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A random variable X is said to have the skew-Laplace probability distribution if its pdf is given by $f(x) = 2g(x)G(\lambda x)$, where g (.) and G (.), respectively, denote the pdf and the cdf of the Laplace distribution. When the skew Laplace distribution is truncated on the left at 0 it is called it the truncated skew Laplace (TSL) distribution. This article provides a comparison of TSL distribution with two-parameter gamma model and the hypoexponential model, and an application of the subject model in maintenance system is studied.

Key words: Probability Distribution; Truncation; Simulation; Reliability, Renewal Process.

Introduction

Very few real world phenomena studied statistically are symmetrical in nature, thus, the symmetric models would not be useful for studying every phenomenon. The normal model is, at times, a poor description of observed phenomena. Skewed models, which exhibit varying degrees of asymmetry, are a necessary component of the modeler's tool kit. The term skew Laplace (SL) means a parametric class of probability distributions that extends the Laplace probability density function (pdf) by an additional shape parameter that regulates the degree of skewness, allowing for a continuous variation from Laplace to non-Laplace.

The skew Laplace distribution as a generalization of the Laplace law should be a natural choice in all practical situations in which some skewness is present. Several asymmetric forms of the skewed Laplace distribution have appeared in the literature with different formulations. Aryal *et al.* (2005b) studied extensively the mathematical properties of a skew Laplace distribution. This distribution was developed using the idea introduced by O'Hagan and studied by Azzalini (1985). A random variable X is said to have the skew symmetric distribution if its probability density function (pdf) is given by

$$f(x) = 2g(x)G(\lambda x) \tag{1.1}$$

where, $-\infty< x< \infty$, $-\infty< \lambda< \infty$, $g(x)$ and $G(x)$ are the corresponding pdf and the cumulative distribution function (cdf) of the symmetric distributions.

The Laplace distribution has the pdf and cdf specified by

$$g(x) = \frac{1}{2\varphi}\exp\left(-\frac{|x|}{\varphi}\right) \tag{1.2}$$

and

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$$G(x) = \begin{cases} \frac{1}{2}\exp\left(\frac{x}{\varphi}\right) & if\ x \le 0, \\ 1 - \frac{1}{2}\exp\left(-\frac{x}{\varphi}\right) & if\ x \ge 0 \end{cases} \tag{1.3}$$

respectively, where $-\infty < x < \infty$ and $\varphi > 0$. Hence, the pdf $f(x)$ and the cdf $F(x)$ of the skew Laplace random variable is given, respectively, by

$$f(x) = \begin{cases} \frac{1}{2\varphi}\exp\left\{-\frac{(1+|\lambda|)|x|}{\varphi}\right\}, if\ \lambda x \le 0, \\ \frac{1}{\varphi}\exp\left(-\frac{|x|}{\varphi}\right)\left\{1 - \frac{1}{2}\exp\left(-\frac{\lambda x}{\varphi}\right)\right\}, if\ \lambda x \ge 0, \end{cases} \tag{1.4}$$

and

$$F(x) = \begin{cases} \frac{1}{2} + \frac{sign(\lambda)}{2}\left[\frac{1}{1+|\lambda|}\exp\left\{-\frac{(1+|\lambda|)|x|}{\varphi}\right\} - 1\right], \\ \quad if\ \lambda x \le 0, \\ \frac{1}{2} + sign(\lambda)\left[\frac{1}{2} - \exp\left(-\frac{|x|}{\varphi}\right)\Phi(\lambda)\right], \\ \quad if\ \lambda x \ge 0. \end{cases} \tag{1.5}$$

where,

$$\Phi(\lambda) = 1 - \frac{1}{2(1+|\lambda|)}\exp\left(-\frac{\lambda x}{\varphi}\right).$$

Aryal et al. (2005a) proposed a reliability model that can be derived from the skew Laplace distribution on truncating it at 0 on the left. This is called the truncated skew Laplace (TSL) probability distribution. The cdf of this reliability model for $\lambda > 0$ is given by

$$F^*(x) = 1 + \frac{\exp\left(-\frac{(1+\lambda)x}{\varphi}\right) - 2(1+\lambda)\exp\left(-\frac{x}{\varphi}\right)}{(2\lambda+1)} \tag{1.6}$$

and the corresponding pdf is given by

$$f^*(x) = \begin{cases} \frac{(1+\lambda)}{\varphi(2\lambda+1)}\left\{2\exp\left(-\frac{x}{\varphi}\right) - \exp\left(-\frac{(1+\lambda)x}{\varphi}\right)\right\} & if\ x \ge 0 \\ 0 & otherwise \end{cases} \tag{1.7}$$

