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Auxiliary information is used along with ranking information to derive several classes of estimators to estimate the population mean of a variable of interest based on RSS (ranked set sample). The properties of these newly suggested estimators were examined. Comparisons between special cases of these estimators and other known estimators are made using a real data set. Some of the new estimators are superior to the old ones in terms of bias and mean square error.

Keywords: Auxiliary variables, efficiency, ranking, ranked set sample.

Introduction

Many authors have discussed the use of supplementary information of auxiliary variables in survey sampling to improve the existing estimators (for example, Cochran, 1977). The ratio estimator is among the most commonly adopted to estimate: (1) population means, or (2) the total of some variable of interest from a finite population with the help of an auxiliary variable when the correlation coefficient between the two variables is positive. When the correlation coefficient between the two variables is negative, the product estimator is used. These estimators are more efficient, i.e. have smaller

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variances than the usual estimators of the population mean based on the sample mean of a simple random sample (SRS).

Ranked set sampling (RSS) can be used when the measurement of sample units drawn from a population of interest is very laborious or costly, but several elements can be easily arranged (ranked) in the order of magnitude. Takahasi and Wakimoto (1968) established the theory of RSS. They showed that the mean of the RSS is an unbiased estimator for the population mean and is more efficient than the mean of SRS. Dell and Clutter (1972) studied the effect of ranking error on the efficiency of RSS. The RSS has many statistical applications in biology and environmental studies (Barabesi & El-Sharaawi, 2001), for example, McIntyre (1952) first suggested using RSS to estimate the yield of pasture. In addition, RSS has been investigated by many researchers (Stokes, 1977; Stokes & Sager, 1988; Lam, et al., 1994, 1980; Mode, et al., 1999; Al-Saleh & Al-Shrafat, 2001; Al-Saleh & Zheng, 2000; Al-Saleh & Al-Omary, 2000), for more details about RSS, see Kaur, et al., 1995.

The RSS method can be summarized as follows: Select m random samples of size m units each and rank the units within each sample with respect to the variable of interest by a visual inspection or some other simple method.

Next, select for actual measurement the ith smallest unit from the ith sample for $i = 1, 2,...,$ m. In this way, a total of m measured units are obtained, one from each sample. The cycle may be repeated r times to get a sample of size $n = rm$. These $n = rm$ units form the RSS data. Note that in RSS, rm^2 elements are identified, but only rm of them are quantified. Thus, comparing this sample with a simple random sample (SRS) of size rm is reasonable.

Some Notions and Preliminaries

Let Y denote the variable of interest whose population mean and variance are μ_y and σ_y^2 respectively. Estimate μ_y using the information provided by one or two auxiliary variables X_1 and X_2 based on SRS and RSS will be considered. Let μ_{x_i} and $\sigma_{x_i}^2$ be the population mean and variance for X_i, $i = 1,2$. Let Y_j, X_{1j} and X_{2j} denote the values of the variables Y, X_1 and X_2 respectively, on the jth unit of the population. The population means μ_{x_1} and μ_{x_2} of the auxiliary variables are assumed to be known.

Let $Y_{(i)j}$, $X_{1(i)j}$ and $X_{2(i)j}$ represent the ith order statistics of a sample of size m in the jth cycle of the variables Y, X_1 and X_2 respectively based on a RSS of size $n = rm$ drawn from the population. The sample mean for each variable using RSS data are defined as follows:

$$\overline{Y}_{(n)} = \frac{1}{n} \sum_{j=1}^{r} \sum_{i=1}^{m} Y_{(i)j},$$

$$\overline{X}_{1(n)} = \frac{1}{n} \sum_{j=1}^{r} \sum_{i=1}^{m} X_{1(i)j},$$

and

$$\overline{X}_{2(n)} = \frac{1}{n} \sum_{j=1}^{r} \sum_{i=1}^{m} X_{2(i)j}.$$

Consider the following notations:

$$T_{y_{(i)}} = \left(\mu_{y_{(i)}} - \mu_y \right), \quad T_{x_{1(i)}} = \left(\mu_{x_{1(i)}} - \mu_{x_1} \right),$$

$$T_{x_{2(i)}} = \left(\mu_{x_{2(i)}} - \mu_{x_2} \right),$$

$$T_{yx_{1(i)}} = \left(\mu_{y_{(i)}} - \mu_y \right)\left(\mu_{x_{1(i)}} - \mu_{x_1} \right),$$

$$T_{yx_{2(i)}} = \left(\mu_{y_{(i)}} - \mu_y \right)\left(\mu_{x_{2(i)}} - \mu_{x_2} \right),$$

$$T_{x_1 x_{2(i)}} = \left(\mu_{x_{1(i)}} - \mu_{x_1} \right)\left(\mu_{x_{2(i)}} - \mu_{x_2} \right),$$

$$\sigma_{yx_{1(i)}} = E\left(Y_{(i)} - \mu_{y(i)} \right)\left(X_{1(i)} - \mu_{x_{1(i)}} \right),$$

$$\sigma_{yx_{2(i)}} = E\left(Y_{(i)} - \mu_{y(i)} \right)\left(X_{2(i)} - \mu_{x_{2(i)}} \right),$$

