^2)}, \tag{12}$$

where

$$b_2 = \frac{\phi_2^{(-1)}(\sigma^q_{vb} + \sigma^q_{ub}\tan A_{H_1})}{\mu^q_{ub}\tan A_{H_1} - \mu^q_{vb}}.$$

Case III: Values of $\cos\theta^{fq}$ and $\sin\theta^{fq}$ are, respectively, negative and positive

$$\varphi_1^{sign(\cos\theta^{fq})} = \varphi_1^{(-1)}$$
$$\varphi_2^{sign(\sin\theta^{fq})} = \varphi_2^{(+1)} \tag{13}$$

Using equation (13), obtain the angle θ^{fq} from equation (8) as

$$\theta^{fq} = \arcsin\frac{2b_1 \pm \sqrt{4a_2^2(a_2^2 - 1 + b_1^2)}}{2(a_2^2 + b_1^2)}, \quad (14)$$

where

$$a_2 = \frac{\phi_1^{(-1)}(\sigma_{vb}^q - \sigma_{ub}^q \tan A_{H_1})}{\mu_{ub}^q \tan A_{H_1} - \mu_{vb}^q}.$$

Case IV: Values of $\cos\theta^{fq}$ and $\sin\theta^{fq}$ are both negative

$$\varphi_1^{sign(\cos\theta^{fq})} = \varphi_1^{(-1)}$$
$$\varphi_2^{sign(\sin\theta^{fq})} = \varphi_2^{(-1)}. \quad (15)$$

In this final case, evaluate the angle θ^{fq} from equation (8) by using equation (15), to get

$$\theta^{fq} = \arcsin\frac{2b_2 \pm \sqrt{4a_2^2(a_2^2 - 1 + b_2^2)}}{2(a_2^2 + b_2^2)}. \quad (16)$$

Upon achieving the solutions of the angle θ^{fq} from all four cases in equations (10), (12), (14) and (16), these solutions need to be adjusted by considering all four quadrants as

$$\theta^* = \begin{cases} \theta^* \text{ and } 180^0 - \theta^{fq}, \text{if } \theta^{fq} > 0 \\ 180^0 - \theta^{fq} \text{ and } 360^0 - \theta^{fq}, \quad \text{otherwise.} \end{cases} \quad (17)$$

Angle θ^{fq} has two solutions for each case but imposing equation (17) means there are four solutions for θ^{fq} for each case, which in turn gives sixteen solutions from all four cases. In practice, only one of the solutions of θ^{fq} is accountable for the angle A_{H_1}. In order to obtain this particular solution, use all sixteen solutions in equation (5) to find the corresponding x and y coordinates for the boundary points based on equation (6). Consequently, find an angle for this boundary point for each solution of θ^{fq} deeming all four quadrants. Assume $P_{H_0}^{fq}(X^{fq}, Y^{fq})$ is a boundary point having an angle $A_{H_0}^{fq}$ for a specific value of θ^{fq}. Now, finding this particular value of θ^{fq}, which applies to A_{H_1} requires finding which $A_{H_0}^{fq}$ is such that $\left|A_{H_1} - A_{H_0}\right| \approx 0$, that is, $A_{H_0}^{fq}$ is equal or close to A_{H_1}. As a result, $P_{H_0}^{fq}(X^{fq}, Y^{fq})$ would be a boundary point of a full quel for angle A_{H_1} (see, Figures 2a and 2c). In a manner similar to the case for the full quel, the angle θ^{fq} may be solved for, which is responsible for an angle A_{H_1} and boundary point $P_{H_0}^{fq}(X^{fq}, Y^{fq})$ of a half quel for angle A_{H_1} found (see, Figures 2b and 2d) for the particular solution of θ^{fq}.