It is immediate that the reliability function $R(t)$ and the hazard rate function $h(t)$ of a TSL random variable is given, respectively, by

$$R(t) = \frac{2(1+\lambda)\exp\left(-\frac{t}{\varphi}\right) - \exp\left(-\frac{(1+\lambda)t}{\varphi}\right)}{(2\lambda+1)} \tag{1.8}$$

and

$$h(t) = \frac{(1+\lambda)}{\varphi}\frac{\left\{2 - \exp\left(\frac{-\lambda t}{\varphi}\right)\right\}}{\left\{2 + 2\lambda - \exp\left(\frac{-\lambda t}{\varphi}\right)\right\}}. \tag{1.9}$$

Also, note that the mean residual lifetime (MRLT) of a TSL random variable is given by

$$m(t) = \frac{\varphi}{(1+\lambda)}\left\{\frac{2(1+\lambda)^2 - \exp\left(-\frac{\lambda t}{\varphi}\right)}{2(1+\lambda) - \exp\left(-\frac{\lambda t}{\varphi}\right)}\right\}. \tag{1.10}$$

This article provides a comparison of this reliability model with other competing models, namely, the two parameter gamma and hypoexponential distribution. We also study an application of the TSL probability model in preventive maintenance and cost optimization.

TSL vs. Gamma Distribution

A random variable X is said to have a gamma probability distribution with parameters α and β denoted by $G(\alpha, \beta)$ if it has a probability density function given by

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right),\ \alpha,\beta,x \geq 0, \tag{2.1}$$

where $\Gamma(.)$ denotes the gamma function. The parameters α and β are the shape and scale parameters, respectively. The reliability and hazard functions are not available in closed form unless α is an integer; however, they may be expressed in terms of the standard incomplete gamma function $\Gamma(a,z)$ defined by

$$\Gamma(a,z) = \int_0^z y^{a-1} \exp(-y)dy,\ a > 0.$$

In terms of $\Gamma(a,z)$ the reliability function for random variable Gamma is given by

$$R(t;\alpha,\beta) = \frac{\Gamma(\alpha) - \Gamma(\alpha, t/\beta)}{\Gamma(\alpha)}. \tag{2.2}$$

If α is an integer, then the reliability function is given by

$$R(t;\alpha,\beta) = \sum_{k=0}^{\alpha-1} \frac{(t/\beta)^k \exp(-t/\beta)}{k!}. \tag{2.3}$$

The hazard rate function is given by

$$h(t;\alpha,\beta) = \frac{t^{\alpha-1}\exp(-t/\beta)}{\beta^{\alpha}\left[\Gamma(\alpha) - \Gamma(\alpha, t/\beta)\right]}, \tag{2.4}$$

for any $\alpha > 0$, however, if α is an integer it becomes

$$h(t;\alpha,\beta) = \frac{t^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)\sum_{k=0}^{\alpha-1}(t/\beta)^k / k!}. \tag{2.5}$$

The shape parameter α is of special interest, since whether $\alpha - 1$ is negative, zero or positive, corresponds to a decreasing failure rate (DFR), constant, or increasing failure rate (IFR), respectively.

It is clear that the gamma model has more flexibility than the TSL model as the former can be used even if the data has DFR. In fact, the standard exponential distribution is $TSL(0,1)$ as well as $Gamma(1,1)$. However, if in the gamma model $\alpha > 1$, it has IFR which appears to be the same as that of the TSL model, but a careful study has shown a significance difference between these two models, this is the case for which real world data - where the TSL model gives a better fit than the competing gamma model - could be presented.

According to Pal *et al.* (2006) the failure times (in hours) of pressure vessels constructed of fiber/epoxy composite materials wrapped around metal lines subjected to a certain constant pressure, studied by Keating *et al.* (1990), can be described using $Gamma(1.45, 300)$ model. The subject data was studied using TSL model. It was observed that TSL (5939.8, 575.5) fits the subject data better than the gamma distribution The Kolmogorov-Smirnov goodness of fit indicated that, the D-statistic for Gamma (1.45, 300) and TSL (5939.8, 575.5) distribution are $D_{Gamma} = 0.2502$ *and* $D_{TSL} = 0.200$ respectively. Since the smaller D-statistic, the better is the fit so it is concluded that the TSL model fits better than the gamma model.

Figure 1 displays the P-P plot of the fits of the pressure vessels data assuming the TSL and the gamma models. It is clear that the TSL pdf is a better fit than the gamma model. Thus, the TSL is recommended for the pressure vessel data. Table 1 gives the reliability estimates using TSL and gamma models. It is observed that there is a significant difference in these estimates.