$$\sigma_{x_1 x_{2(i)}} = E\left(X_{1(i)} - \mu_{x_{1(i)}} \right)\left(X_{2(i)} - \mu_{x_{2(i)}} \right).$$

then:

$$\sum_{i=1}^{n} T_{y_{(i)}} = 0, \quad \sum_{i=1}^{n} T_{x_1(i)} = 0, \quad \sum_{i=1}^{n} T_{x_2(i)} = 0,$$

$$\sum_{i=1}^{n} \sigma_{y_{(i)}}^2 = n\sigma_y^2 - \sum_{i=1}^{n} T_{y_{(i)}}^2,$$

$$\sum_{i=1}^{n} \sigma_{x_{1(i)}}^2 = n\sigma_{x_1}^2 - \sum_{i=1}^{n} T_{x_{1(i)}}^2,$$

$$\sum_{i=1}^{n} \sigma_{yx_{1(i)}} = n\sigma_{yx_2} - \sum_{i=1}^{n} T_{yx_{1(i)}},$$

$$\sum_{i=1}^{n} \sigma_{yx_{2(i)}} = n\sigma_{yx_2} - \sum_{i=1}^{n} T_{yx_{2(i)}}$$

$$\sum_{i=1}^{n} \sigma_{x_1 x_{2(i)}} = n\sigma_{x_1 x_2} - \sum_{i=1}^{n} T_{x_1 x_{2(i)}}$$

and

$$\sum_{i=1}^{n} \sigma_{x_{2(i)}}^2 = n\sigma_{x_2}^2 - \sum_{i=1}^{n} T_{x_{2(i)}}^2.$$

The following classes of estimators of the mean of the variable Y based on RSS are:

$$\tilde{Y}_{a_1,a_2} = \overline{Y}_{(n)} \left(\frac{\overline{X}_{1(n)}}{\mu_{x_1}} \right)^{a_1} \left(\frac{\overline{X}_{2(n)}}{\mu_{x_2}} \right)^{a_2} \quad (2.1)$$

and

$$\tilde{Y}_{w_1, w_2} =$$

$$\overline{Y}_{(n)} \left[w_1 \left(\frac{\overline{X}_{1(n)}}{\mu_{x_1}} \right)^{a_1} + w_2 \left(\frac{\overline{X}_{2(n)}}{\mu_{x_2}} \right)^{a_2} \right] \quad (2.2)$$

where a_1, a_2, w_1, w_2 are constants and $w_1 + w_2 = 1$.

Estimators Based on RSS and One or Two Auxiliary Variables

It is not possible to rank two or more dimensional data, therefore, ranking one of the variables and taking the corresponding values of other variables is an option. Assuming that the variable can be ranked perfectly - there are no errors in ranking the units, there will be errors in ranking the other variables.

Ranking on Study Variable Y

Assume that the ranking on variable Y is perfect while the ranking on variables X_1 and X_2 will have errors; the estimators (2.1) and (2.2) are respectively given by:

$$\tilde{Y}_a = \overline{Y}_{(n)} \left(\frac{\overline{X}_{1[n]}}{\mu_{x_1}} \right)^{a_1} \left(\frac{\overline{X}_{2[n]}}{\mu_{x_2}} \right)^{a_2}, \quad (3.1)$$

and

$$\tilde{Y}_w =$$

$$w_1 \overline{Y}_{(n)} \left(\frac{\overline{X}_{1[n]}}{\mu_{x_1}} \right)^{a_1} + w_2 \overline{Y}_{(n)} \left(\frac{\overline{X}_{2[n]}}{\mu_{x_2}} \right)^{a_2}. \quad (3.2)$$

where

$$\overline{X}_{1[n]} = \frac{1}{n} \sum_{j=1}^{r} \sum_{i=1}^{m} X_{1[i]j}$$

and

$$\overline{X}_{2[n]} = \frac{1}{n} \sum_{j=1}^{r} \sum_{i=1}^{m} X_{2[i]j}$$

are the sample means of the RSS for X_1 and X_2 respectively and $X_{1[i]j}$ and $X_{2[i]j}$ are the ith judgment order statistic of the ith sample of

the jth cycle, of the variables X_1 and X_2 respectively.

Let

$$e_0 = \frac{\overline{Y}_{(n)} - \mu_y}{\mu_y}, \quad e_1 = \frac{\overline{X}_{1[n]} - \mu_{x_1}}{\mu_{x_1}}$$

and

$$e_2 = \frac{\overline{X}_{2[n]} - \mu_{x_2}}{\mu_{x_2}}.$$

Obtain the bias and the *MSE* of the estimators $\tilde{Y}_a$ and $\tilde{Y}_w$ respectively up to the order of n^{-1} as follows:

$$B\left(\tilde{Y}_a\right) =$$

$$\frac{a_1}{rm^2\mu_{x_1}}\left(m\sigma_{yx_1} - \sum_{i=1}^{m}T_{yx_{1[i]}}\right) + \frac{a_2}{rm^2\mu_{x_2}}\left(m\sigma_{yx_2} - \sum_{i=1}^{m}T_{yx_{2[i]}}\right)$$