Outline of the New Tests

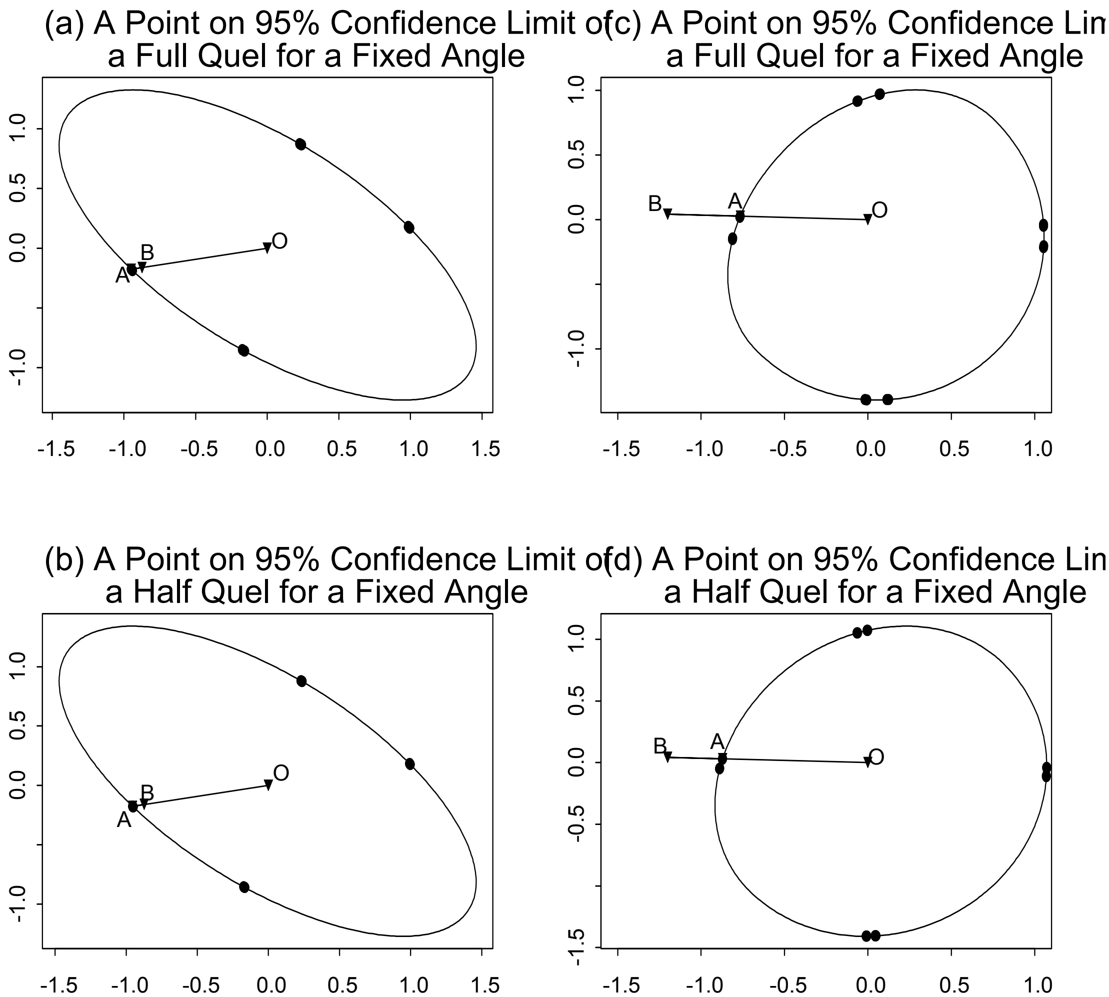
An outline of the new test procedure is as follows:

1. Estimate the parameter vector θ for the given data set, $\hat{\theta} = (\hat{\beta}', \hat{\eta}')'$. Assume P_{H_1} is the sample point for $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)'$ and compute $D_{H_1} = \sqrt{\hat{\beta}_1 + \hat{\beta}_2}$ and $A_{H_1} = \arctan(\hat{\beta}_1 + \hat{\beta}_2)$.

2. Utilizing the estimate of η in step 1 and the null values of β, construct $\hat{\theta}_0 = (\beta_0', \hat{\eta}')'$. Generate a sample of size *n* under the null from the density function $f(y_t \mid x_t, \theta)$ by setting $\theta = \hat{\theta}_0$ and estimate $\beta = (\beta_1, \beta_2)'$ for this sample. Repeat this process *N* times and let (u_i, v_i), $i = 1, 2, \cdots, N$, be the estimates of $\beta = (\beta_1, \beta_2)'$ for the i^{th} sample under the null.