TSL vs. Hypoexponential Probability Distribution

Observing the probability structure of the truncated skew Laplace pdf it is of interest to seek an existing probability distribution, which can be written as a difference of two exponential functions. Since the hypoexponential distribution has this characteristic the TSL pdf will be compared with the hypoexponential pdf. Many natural phenomena can be divided into

Figure 1: P-P Plots of Vessel Data Using TSL and Gamma Distribution

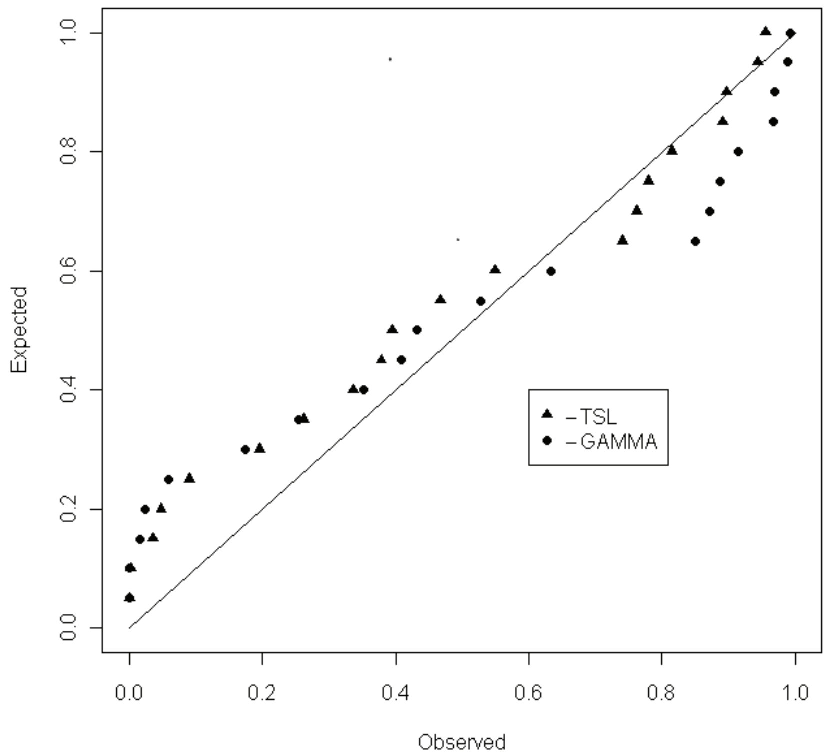


Table 1: Reliability Estimates of the Pressure Vessels Data

t	$\hat{R}_{TSL}(t)$	$\hat{R}_{GAMMA}(t)$	t	$\hat{R}_{TSL}(t)$	$\hat{R}_{GAMMA}(t)$
0.75	0.999	0.999	363	0.532	0.471
1.70	0.997	0.999	458	0.451	0.365
20.80	0.965	0.984	776	0.260	0.150
28.50	0.952	0.976	828	0.237	0.129
54.90	0.909	0.940	871	0.220	0.113
126.0	0.803	0.826	970	0.185	0.085
175.0	0.738	0.745	1278	0.108	0.034
236.0	0.664	0.647	1311	0.102	0.030
274.0	0.621	0.590	1661	0.056	0.010
290.0	0.604	0.567	1787	0.045	0.007

sequential phases. If the time the process spent in each phase is independent and exponentiallydistributed, then it can be shown that overall time is hypoexponentially distributed. It has been empirically observed that service times for input-output operations in a computer system often possess this distribution (see Trivedi, 1982) and will have n parameters one for each of its distinct phases. Interest then lies in a two-stage hypoexponential process, that is, if X is a random variable with parameters λ_1 and λ_2 $(\lambda_1 \neq \lambda_2)$ then its pdf is given by

$$f(x) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\{\exp(-\lambda_1 x) - \exp(-\lambda_2 x)\},\ x > 0 \tag{3.1}$$

The notation $Hypo(\lambda_1, \lambda_2)$ denotes a hypoexponential random variable with parameters λ_1 and, λ_2 . The corresponding cdf is given by

$$F(x) = 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1}\exp(-\lambda_1 x) + \frac{\lambda_1}{\lambda_2 - \lambda_1}\exp(-\lambda_2 x),\quad x \geq 0 \tag{3.2}$$

The reliability function $R(t)$ of a $Hypo(\lambda_1, \lambda_2)$ random variable is given by

$$R(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1}\exp(-\lambda_1 t) - \frac{\lambda_1}{\lambda_2 - \lambda_1}\exp(-\lambda_2 t) \tag{3.3}$$

The hazard rate function $h(t)$ of a $Hypo(\lambda_1, \lambda_2)$ random variable is given by

$$h(t) = \frac{\lambda_1 \lambda_2 [\exp(-\lambda_1 t) - \exp(-\lambda_2 t)]}{\lambda_2 \exp(-\lambda_1 t) - \lambda_1 \exp(-\lambda_2 t)}. \tag{3.4}$$