$$+ \frac{\mu_y a_1(a_1-1)}{2rm^2\mu_{x_1}^2}\left(m\sigma_{x_1}^2 - \sum_{i=1}^{m}T_{x_{1[i]}}^2\right) +$$

$$\frac{\mu_y a_2(a_2-1)}{2rm^2\mu_{x_2}^2}\left(m\sigma_{x_2}^2 - \sum_{i=1}^{m}T_{x_{2[i]}}^2\right)$$

$$+ \frac{\mu_y a_1 a_2}{rm^2\mu_{x_1}\mu_{x_2}}\left(m\sigma_{x_1x_2} - \sum_{i=1}^{m}T_{x_1x_{2[i]}}\right)$$

$$(3.3)$$

The *MSE* of $\tilde{Y}_a$ when ranking on variable *Y* is:

$$MSE\left(\tilde{Y}_a\right) =$$

$$\frac{m\sigma_y^2 - \sum_{i=1}^{m}T_{y_{(i)}}^2}{rm^2} + \frac{\mu_y^2 a_1^2\left(m\sigma_{x_1}^2 - \sum_{i=1}^{m}T_{x_{1[i]}}^2\right)}{rm^2\mu_{x_1}^2} +$$

$$\frac{\mu_y^2 a_2^2\left(m\sigma_{x_2}^2 - \sum_{i=1}^{m}T_{x_{2[i]}}^2\right)}{rm^2\mu_{x_2}^2} + \frac{2\mu_y^2 a_1\left(m\sigma_{yx_1} - \sum_{i=1}^{m}T_{yx_{1[i]}}\right)}{rm^2\mu_y\mu_{x_1}}$$

$$+ \frac{2\mu_y^2 a_2\left(m\sigma_{yx_2} - \sum_{i=1}^{m}T_{yx_{2[i]}}\right)}{rm^2\mu_y\mu_{x_2}} + \frac{2\mu_y^2 a_1 a_2\left(m\sigma_{x_1x_2} - \sum_{i=1}^{m}T_{x_1x_{2[i]}}\right)}{rm^2\mu_{x_1}\mu_{x_2}}$$

$$(3.4)$$

up to the order of n^{-1}. The optimum values of a_1 and a_2, which minimize the *MSE* of $\widetilde{Y}_a$, are obtained by the derivation of (3.4) with respect to a_1 and a_2 respectively

$$a_1^* =$$

$$\frac{\mu_{x_1}(m\sigma_{yx_1} - \sum_{i=1}^{m}T_{yx_{1[i]}})(m\sigma_{x_2}^2 - \sum_{i=1}^{m}T_{x_{2[i]}}^2) - }{}$$

$$-\frac{(m\sigma_{yx_2} - \sum_{i=1}^{m}T_{yx_{2[i]}})(m\sigma_{x_1x_2} - \sum_{i=1}^{m}T_{x_1x_{2[i]}})}{\mu_y(m\sigma_{x_1}^2 - \sum_{i=1}^{m}T_{x_{1[i]}}^2)(m\sigma_{x_2}^2 - \sum_{i=1}^{m}T_{x_{2[i]}}^2) - }$$

$$(m\sigma_{x_1x_2} - \sum_{i=1}^{m}T_{x_1x_{2[i]}})$$

$$(3.5)$$

$$a_2^* =$$

$$\frac{\mu_{x_2}(m\sigma_{yx_2} - \sum_{i=1}^{m}T_{yx_{2[i]}})(m\sigma_{x_1}^2 - \sum_{i=1}^{m}T_{x_{1[i]}}^2) - }{}$$

$$-\frac{(m\sigma_{yx_1} - \sum_{i=1}^{m}T_{yx_{1[i]}})(m\sigma_{x_1x_2} - \sum_{i=1}^{m}T_{x_1x_{2[i]}})}{\mu_y(m\sigma_{x_1}^2 - \sum_{i=1}^{m}T_{x_{1[i]}}^2)(m\sigma_{x_2}^2 - \sum_{i=1}^{m}T_{x_{2[i]}}^2) - (m\sigma_{x_1x_2} - \sum_{i=1}^{m}T_{x_1x_{2[i]}})}$$

$$(3.6)$$

The minimum *MSE* up to terms of n^{-1} for the class $\widetilde{Y}_{a^*}$ is:

$$MSE_{\min}\left(\widetilde{Y}_{a^*}\right) =$$

$$\frac{1}{rm^2}[(m\sigma_{yx_1} - \sum_{i=1}^{m}T_{yx_{1[i]}})^2(m\sigma_{x_2}^2 - \sum_{i=1}^{m}T_{x_2}^2)$$

$$+(m\sigma_{yx_2} - \sum_{i=1}^{m}T_{yx_{2[i]}})^2(m\sigma_{x_1}^2 - \sum_{i=1}^{m}T_{x_1}^2) - (m\sigma_y^2 - \sum_{i=1}^{m}T_{y_{(i)}}^2)$$