3. Based on the values (u_i, v_i), $i = 1, 2, \cdots, N$, in step 2, obtain the contour points of the 100(1 - α)% confidence limit of the quel

Figure 2: Point A on 95% Confidence Contour of a Full or Half Quel for a Fixed Angle A_{H_1} Obtained from a Point B when $N = 200$, where the Points A and B Are the Points P_{H_9} and P_{H_1} as Defined so that $D_{H_9} = AO$ and $D_{H_1} = BO$.*



*The underlying distribution was the two-regressor binary logit model of size n = 30 using design matrix X_3 [(a), (b)] and the three-regressor binary logit model of size n = 30 using design matrix X_2 [(c), (d)]. H_0 is not rejected for [(a), (b)] and is rejected for [(c), (d)].

(half or full) by considering $\xi_{percentile}^{fq} = \xi_{1-\alpha}^{fq}$ or $\xi_{percentile}^{hq} = \xi_{1-\alpha}^{hq}$ for a full or half quel, respectively.

4. Corresponding to the angle A_{H_1} in step 1, obtain a point $P_{H_1}(X,Y)$ on the contour of the 100(1 - α)% confidence limit of the quel (full or half) following subsection 3.2 and compute $D_{H_0} = \sqrt{X^2 + Y^2}$. Reject H_0 if $D_{H_1} > D_{H_0}$.

Examples

Consider the problems of testing two linear restrictions associated with the two-regressor binary logit and the three-regressor binary logit models.

A binary logit model associated with two regressors $x_t = (x_{1t}, x_{2t})'$ and errors ς_t is given by

$$\Pr(y_t = 1) = \nu(x_t'\theta) = \frac{1}{[1+\exp(-x_t'\theta)]}, t = 1,2,\cdots,n \quad (18)$$

where

$$y_t = \begin{cases} 1 & \text{if } x_t'\theta + \xi_t > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$\theta = \beta$ and ξ_t is a standard logistic distribution with $\nu(u) = [1+\exp(-u)]$ being the standard logistic distribution function. Let $\hat{\theta}$ be the ML estimate of θ. The covariance matrix of $\hat{\theta} = \hat{\beta}$ for this model is given by $V(\hat{\theta}) = V(\hat{\beta}) = \left[\sum_t \nu_t(1-\nu_t)x_t x_t'\right]^{-1}$, where $\hat{\nu}_t = \nu(x_t'\hat{\theta})$. Thus, the Wald test statistic is similar to (2) with $V(\hat{\theta})$ and $R = I_2$. The LR test statistic for this model is given by (3), where $\hat{\theta} = \hat{\beta}$, $\hat{\theta}_0 = \beta_0$ and

$$l(\theta) = \sum_t [y_t \ln \hat{\nu}_t + (1-y_t)\ln(1-\nu_t)]$$

with

$$\nu_t = \nu(x_t'\theta).$$

Also, consider the three regressors binary logit model having the same form as (18) but with $x_t = (x_{1t}, x_{2t}, x_{3t})'$ and $\theta = (\beta', \eta)'$. Here η is a scalar nuisance parameter. Let $\hat{\theta} = (\hat{\beta}', \hat{\eta})'$ be the ML estimate of θ under the alternative hypothesis. In this three-regressor binary logit model, the Wald test statistic is defined in (2) with the covariance matrix of $\hat{\theta}$ as,

$$V(\hat{\theta}) = V(\hat{\beta}) = \left[\sum_t \nu_t(1-\nu_t)x_t x_t'\right]^{-1},$$

where $\hat{\nu}_t = \nu(x_t'\hat{\theta})$ and $R = (I_2 : 0)$. The LR test statistic of this model is given by (3), where $\hat{\theta}_0 = (\beta_0', \hat{\eta})'$ is the ML estimate of θ under the null hypothesis.

In these two linear restrictions testing problems, all the test statistics defined follow the asymptotic Chi-squared distribution with two degrees of freedom under the null and standard regularity conditions.