It is clear that $h(t)$ is an increasing function of the parameter λ_2; it increases from 0 to $\min\{\lambda_1, \lambda_2\}$. Note that the mean residual life time (MRLT) at time t for $Hypo(\lambda_1, \lambda_2)$ is given by

$$m_{Hypo}(t) = \frac{1}{\lambda_1 \lambda_2}\frac{\lambda_2^2 \exp(-\lambda_1 t) - \lambda_1^2 \exp(-\lambda_2 t)}{[\lambda_2 \exp(-\lambda_1 t) - \lambda_1 \exp(-\lambda_2 t]}. \tag{3.5}$$

To compare the TSL and hypoexponential pdf in terms of reliability and mean residual life times, random samples of size 50, 100 and 500 are generated from a hypoexponential pdf with parameters λ_1=1 and $\lambda_2 = 2, 5, 10 \& 20$ for each sample size. Numerical iterative procedure, Newton-Raphson algorithm, is used to estimate the maximum likelihood estimates of λ_1 & λ_2. To compare these results, the parameters φ and λ of a TSL distribution are estimated (See Table 2). In addition the mean residual life times were computed for both the models at $t = t_{n/2}$.

In Table 2, M_{TSL} and M_{HYPO} denote the MRLT of TSL and hypoexponential models respectively. Table 2 shows that if the sample size is large and the difference between the two parameters λ_1 and λ_2 is large both the TSL and hypoexponential model will produce the same result. However, for a small sample size and a small difference between λ_1 and λ_2 a significant difference is observed between the two models. Figures 2-5 illustrate the plotted reliability graphs and provide the support for these findings.

TSL Distribution and Preventive Maintenance

In many situations, failure of a system or unit during actual operation can be very costly or in some cases dangerous if the system fails, thus, it may be better to repair or replace before it fails. However, it is not typically feasible to make frequent replacements of a system. Thus, developing a replacement policy that balances the cost of failures against the cost of planned replacement or maintenance is necessary. Suppose a unit that is to operate over a time 0 to time t, [0, t], is replaced upon failure (with failure probability distribution F). Assume

Table 2: Mean Residual Lifetimes (MRLT) of TSL and Hypoexponential Models Computed by Using (1.10) and (3.5) for Different Sample Sizes

n	λ_1	λ_2	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\varphi}$	$\hat{\lambda}$	M_{TSL}	M_{HYPO}
50	1	2	0.934	2.325	2.745	1.349	1.362	1.129
50	1	5	0.975	5.133	2.779	1.097	1.108	1.029
50	1	10	0.979	12.223	1.042	6.968	1.042	1.021
50	1	20	0.940	26.742	1.069	15.349	1.069	1.063
100	1	2	0.876	2.565	1.376	2.403	1.391	1.184
100	1	5	0.903	6.835	1.178	6.216	1.179	1.108
100	1	10	0.950	9.838	1.098	8.439	1.099	1.052
100	1	20	1.029	26.322	0.892	0.242	0.982	0.971
500	1	2	0.915	2.576	1.339	3.076	1.348	1.132
500	1	5	0.961	6.489	1.088	3.453	1.093	1.042
500	1	10	0.881	10.224	1.174	8.355	1.173	1.135
500	1	20	1.016	27.411	0.988	14.044	0.988	0.983

Figure 2: Reliability of TSL and Hypoexponential Distributions for n=50, λ_1=1, λ_2=2