$$-\frac{2(m\sigma_{x_1}^2 - \sum_{i=1}^{m}T_{x_1}^2)(m\sigma_{x_2}^2 - \sum_{i=1}^{m}T_{x_2}^2)(m\sigma_{x_1x_2} - \sum_{i=1}^{m}T_{x_1x_{2[i]}})}{(m\sigma_{x_1}^2 - \sum_{i=1}^{m}T_{x_1}^2)(m\sigma_{x_2}^2 - \sum_{i=1}^{m}T_{x_2}^2) - (m\sigma_{x_1x_2} - \sum_{i=1}^{m}T_{x_1x_{2[i]}})^2}]$$

$$(3.7)$$

If a_1 and a_2 take the values in (3.5) and (3.6) respectively, the bias of $\widetilde{Y}_a$ from (3.3) is given by:

$$B_{\min}\left(\widetilde{Y}_{a^*}\right) =$$

$$\frac{g_1}{rm^2\mu_y^2\{[m\sigma_{x_1}^2 - \sum_{i=1}^{m}T_{x_{1[i]}}^2][m\sigma_{x_2}^2 - \sum_{i=1}^{m}T_{x_{2[i]}}^2] - }$$

$$[m\sigma_{x_1x_2} - \sum_{i=1}^{m}T_{x_1x_{2[i]}}]^2\}$$

$$(3.8)$$

where g_1 is given in the Appendix.

The bias and the *MSE* of the estimators of (3.2) are given by:

$$B\left(\widetilde{Y}_w\right) = \mu_y\{\frac{a_1w_1}{rm^2\mu_y\mu_{x_1}}[m\sigma_{yx_1} - \sum_{i=1}^{m}T_{yx_{1[i]}}]$$

$$+\frac{a_2w_2}{rm^2\mu_y\mu_{x_2}}[m\sigma_{yx_2} - \sum_{i=1}^{m}T_{yx_{2[i]}}]$$

$$+\frac{w_1a_1(a_1-1)}{2rm^2\mu_{x_2}^2}[m\sigma_{x_1}^2 - \sum_{i=1}^{m}T_{x_{1[i]}}^2]$$

$$+\frac{w_2a_2(a_2-1)}{2rm^2\mu_{x_2}^2}[m\sigma_{x_2}^2 - \sum_{i=1}^{m}T_{x_{2[i]}}^2]\},$$

$$(3.9)$$

up to the order of n^{-1}. The *MSE* of the estimator $\widetilde{Y}_w$ if ranking on variable Y is:

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$$MSE\ (\tilde{Y}_w) = \mu_y^2 \left\{ \frac{m\sigma_y^2 - \sum_{i=1}^{m} T_{y_{(i)}}^2}{rm^2\mu_y^2} + \frac{w_1^2 a_1^2 [m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_{1[i]}}^2]}{rm^2\mu_{x_1}^2} + \right.$$

$$\frac{w_2^2 a_2^2 [m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_{2[i]}}^2]}{rm^2\mu_{x_2}^2} + \frac{2w_1 a_1 [m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_{1[i]}}]}{rm^2\mu_y\mu_{x_1}} +$$

$$\left. \frac{2w_2 a_2 [m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_{2[i]}}]}{rm^2\mu_y\mu_{x_2}} + \frac{2w_1 w_2 a_1 a_2 [m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_{2[i]}}]}{rm^2\mu_{x_1}\mu_{x_2}} \right\}$$

$$(3.10)$$

if a_1 and a_2 are both known and take the values in (3.5) and (3.6) respectively, up to order n^{-1}.

The optimum values of w_1 and w_2, which minimize the *MSE* of $\widetilde{Y}_w$, obtained by the derivation of equation (3.10) with respect to w_1 under the restriction $w_1 + w_2 = 1$, are given by:

$$w_1^* =$$

$$\frac{\begin{array}{c} a_2^2 \left(m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_{2[i]}}^2 \right) - a_1 \left(m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_{1[i]}} \right) + \\ a_2 \left(m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_{2[i]}} \right) - a_1 a_2 \left(m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_{2[i]}} \right) \end{array}}{\begin{array}{c} \left(a_1^2 m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_{1[i]}}^2 \right) + a_2^2 \left(m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_{2[i]}}^2 \right) - \\ 2a_1 a_2 \left(m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_{2[i]}} \right) \end{array}}$$

and the *MSE* of

$$\tilde{Y}_{w^*} = w_1^* \bar{Y}_{(n)} \left(\frac{\bar{X}_{1[n]}}{\mu_{x_1}} \right)^{a_1} + w_2^* \bar{Y}_{(n)} \left(\frac{\bar{X}_{2[n]}}{\mu_{x_2}} \right)^{a_2}$$

$$(3.12)$$

is the minimum *MSE* up to terms of n^{-1}. As for the class $\widetilde{Y}_w$:

$$MSE_{\min} \left(\tilde{Y}_{w^*} \right) =$$

$$\frac{1}{rm^2} \left\{ \left[m\sigma_y^2 - \sum_{i=1}^{m} T_{y_{(i)}}^2 \right] + w_1^{*2} \left(a_1^2 \left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_{1[i]}}^2 \right] \right. \right.$$