Simulation Study

The object of the simulation study is to investigate the small-sample properties of the Wald, LR, and quel (full or half) tests for hypothesis testing problems involving two linear restrictions in both models discussed in terms of size and power. For testing two linear restrictions, $\beta_0 = (0,0)'$ was used so that the null hypothesis is $H_0 : \beta_1 = \beta_2 = 0$ and the alternative hypothesis is H_1 : at least one of β_1 or $\beta_2 \neq 0$. In this case, four design matrices for x_t were used as follows: X_1: Two independent series of independent *N*(0, 1) random drawings (x_{1t} and x_{2t}), X_2: Three independent series of independent *N*(0, 1) random drawings (x_{1t}, x_{2t} and x_{3t}), X_3: Quarterly Australian private capital movements ($'000 million) ($x_{1t}$) and government capital movements ($'00 million)

(x_{2t}) beginning the first quarter of 1968, and X_4: Quarterly Australian private capital movements ($'000 million) ($x_{1t}$), the same capital movements lagged one quarter (x_{2t}) and government capital movements ($'00 million) ($x_{3t}$) beginning the first quarter of 1968. All tests were performed at the 5% nominal level using the sample sizes, n = 30 and 80. In the three-regressor binary logit model, value of the nuisance parameter was set at η = 0.1. Each experiment was based on 1,000 replications. Empirical sizes and powers of the Wald and LR tests were estimated using asymptotic critical values. The empirical sizes and powers of the quel (full and half) test were also computed for both models. In each replication of the experiment, N = 200 samples were drawn from the data generating process under the null hypothesis to find the contour points of the percentile confidence limit of a quel (full or half).

Empirical sizes and powers of the Wald, LR and quel (full or half) tests are reported in Tables 1-4 for the selected small-sample and experiments noted above. In the analysis of size, the rejection probabilities of the tests under the null, which are outside the range [0.0322, 0.0678], were significantly different from 5% at the 0.01 level for 1,000 replications. Based on these rejection probabilities, Tables 1 and 2 show that estimated sizes for the Wald and LR tests are significantly different from 5% at the 0.01 level in both models for sample size n = 30 and Table 3 shows that estimated sizes for the Wald test were significantly different from 5% at the 0.01 level in the two-regressor binary logit model for sample size n = 80. Of these tests, the LR test in general shows extreme liberal behavior, whereas the Wald test shows extreme conservative behavior in most data situations. Both proposed tests reported in Tables 1-4 perform extremely well and hold nominal level reasonably well in all instances. However, the performance of the full quel test is uniformly best in that it holds nominal level well in all data distribution situations with no apparent anti-conservative behavior.

Empirical powers of all tests were computed for the parameter space around the null that divided into five different regions based on the signs of the parameter values. In both models, the powers of the Wald test are non-monotonic at non-local alternatives in most of the regions (see, for example, regions 1 and 5 in Table 1). The Wald power function becomes almost flat at zero, for example, region 5 in Table 2. In the most serious case of region 4 in Table 2, the power for the Wald test is below 26% whereas at the same point in the parameter space, our proposed tests attain a power of 100%. The LR test, as well as the proposed new tests, has monotonic power functions in all the five regions of the parameter space in all cases for both of the models considered here. In some regions, the LR test perform better than all other tests, for example, region 2 in Table 2. However, power estimates of the LR test are erroneous because this test is liberal. The proposed quel tests show excellent power properties in most data situations. Of these new tests, power of the full quel test is better than that of the half quel test in all five regions except for a few points of some regions in the parameter space. Moreover, the full quel test has more balanced power compared to that of the half quel test at the same local alternatives in most of the regions.

Overall, the full quel test has consistently higher power and holds its level quite well. In some situations, the half quel test showed good power property and well controlled level. Among the LR, half quel and full quel tests, the full quel test can be recommended for testing two linear restrictions in these nonlinear models.

Conclusion

The Wald test requires an analytical form of the variance-covariance matrix of the ML estimators of the parameters, and it shows extreme conservative and non-monotonic power behavior caused by inaccuracy of the estimated covariance matrix of the estimator. In this article an alternative Wald type test was proposed to resolve this problem of small-sample local biasedness and non-monotonic power behavior of the Wald test for two linear restrictions. The proposed new tests have desirable size with

Table 1: Estimates of Size and Power for the Wald, LR, Full Quel, and Half QuelTests at the α = 5% Level of Significance for Testing $H_0 : \beta_1 = \beta_2 = 0$ in the Two-Regressor Binary Logit Model Using Design Matrix X_1 when *n* = 30.