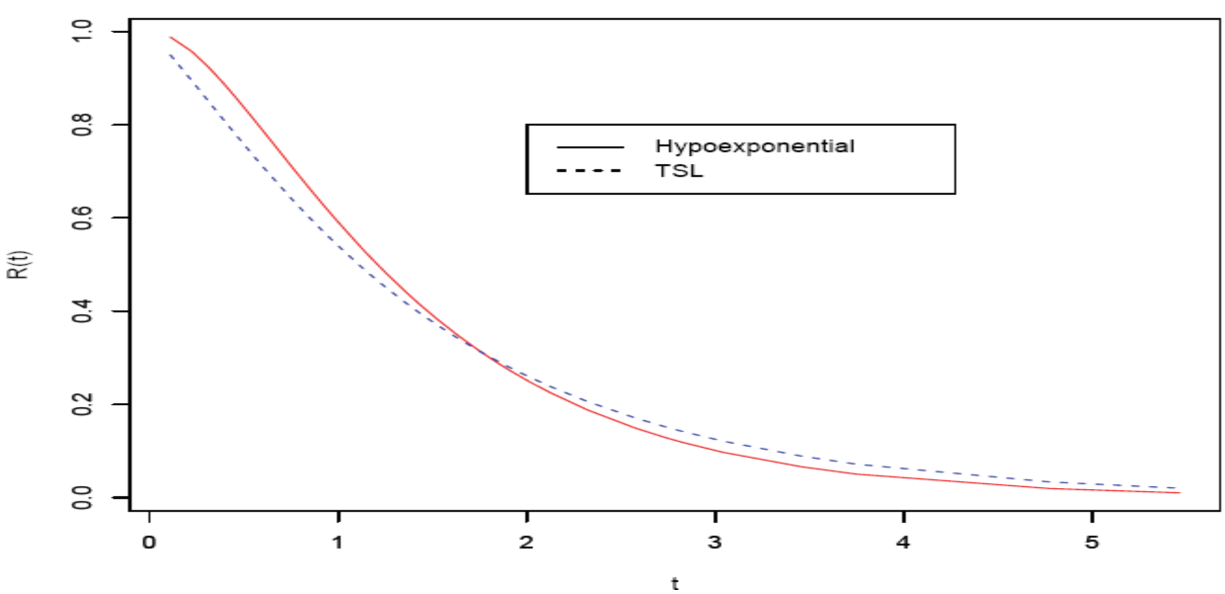


Figure 3: Reliability of TSL and Hypoexponential Distributions for n=50, λ_1=1, λ_2=5

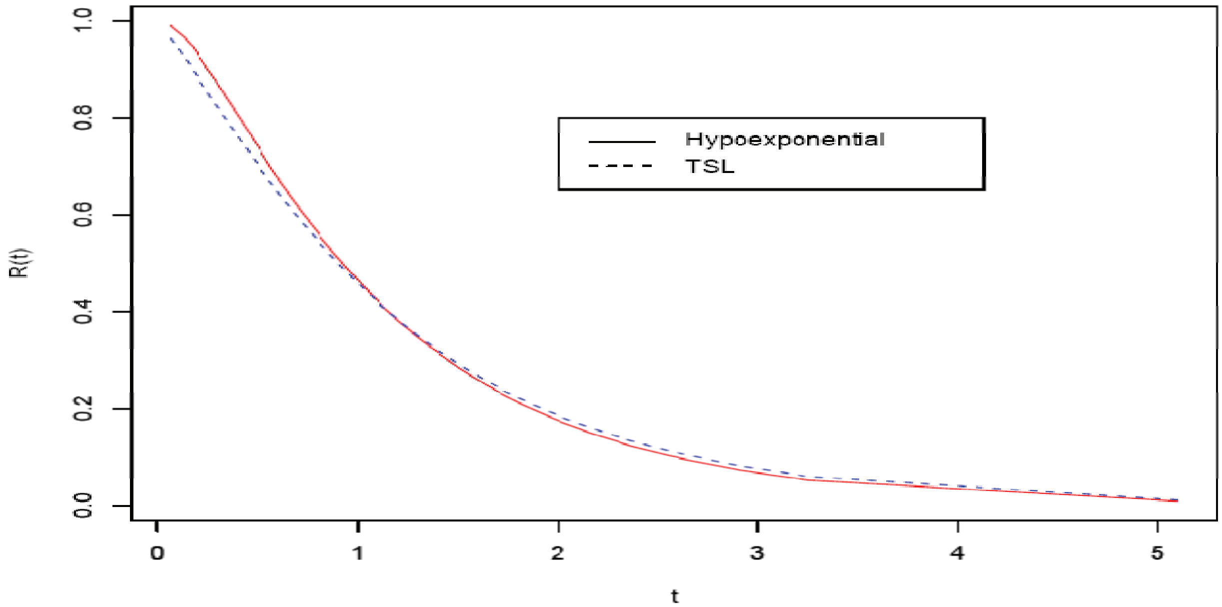


Figure 4: Reliability of TSL and Hypoexponential Distributions for n=50, λ_1=1, λ_2=10