$$\left. + a_2 \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_{2[i]}}^2 \right] - 2a_1 a_2 \left[m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_{2[i]}} \right] \right)$$

$$-2w_1^* \left(a_2^2 \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_{2[i]}}^2 \right] - a_1 \left[m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_{1[i]}} \right] \right.$$

$$\left. + a_2 \left[m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_{2[i]}} \right] - a_1 a_2 \left[m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_{2[i]}} \right] \right)$$

$$\left. + a_2^2 \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_{2[i]}}^2 \right] + 2a_2 \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_{2[i]}}^2 \right] \right\}$$

$$(3.13)$$

If *w* takes the value in (3.11), then bias of $\widetilde{Y}_w$ from (3.9) is given by:

$$B \left(\tilde{Y}_{w^*} \right) =$$

$$\mu_y \left\{ w_1^* \left(a_1 \left[m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_{1[i]}} \right] - a_2 \left[m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_{2[i]}} \right] \right. \right.$$

$$\left. + \frac{a_1(a_1-1)}{2} + \left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_{1[i]}}^2 \right] - \frac{a_2(a_2-1)}{2} \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_{2[i]}}^2 \right] \right)$$

$$\left. + \frac{a_2(a_2-1)}{2} + \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_{2[i]}}^2 \right] + a_2 \left[m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_{2[i]}} \right] \right\}$$

$$(3.14)$$

Ranking on One Auxiliary Variable

If the ranking of X_1 is perfect, then the two estimators (2.1) and (2.2) are given by:

$$\tilde{Y}_a = \bar{Y}_{[n]} \left(\frac{\bar{X}_{1(n)}}{\mu_{x_1}} \right)^{a_1} \left(\frac{\bar{X}_{2[n]}}{\mu_{x_2}} \right)^{a_2} \quad (3.15)$$

and

$$\tilde{Y}_w = w_1 \bar{Y}_{[n]} \left(\frac{\bar{X}_{1(n)}}{\mu_{x_1}} \right)^{a_1} + w_2 \bar{Y}_{(n)} \left(\frac{\bar{X}_{2[n]}}{\mu_{x_2}} \right)^{a_2}$$
(3.16)

The formulas for the bias and *MSE* of estimators (3.15) and (3.16) respectively will be the same as in 3.1 except for the current estimators replace $[\ ]$ by $(\ )$ in X_1, and $(\ )$ by $[\ ]$ in Y.

Similarly if the ranking on X_2 is perfect, then the estimators:

$$\tilde{Y}_a = \bar{Y}_{[n]} \left(\frac{\bar{X}_{1(n)}}{\mu_{x_1}} \right)^{a_1} \left(\frac{\bar{X}_{2[n]}}{\mu_{x_2}} \right)^{a_2}$$
(3.17)

$$\tilde{Y}_w = w_1 \bar{Y}_{[n]} \left(\frac{\bar{X}_{1(n)}}{\mu_{x_1}} \right)^{a_1} + w_2 \bar{Y}_{(n)} \left(\frac{\bar{X}_{2[n]}}{\mu_{x_2}} \right)^{a_2}$$
(3.18)

result.

The formulas for the bias and the *MSE* of estimators (3.17) and (3.18) respectively, will be the same as in 3.1 except for the case of replacing $[\ ]$ by $(\ )$ in X_2, and $(\ )$ by $[\ ]$ in Y ($a_1 = 0$ or $a_2 = 0$ in (2.1) and $w_1 = 0$ or $w_2 = 0$ correspond to the case of one auxiliary variable).

Comparisons of Estimators

Consider the following known estimators. The RSS sample mean of the data:

$$\bar{Y}_{(n)} = \frac{1}{n} \sum_{j=1}^{r} \sum_{i=1}^{m} Y_{(i)j}$$

is an unbiased estimator for the population mean and its variance is given by:

$$Var\left(\bar{Y}_{(n)} \right) = \frac{1}{rm^2} [m\sigma_y^2 - \sum_{i=1}^{m} T_{y_{(i)}}^2]$$

(Takahasi & Wakimoto, 1968).

The Ratio estimator using RSS data is defined as:

$$\bar{Y}_R = \bar{Y}_{(n)} \frac{\mu_x}{\bar{X}_{[n]}} .$$

This estimator is a special case of the estimator in equation (1) where $a_1 = -1$ *and* $a_1 = 0$. The bias and the *MSE* of this estimator are respectively given by:

$$B\left(\bar{Y}_R \right) =$$

$$\frac{\mu_y}{rm^2 \mu_x} \left[\left(m\sigma_{yx} - \sum_{i=1}^{m} T_{yx_{[i]}} \right) - \frac{1}{\mu_x} \left(m\sigma_x^2 - \sum_{i=1}^{m} T_{x_{[i]}}^2 \right) \right]$$

$$MSE\ \left(\bar{Y}_R \right) =$$

$$\frac{\mu_y^2}{rm^2} \left[\frac{1}{\mu_y^2} \left(m\sigma_y^2 - \sum_{i=1}^{m} T_{y_{(i)}}^2 \right) + \frac{1}{\mu_x^2} \left(m\sigma_x^2 - \sum_{i=1}^{m} T_{x_{[i]}}^2 \right) - \frac{2}{\mu_y \mu_x} \left(m\sigma_{yx} - \sum_{i=1}^{m} T_{yx_{[i]}} \right) \right]$$

(Samawi & Muttlak, 1996).