Region	β_1	β_2	Asymptotic Tests: Wald	Asymptotic Tests: LR	New tests: Full Quel	New tests: Half Quel
	0.00	0.000	0.015*	0.058	0.048	0.046
1	-0.30	0.000	0.068	0.114	0.135	0.131
1	-0.60	0.000	0.221	0.313	0.354	0.356
1	-0.90	0.000	0.504	0.633	0.690	0.688
1	-1.50	0.000	0.875	0.949	0.965	0.965
1	-2.00	0.000	0.890	1.000	0.994	0.994
1	-3.50	0.000	0.467	1.000	1.000	1.000
2	0.30	0.000	0.067	0.146	0.154	0.149
2	0.60	0.000	0.232	0.401	0.410	0.405
2	0.90	0.000	0.504	0.691	0.704	0.697
2	1.50	0.000	0.877	0.960	0.965	0.963
2	2.00	0.000	0.859	0.994	0.998	0.998
2	3.50	0.000	0.462	1.000	1.000	1.000
3	-0.30	-0.300	0.087	0.200	0.193	0.183
3	-0.50	-0.500	0.231	0.405	0.415	0.416
3	-0.90	-0.900	0.692	0.863	0.870	0.869
3	-1.40	-1.400	0.802	0.986	0.990	0.991
3	-1.90	-1.900	0.580	0.999	1.000	1.000
3	-3.35	-3.350	0.118	1.000	1.000	1.000
4	0.30	0.300	0.097	0.202	0.209	0.204
4	0.50	0.500	0.253	0.418	0.439	0.435
4	0.90	0.900	0.659	0.847	0.863	0.859
4	1.40	1.400	0.788	0.991	0.996	0.995
4	1.90	1.900	0.561	0.999	1.000	1.000
4	3.35	3.350	0.117	1.000	1.000	1.000
5	-0.30	0.300	0.097	0.176	0.183	0.178
5	-0.50	0.500	0.268	0.418	0.425	0.425
5	-0.95	0.950	0.779	0.915	0.924	0.917
5	-1.55	1.550	0.815	1.000	0.998	0.998
5	-2.25	2.250	0.535	1.000	1.000	1.000
5	-3.35	3.350	0.230	1.000	1.000	1.000

Note: * Size is significantly different from 5% at the 1% level.

Table 2: Estimates of Size and Power for the Wald, LR, Full Quel, and Half Quel Tests at the α = 5% Level of Significance for Testing $H_0 : \beta_1 = \beta_2 = 0$ in the Two-Regressor Binary Logit Model Using Design Matrix X_4 when *n* = 30.

Region	β_1	β_2	Asymptotic tests: Wald	Asymptotic tests: LR	New tests: Full Quel	New tests: Half Quel
	0.00	0.00	0.019*	0.076*	0.053	0.044
1	-0.40	0.00	0.022	0.088	0.092	0.089
	-0.75	0.00	0.030	0.116	0.134	0.121
	-0.95	0.00	0.039	0.131	0.156	0.141
	-1.25	0.00	0.060	0.182	0.197	0.185
	-1.90	0.00	0.160	0.311	0.349	0.328
	-3.25	0.00	0.522	0.709	0.769	0.759
2	0.15	0.00	0.023	0.077	0.075	0.066
	0.55	0.00	0.037	0.109	0.097	0.082
	0.90	0.00	0.063	0.150	0.127	0.116
	1.55	0.00	0.125	0.253	0.237	0.229
	2.25	0.00	0.289	0.455	0.416	0.409
	3.10	0.00	0.502	0.689	0.667	0.645
3	-0.10	-0.10	0.017	0.077	0.060	0.051
	-0.35	-0.35	0.026	0.098	0.081	0.077
	-0.75	-0.75	0.085	0.201	0.170	0.153
	-1.15	-1.15	0.215	0.384	0.305	0.295
	-1.45	-1.45	0.355	0.538	0.460	0.442
	-2.25	-2.25	0.756	0.876	0.771	0.791
4	0.10	0.10	0.022	0.079	0.094	0.095
	0.35	0.35	0.046	0.114	0.774	0.775
	0.75	0.75	0.107	0.215	0.998	0.999
	1.15	1.15	0.259	0.415	1.000	1.000
	1.45	1.45	0.393	0.570	1.000	1.000
	2.25	2.25	0.751	0.881	1.000	1.000
5	-0.10	0.10	0.019	0.077	0.087	0.078
	-0.35	0.35	0.021	0.085	0.092	0.095
	-0.75	0.75	0.019	0.099	0.113	0.106
	-1.15	1.15	0.022	0.122	0.143	0.134
	-1.45	2.45	0.029	0.147	0.184	0.165
	-2.25	2.25	0.040	0.247	0.296	0.265

Note: * Size is significantly different from 5% at the 1% level.