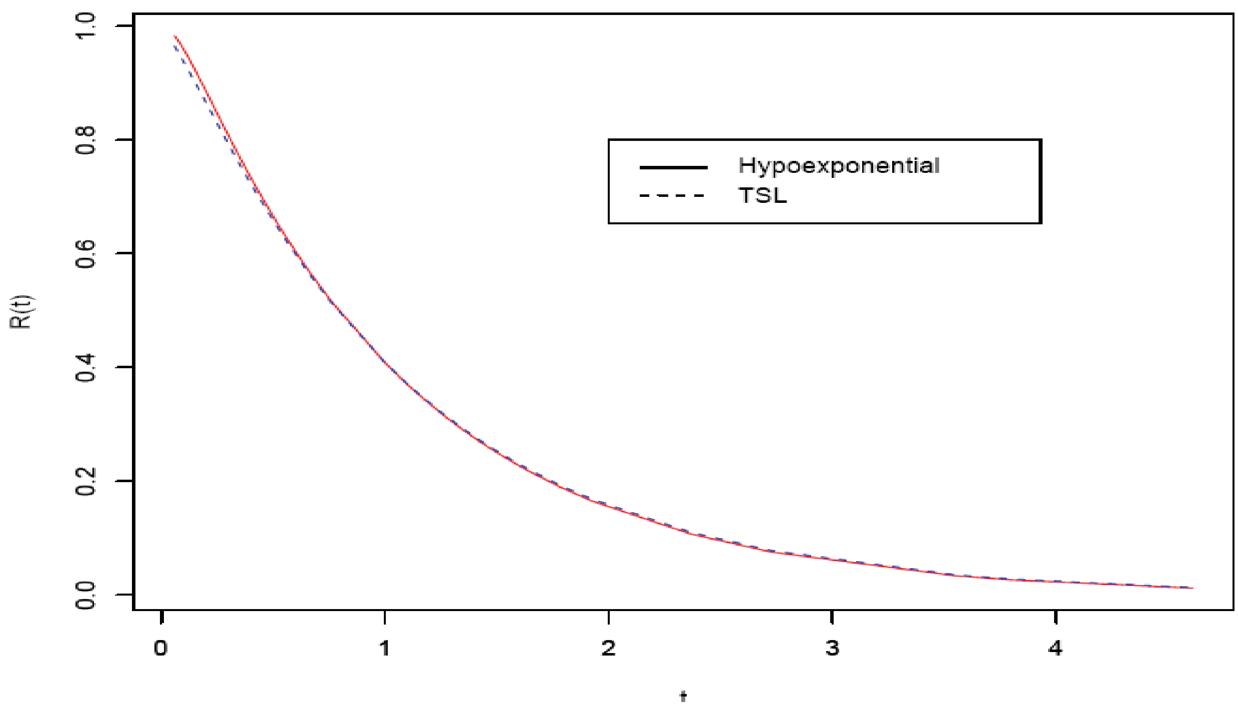
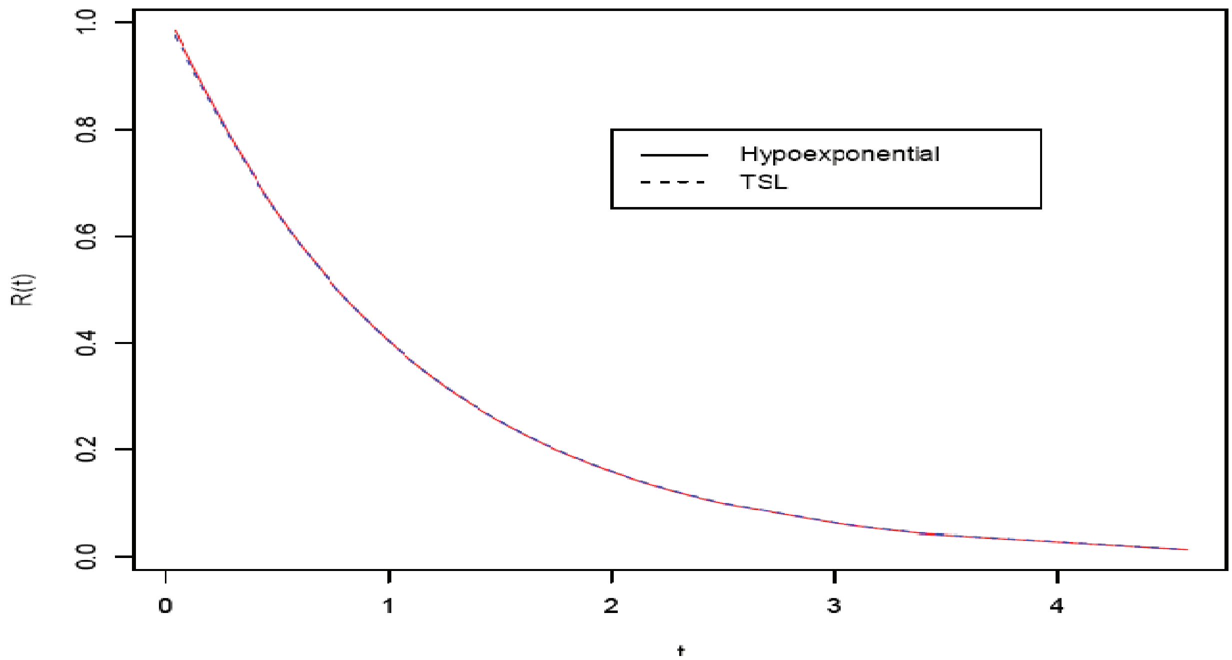