The product estimator using RSS data is defined as:

$$\bar{Y}_P = \bar{Y}_{(n)} \frac{\bar{X}_{[n]}}{\mu_x}$$

This estimator is a special case for the estimator in equation (1) where $a_1 = 1$ *and* $a_1 = 0$. The new estimator is called the product estimator and its bias and *MSE* respectively are given by

$$B\left(\bar{Y}_P \right) = \frac{\mu_y}{rm^2} [m\sigma_{yx} - \sum_{i=1}^{m} T_{yx_{[i]}}]$$

$$MSE\left(\bar{Y}_P\right)=\frac{\mu_y^2}{rm^2}\{\frac{1}{\mu_y^2}[m\sigma_y^2-\sum_{i=1}^{m}T_{y_{(i)}}^2]+$$

$$\frac{1}{\mu_x^2}[m\sigma_x^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2]+\frac{2}{\mu_y\mu_x}[m\sigma_{yx}-\sum_{i=1}^{m}T_{yx_{1[i]}}]\}$$

If $a_1=a_2=-1$ is set in the estimator of equation (2) the following new estimator results:

$$\hat{Y}_a=\bar{Y}_{(n)}\frac{\mu_{x_1}}{\bar{X}_{1[n]}}\frac{\mu_{x_2}}{\bar{X}_{2[n]}}.$$

The bias and the MSE are respectively given by

$$B\left(\hat{Y}_a\right)=\frac{\mu_y}{rm^2}\left\{\frac{1}{\mu_{x_1}^2}\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]\right.$$

$$+\frac{1}{\mu_{x_2}^2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]+\frac{1}{\mu_{x_1}\mu_{x_2}}\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]$$

$$-\frac{1}{\mu_y\mu_{x_1}}\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]-\frac{1}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]\right\}$$

and

$$MSE\left(\hat{Y}_a\right)=\frac{\mu_y^2}{rm^2}\left\{\frac{1}{\mu_y^2}\left[m\sigma_y^2-\sum_{i=1}^{m}T_{y_{(i)}}^2\right]\right.$$

$$+\frac{1}{\mu_{x_1}^2}\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]+\frac{1}{\mu_{x_2}^2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]$$

$$-\frac{2}{\mu_y\mu_{x_1}}\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]-\frac{2}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]$$

$$+\frac{2}{\mu_{x_1}\mu_{x_2}}\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]\right\}$$

Setting $a_1=a_2=1$ in the estimator of equation (2) results in a new estimator defined as:

$$\ddot{Y}_a=\bar{Y}_{(n)}\frac{\bar{X}_{1[n]}}{\mu_{x_1}}\frac{\bar{X}_{2[n]}}{\mu_{x_2}}.$$

The bias and the MSE are respectfully given by

$$B\left(\ddot{Y}_a\right)=$$

$$\frac{\mu_y}{rm^2}\left\{\frac{1}{\mu_{x_1}\mu_{x_2}}\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]\right.$$

$$+\frac{1}{\mu_y\mu_{x_1}}\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]+\frac{1}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]\right\}$$

and

$$MSE\left(\ddot{Y}_a\right)=$$

$$\frac{\mu_y^2}{rm^2}\left\{\frac{1}{\mu_y^2}\left[m\sigma_y^2-\sum_{i=1}^{m}T_{y_{(i)}}^2\right]+\frac{1}{\mu_{x_1}^2}\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_{1[i]}}^2\right]\right.$$

$$+\frac{1}{\mu_{x_2}^2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_{2[i]}}^2\right]+\frac{2}{\mu_y\mu_{x_1}}\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]$$

$$+\frac{2}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]+\frac{2}{\mu_{x_1}\mu_{x_2}}\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_{2[i]}}\right]\right\}$$

If $a_1=a_2=1$ in the estimator of equation 3 is set, a new estimator called the Multivariate ratio estimator using RSS can be defined as

$$\breve{Y}_w=\bar{Y}_{(n)}\left\{w_1\frac{\bar{X}_{1[n]}}{\mu_{x_1}}+w_2\frac{\bar{X}_{2[n]}}{\mu_{x_2}}\right\}.$$