Table 3: Estimates of Size and Power for the Wald, LR, Full Quel, and Half Quel Tests at the α = 5% Level of Significance for Testing $H_0 : \beta_1 = \beta_2 = 0$ in the Two-Regressor Binary Logit Model Using Design Matrix X_3 when n = 80.

Region	β_1	β_2	Asymptotic tests: Wald	Asymptotic tests: LR	New tests: Full Quel	New tests: Half Quel
	0.00	0.00	0.015*	0.061	0.058	0.061
1	-0.40	0.00	0.427	0.573	0.600	0.590
	-0.75	0.00	0.905	0.983	0.976	0.975
	-0.95	0.00	0.983	0.997	0.999	0.999
	-1.25	0.00	1.000	1.000	1.000	1.000
	-1.90	0.00	1.000	1.000	1.000	1.000
	-3.25	0.00	1.000	1.000	1.000	1.000
2	0.15	0.00	0.060	0.144	0.154	0.142
	0.55	0.00	0.732	0.814	0.820	0.821
	0.90	0.00	0.986	0.995	0.997	0.996
	1.55	0.00	1.000	1.000	1.000	1.000
	2.25	0.00	1.000	1.000	1.000	1.000
	3.10	0.00	1.000	1.000	1.000	1.000
3	-0.10	-0.10	0.395	0.710	0.779	0.768
	-0.35	-0.35	0.998	1.000	1.000	1.000
	-0.75	-0.75	1.000	1.000	1.000	1.000
	-1.15	-1.15	1.000	1.000	1.000	1.000
	-1.45	-1.45	1.000	1.000	1.000	1.000
	-2.25	-2.25	1.000	1.000	1.000	1.000
4	0.10	0.10	0.369	0.691	0.694	0.694
	0.35	0.35	1.000	1.000	1.000	1.000
	0.75	0.75	1.000	1.000	1.000	1.000
	1.15	1.15	1.000	1.000	1.000	1.000
	1.45	1.45	1.000	1.000	1.000	1.000
	2.25	2.25	1.000	1.000	1.000	1.000
5	-0.10	0.10	0.200	0.454	0.492	0.492
	-0.35	0.35	0.985	1.000	0.999	0.999
	-0.75	0.75	1.000	1.000	1.000	1.000
	-1.15	1.15	1.000	1.000	1.000	1.000
	-1.45	2.45	1.000	1.000	1.000	1.000
	-2.25	2.25	1.000	1.000	1.000	1.000

Note: * Size is significantly different from 5% at the 1% level.

Table 4: Estimates of Size and Power for the Wald, LR, Full Quel, and Half QuelTests at the α = 5% Level of Significance for Testing $H_0 : \beta_1 = \beta_2 = 0$ in the Two-Regressor Binary Logit Model Using Design Matrix X_2 when n = 80.