Figure 5: Reliability of TSL and Hypoexponential Distributions for n=50, λ_1=1, λ_2=20



that the failures are easily detected and instantly replaced and that cost c_1 includes the cost resulting from planned replacement and cost c_2 that includes all costs invested resulting from failure then the expected cost during the period [0, t] is given by

$$C(t) = c_1 E(N_1(t)) + c_2 E(N_2(t)), \quad (4.1)$$

where, $E(N_1(t))$ and $E(N_2(t))$ denote the expected number of planned replacement and the expected number of failures respectively. The goal is to determine the policy minimizing $C(t)$ for a finite time span or minimizing $\lim_{t\to\infty}\frac{C(t)}{t}$ for an infinite time span. Because the TSL probability distribution has an increasing failure rate it is expected that this model would be useful in a maintenance system.

Age Replacement Policy and TSL Probability Distribution

Consider the so-called age replacement policy; in this policy an item is always replaced exactly at the time of failure, or at t^* time after installation, whichever occurs first. This age replacement policy for infinite time spans seems to have received the most attention in the literature. Morese (1958) showed how to determine the replacement interval minimizing cost per unit time, while Barlow and Proschen (1962) proved that if the failure distribution, F, is continuous then a minimum-cost age replacement exists for any infinite time span.

In this article, the goal is to determine the optimal time t^* at which preventive replacement should be performed. The model should determine the time t^* that minimizes the total expected cost of preventive and failure maintenance per unit time. The total cost per cycle consists of the cost of preventive maintenance in addition to the cost of failure maintenance. Hence,

$$EC(t^*) = c_1\left(R(t^*)\right) + c_2\left(1 - R(t^*)\right) \quad (4.1.1)$$

where c_1 and c_2 denote the cost of preventive maintenance and failure maintenance respectively, and $R(t^*)$ is the probability that the equipment survives until age t^*. The expected cycle length consists of the length of a preventive cycle plus the expected length of a failure cycle. Thus, we have

$$\text{Expected Cycle Length} = t^* R(t^*) + M(t^*)\left(1 - R(t^*)\right)$$

(4.1.2)

where

$$M(t^*)\left(1 - R(t^*)\right) = \int_{-\infty}^{t^*} tf(t)dt$$

is the mean of the truncated distribution at time t^*. Hence, the expected cost per unit time is equal to:

$$\frac{c_1 R(t^*) + c_2\left[1 - R(t^*)\right]}{t^* R(t^*) + M(t^*)\left[1 - R(t^*)\right]} \quad (4.1.3)$$

Assume that a system has a time to failure distribution of the truncated skew Laplace pdf; the goal is to compute the optimal time t^* of preventive replacement. Because the reliability function of a TSL random variable is given by

$$R(t^*) = \frac{2(1+\lambda)\exp\left(-\frac{t^*}{\varphi}\right) - \exp\left(-\frac{(1+\lambda)t^*}{\varphi}\right)}{(2\lambda+1)}$$

and

$$M(t^*) = \frac{1}{1 - R(t*)}\int_0^{t^*} tf(t)dt$$

thus,

$$\begin{aligned}\int_0^{t^*} tf(t)dt = {} & \frac{2(1+\lambda)\varphi}{2\lambda+1}\left[1 - \exp(-t^*/\varphi)\right] \\ & - \frac{2(1+\lambda)}{2\lambda+1} t^* \exp(-t^*/\varphi) \\ & + \frac{t^*}{(2\lambda+1)}\exp\left(-(1+\lambda)t^*/\varphi\right) \\ & - \frac{\varphi}{(2\lambda+1)(1+\lambda)t^*/\varphi}\left[1 - \exp\left(-(1+\lambda)t^*/\varphi\right)\right]\end{aligned}$$

Substituting and simplifying the expressions, the expected cost per unit time (ECU) is given by:

$$ECU(t^*) = \frac{(1+\lambda)}{\varphi} \frac{\left\{\begin{array}{l} 2(c_2 - c_1)(1+\lambda)\exp(-t^*/\varphi) \\ -(c_2 - c_1)\exp\left(-(1+\lambda)t^*/\varphi\right) \\ -c_2(2\lambda+1) \end{array}\right\}}{\left\{\begin{array}{l} 2(1+\lambda)^2 \exp\left(-t^*/\varphi\right) \\ -(1+4\lambda+2\lambda^2) \\ -\exp\left(-(1+\lambda)t^*/\varphi\right) \end{array}\right\}}$$

(4.1.4)

Methodology

In order to minimize a function $g(t)$ subject to $a \le t \le b$ the Golden Section Method, which employs the following steps to calculate the optimum value may be used.