The bias and the MSE are respectively given by

$$B\left(\breve{Y}_w\right)=\mu_y\left\{\frac{w_1}{\mu_y\mu_{x_1}}\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_{1[i]}}\right]+\right.$$

$$\frac{w_2}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_{2[i]}}\right]\right\}$$

and

$$MSE\ \left(\breve{Y}_w\right)=\frac{\mu_y^2}{rm^2}\left\{\frac{1}{\mu_y^2}\left[m\sigma_y^2-\sum_{i=1}^{m}T_{y(i)}^2\right]\right.$$

$$+w_1^2\left(\frac{1}{\mu_{x_1}^2}\left[m\sigma_{x_1}^2-\sum_{i=1}^{m}T_{x_1[i]}^2\right]+\frac{1}{\mu_{x_2}^2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2[i]}^2\right]\right.$$

$$\left.-\frac{2}{\mu_{x_1}\mu_{x_2}}\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_2[i]}\right]\right)-2w_1\left(\frac{1}{\mu_{x_2}^2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2[i]}^2\right]\right.$$

$$-\frac{1}{\mu_y\mu_{x_1}}\left[m\sigma_{yx_1}-\sum_{i=1}^{m}T_{yx_1[i]}\right]+\frac{1}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_2[i]}\right]$$

$$\left.-\frac{1}{\mu_{x_1}\mu_{x_2}}\left[m\sigma_{x_1x_2}-\sum_{i=1}^{m}T_{x_1x_2[i]}\right]\right)$$

$$\frac{1}{\mu_{x_2}^2}\left[m\sigma_{x_2}^2-\sum_{i=1}^{m}T_{x_2[i]}^2\right]+\frac{2}{\mu_y\mu_{x_2}}\left[m\sigma_{yx_2}-\sum_{i=1}^{m}T_{yx_2[i]}\right]\right\}.$$

The comparison between the estimators proposed is illustrated by using a real data set. The data for the illustration was taken from Ahmed (1995); the population consists of 332 villages. Consider the variables, Y, X_1 and X_2 where Y is number of cultivators, X_1 is the area of the village and X_2 is the number of household in the village.

The following steps summarize the simulation procedure to find the bias and MSE of an estimator for the population mean using perfect ranking on the variable of interest Y.

Step 1:

Simulate rm^2 observations from the 332 real data values with replacement and perform the RSS procedure with m =5 and r = 16 to get sample of size $n=rm=80$.

Step 2:
Use the data in Step 1 to calculate

$$\hat{Y}_{(n)}=\frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m}\hat{Y}_{(i:m)j}=\frac{1}{80}\sum_{j=1}^{16}\sum_{i=1}^{5}\hat{Y}_{(i:m)j}$$

where $\hat{Y}_{(i:m)j}$ is the ith smallest in the sample of size m =5 in the jth cycle.

Step 3:
Repeat steps 1 and 2 (30,000) times, using these 30,000 values to obtain

$$\overline{\hat{Y}}_{(n)}=\frac{1}{30000}\sum_{i=1}^{30000}\hat{Y}_{(n)i}$$

Step 4:
Find the approximate bias and MSE for $\hat{Y}_{(n)}$. The bias is obtained by

$$B\left(\hat{Y}_{(n)}\right)=\frac{1}{30000}\sum_{i=1}^{30000}\hat{Y}_{(n)i}-\mu_y,$$

and the MSE of $\hat{Y}_{(n)}$ is obtained as

$$MSE\left(\hat{Y}_{(n)}\right)=\frac{1}{30000}\sum_{i-1}^{30000}\left(\hat{Y}_{(n)i}-\overline{\hat{Y}}_{(n)}\right)^2.$$

The above simulation was preformed for all other estimators suggested ranking on one of the variables Y, X_1 or X_2. Calculate the efficiency of these estimators with respect to the $MSE\left(\hat{Y}_{(n)}\right)=Var(\overline{Y}_{(n)})$ estimator using

$$e\left(\hat{Y}\right)=\frac{MSE\left(\hat{Y}_{(n)}\right)}{MSE\left(\hat{Y}\right)},$$

where $\hat{Y}$ represents any of the estimators given.

In Tables 1-3, MSE, bias, and efficiency have been calculated for each of the suggested estimators. In Table 1, ranking on the variable Y is shown (i.e., the ranking of variable Y will be perfect while the ranking of the other variables will be with errors in ranking). Tables 2 and 3 show the ranking on the variables X_1 and X_2 respectively.

Considering the results of Tables 1-3 it is observed that $\widetilde{Y}_{a*}$ dominates all other estimators and achieved the highest efficiency. Its efficiency is more than 22 times higher than the

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Table 1: The Bias, MSE and the Efficiency for all Estimators Based on Ranking of the Variable Y

Estimator	Auxiliary variable	MSE	Efficiency	Bias
$\overline{Y}_{(n)}$	None	8374.579	1	0
$\widetilde{Y}^*$	x_1	820.252	10.2041	0.843
$\widetilde{Y}^*$	x_2	909.625	9.17431	0.752
$\widetilde{Y}_{a^*}$	x_1, x_2	379.579	22.2222	0.253
$\widetilde{Y}_{w^*}$	x_1, x_2	582.065	14.4928	-0.521
$\overline{Y}_R$	x_1	4403.464	1.8939	0.422
$\overline{Y}_R$	x_2	2217.261	3.7594	0.998
$\overline{Y}_P$	x_1	33092.8	0.25163	12.522
$\overline{Y}_P$	x_2	30979.3	0.2688	7.121
$\widehat{Y}_a$	x_1, x_2	8479.4	0.98231	14.153
$\ddot{Y}_a$	x_1, x_2	69893.8	0.119147	15.332
$\breve{Y}_w$	x_1, x_2	598.243	14.0845	7.151