			Asymptotic Tests		New tests	
Region	β_1	β_2	Wald	LR	Full Quel	Half Quel
	0	0	0.035	0.058	0.050	0.051
1	-0.30	0.00	0.192	0.244	0.275	0.271
	-0.60	0.00	0.668	0.732	0.754	0.749
	-0.90	0.00	0.948	0.968	0.973	0.973
	-1.50	0.00	0.999	1.000	1.000	1.000
	-2.00	0.00	1.000	1.000	1.000	1.000
	-3.50	0.00	1.000	1.000	1.000	1.000
2	0.30	0.00	0.215	0.258	0.263	0.257
	0.60	0.00	0.664	0.735	0.764	0.745
	0.90	0.00	0.936	0.956	0.968	0.960
	1.50	0.00	1.000	1.000	1.000	0.999
	2.00	0.00	1.000	1.000	1.000	1.000
	3.50	0.00	1.000	1.000	1.000	1.000
3	-0.30	0.30	0.358	0.437	0.460	0.453
	-0.50	-0.50	0.833	0.875	0.875	0.873
	-0.90	-0.90	1.000	1.000	1.000	1.000
	-1.40	-1.40	1.000	1.000	1.000	1.000
	-1.90	-1.90	1.000	1.000	1.000	1.000
	-3.35	-3.35	0.991	1.000	1.000	1.000
4	0.30	0.30	0.424	0.460	0.468	0.464
	0.50	0.50	0.842	0.842	0.859	0.865
	0.90	0.90	0.999	0.999	1.000	1.000
	1.40	1.40	1.000	1.000	1.000	1.000
	1.90	1.90	1.000	1.000	1.000	1.000
	3.35	3.35	0.988	1.000	1.000	1.000
5	-0.30	0.30	0.284	0.332	0.345	0.343
	-0.50	0.50	0.675	0.733	0.756	0.747
	-0.95	0.95	0.992	0.997	0.992	0.995
	-1.55	1.55	1.000	1.000	1.000	1.000
	-2.25	2.25	0.999	1.000	1.000	1.000
	-3.35	3.35	0.995	1.000	1.000	1.000

good power properties, which are developed defining the critical region based on constructing a quel (full or half) utilizing the bootstrap that are known as full and half quel tests. These new test procedures do not suffer from the non-monotonic power and local biasedness behavior of the Wald test. More importantly, the full quel test performs uniformly best in that it holds nominal level quite well and shows comparable power in most instances. In addition, this full quel test can occasionally surpass the LR and half quel test in terms of power over most of the regions of the parameter space. Furthermore, in contrast to the Wald test, this new test does not require an analytical form of the variance-covariance matrix of the ML estimators of the parameters. This adds to its practical advantage when this matrix causes the non-monotonic power of the Wald test or is difficult to obtain. In light of this, the full quel test is best applied with the use of the quel critical region via bootstrap.

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Appendix: Options for a Quel

The confidence contour points for full or half quel are computed by considering the following two options of the percentile of the standardized errors $\xi^{fq}_{percentile}$ and the estimators γ_1 and γ_2.

Option I - Full quell:

Constant ratio: In this case, the percentile of the standardized errors $\xi^{fq}_{percentile}$ and the two additional estimators γ_1^{fq} and γ_2^{fq} are taken into account to construct the full quel, respectively, the same as $\xi^{q}_{percentile}$, γ_1 and γ_2 discussed in subsection 3.2, that is, $\xi^{fq}_{percentile}$ = $\xi^{q}_{percentile}$ and γ_l^{fq} = γ_l for l= 1, 2.

Option II: Half quel

Constant difference in the direction of the angle of the parametric equations: In this option to find those values for the construction of the half quel, first compute the standardized errors as ξ_i^{*q} based on equation (4) in subsection 3.2 with γ_1 and γ_2 replaced by γ_1^a and γ_2^a. Then define $\vartheta = \xi_{percentile}^{*q} \big/ \xi_{median}^{*q}$ and $\gamma_l^a = (2\gamma_l - 1 + \vartheta)/2\vartheta$ for l = 1, 2, where $\xi_{percentile}^{*q}$ and ξ_{median}^{*q} are the percentile and median of the errors ξ_i^{*q}. In this case, $\xi_{percentile}^{*q}$ and γ_l^a are functions of each other. Thus, the solution of γ_l^a for l = 1, 2 can be simply found by the following iterative procedure:

- Start with the initial value of ϑ, ϑ_0.
- Using this initial value ϑ_0, compute γ_l^a using the above formula for l = 1, 2.
- Based on the above values of γ_l^a, compute the standardized errors ξ_i^{*q} from equation (4) in subsection 3.2.
- Compute the percentile and median of the errors ξ_i^{*q} and compute ϑ from the above equation.
- Stop if the convergence condition holds, that is, $|\vartheta - \vartheta_0| < 0.01$, ϑ and ϑ_0 are the current and previous iteration's values, respectively. Store the final solution of γ_l^a as γ_l^{hq} for l = 1, 2 and compute the percentile of the standardized errors obtained from equation (4) with these values, γ_l^{hq} for l = 1, 2, as $\xi_{percentile}^{hq}$ for the construction of a half quel.