Table 2: The Bias, MSE and the Efficiency for all Estimators Based on Ranking of the Variable X_1

Estimator	Auxiliary variable	MSE	Efficiency	Bias
$\overline{Y}_{[n]}$	None	8311.06	1	0
$\widetilde{Y}^*$	x_1	899.012	9.2592	0.899
$\widetilde{Y}^*$	x_2	1250.112	6.6666	0.822
$\widetilde{Y}_{a^*}$	x_1, x_2	378.865	22.2222	0.299
$\widetilde{Y}_{w^*}$	x_1, x_2	581.231	14.4928	-0.675
$\overline{Y}_R$	x_1	4403.511	1.8903	0.533
$\overline{Y}_R$	x_2	2213.96	3.7594	1.228
$\overline{Y}_P$	x_1	33077.6	0.2513	14.532
$\overline{Y}_P$	x_2	30895.8	0.26903	9.217
$\widehat{Y}_a$	x_1, x_2	8711.43	0.98231	15.533
$\ddot{Y}_a$	x_1, x_2	69790.4	0.11909	17.222
$\breve{Y}_w$	x_1, x_2	612.103	13.6986	10.511

Table 3: The Bias, MSE and the Efficiency for all Estimators Based on Ranking of the Variable X_1

Estimator	Auxiliary Variable	MSE	Efficiency	Bias
$\overline{Y}_{[n]}$	None	8311.06	1	0
$\widetilde{Y}^*$	x_1	899.012	9.2592	0.899
$\widetilde{Y}^*$	x_2	1250.112	6.6666	0.822
$\widetilde{Y}_{a^*}$	x_1, x_2	378.865	22.2222	0.299
$\widetilde{Y}_{w^*}$	x_1, x_2	581.231	14.4928	-0.675
$\overline{Y}_R$	x_1	4403.511	1.8903	0.533
$\overline{Y}_R$	x_2	2213.96	3.7594	1.228
$\overline{Y}_P$	x_1	33077.6	0.2513	14.532
$\overline{Y}_P$	x_2	30895.8	0.26903	9.217
$\widehat{Y}_a$	x_1, x_2	8711.43	0.98231	15.533
$\ddot{Y}_a$	x_1, x_2	69790.4	0.11909	17.222
$\breve{Y}_w$	x_1, x_2	612.103	13.6986	10.511

RSS estimator. Some other estimators achieved higher efficiency than $\widehat{Y}_{(n)}$, theses estimators are: $\widetilde{Y}_{w^*}$, $\breve{Y}_w$, $\widetilde{Y}^*$, and $\overline{Y}_R$.

The estimators achieved about the same efficiency no matter which variable was ranked on. This provides greater flexibility in choosing the variable to rank on, since some of the variables are more difficult to rank than others.

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Appendix

$$g_{1=} \left\{ \left[m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_2[i]} \right] \left(2 \left[m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_1[i]} \right] \left[m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_2[i]} \right] \left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_1[i]}^2 \right] \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_2[i]}^2 \right] \right. \right.$$

$$-2 \left[m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_1[i]} \right] \left[m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_2[i]} \right] \left[m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_2[i]} \right]^2 - \left[m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_1[i]} \right] \left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_1[i]}^2 \right]$$

$$\left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_2[i]}^2 \right] \left[m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_2[i]} \right] + \left[m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_2[i]} \right]^2 \left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_1[i]}^2 \right] \left[m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_2[i]} \right] +$$

$$\left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_1[i]}^2 \right]^2 \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_2[i]}^2 \right] \left[m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_2[i]} \right] + \left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_1[i]}^2 \right] \left[m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_2[i]} \right]$$

$$\left[m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_2[i]} \right]^2 - \left[m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_2[i]} \right] \left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_1[i]}^2 \right] \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_2[i]}^2 \right] \left[m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_2[i]} \right] +$$

$$\left[m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_1[i]} \right]^2 \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_2[i]}^2 \right] \left[m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_2[i]} \right] + \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_2[i]}^2 \right]^2 \left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_1[i]}^2 \right]$$

$$\left[m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_1[i]} \right] + \left[m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_1[i]} \right] \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_2[i]}^2 \right] \left[m\sigma_{x_1 x_2} - \sum_{i=1}^{m} T_{x_1 x_2[i]} \right]^2 \right) - \left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_1[i]}^2 \right]$$

$$\left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_2[i]}^2 \right] \left(\left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_2[i]}^2 \right] \left[m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_1[i]} \right]^2 + \left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_1[i]}^2 \right] \left[m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_2[i]} \right]^2 \right.$$

$$+ \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_2[i]}^2 \right] \left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_1[i]}^2 \right] \left[m\sigma_{yx_1} - \sum_{i=1}^{m} T_{yx_1[i]} \right] + \left[m\sigma_{x_2}^2 - \sum_{i=1}^{m} T_{x_2[i]}^2 \right] \left[m\sigma_{x_1}^2 - \sum_{i=1}^{m} T_{x_1[i]}^2 \right]$$

$$\left[m\sigma_{yx_2} - \sum_{i=1}^{m} T_{yx_2[i]} \right] \right) \right\}$$