Step 1:

Select an allowable final tolerance level δ and assume the initial interval where the minimum lies is $[a_1, b_1] = [a, b]$ and let

$$\lambda_1 = a_1 + (1-\alpha)(b_1 - a_1)$$
$$\mu_1 = a_1 + \alpha(b_1 - a_1)$$

Take $\alpha = 0.618$, which is a positive root of $c^2 + c - 1 = 0$, evaluate $g(\lambda_1)$ and $g(\mu_1)$, and let $k = 1$. Go to Step 2.

Step 2:
If $b_k - a_k \le \delta$, stop because the optimal solution is $t^* = (a_k + b_k)/2$, otherwise, if $g(\lambda_k) > g(\mu_k)$ go to Step 3; or if $g(\lambda_k) \le g(\mu_k)$, go to Step 4.

Step 3:
Let $a_{k+1} = a_k$, $b_{k+1} = b_k$, $\lambda_{k+1} = \mu_k$ and $\mu_{k+1} = a_{k+1} + \alpha(b_{k+1} - a_{k+1})$. Evaluate $g(\mu_{k+1})$ and go to Step 5.

Step 4:
Let $a_{k+1} = a_k$, $b_{k+1} = \mu_k$, $\mu_{k+1} = \lambda_k$ and $\lambda_{k+1} = a_{k+1} + (1-\alpha)(b_{k+1} - a_{k+1})$. Evaluate $g(\lambda_{k+1})$ and go to Step 5.

Step 5:
Replace k by $k+1$ and go to Step 1.

Example
To implement this method to find the time t^* subject to the condition that $c_1 = 1$ and $c_2 = 10$ proceed as follows:

Iteration 1:
Consider $[a_1, b_1] = [0,10]$, where $\alpha = 0.618$ so that $1-\alpha = 0.382$.

$$\lambda_1 = a_1 + (1-\alpha)(b_1 - a_1) = 3.82$$
$$\mu_1 = a_1 + \alpha(b_1 - a_1) = 6.18,$$
$$ECU(\lambda_1) = 8.561, \text{ and } ECU(\mu_1) = 8.570.$$

Because $ECU(\lambda_1) < ECU(\mu_1)$ the next interval where the optimal solution lies is $[0, 6.18]$.

Iteration 2:
$[a_2, b_2] = [0, 6.18], \lambda_2 = 2.36$ *and* $\mu_2 = 3.82$.
$ECU(\lambda_2) = 8.533$ *and* $ECU(\mu_2) = 8.561$.

Because $ECU(\lambda_2) < ECU(\mu_2)$ the next interval where the optimal solution lies is $[0, 3.82]$.

Iteration 3:
$[a_3, b_3] = [0, 3.82], \lambda_3 = 1.459$ *and* $\mu_3 = 2.36$
$ECU(\lambda_3) = 8.516$ *and* $ECU(\mu_3) = 8.533$.

Because $ECU(\lambda_3) < ECU(\mu_3)$ the next interval where the optimal solution lies is $[0, 2.36]$.

Iteration 4:
$[a_4, b_4] = [0, 2.36], \lambda_4 = 0.901$ *and* $\mu_4 = 1.459$
$ECU(\lambda_4) = 8.613$ *and* $ECU(\mu_4) = 8.516$.

Because $ECU(\lambda_4) > ECU(\mu_4)$ the next interval where the optimal solution lies is $[0.901, 2.36]$.

Iteration 5:
$[a_5, b_5] = [0.901, 2.36], \lambda_5 = 1.459$ *and* $\mu_5 = 1.803$
$ECU(\lambda_5) = 8.516$ *and* $ECU(\mu_5) = 8.517$.

Because $ECU(\lambda_5) < ECU(\mu_5)$ the next interval where the optimal solution lies is $[0.901, 1.803]$.

Iteration 6:
$[a_6, b_6] = [0.901, 1.803], \lambda_6 = 1.246$ *and* $\mu_6 = 1.459$
$ECU(\lambda_6) = 8.528$ *and* $ECU(\mu_6) = 8.516$.

Because $ECU(\lambda_6) > ECU(\mu_6)$ the next interval where the optimal solution lies is $[1.246, 1.803]$.

Iteration 7:
$[a_7, b_7] = [1.246, 1.803], \lambda_7 = 1.459$ *and* $\mu_7 = 1.590$
$ECU(\lambda_7) = 8.516$ *and* $ECU(\mu_7) = 8.514$.

Because $ECU(\lambda_7) > ECU(\mu_7)$ the next interval where the optimal solution lies is $[1.459,\ 1.803]$.

If the δ level is fixed at 0.5, it can be concluded that the optimum value lies in the interval $[1.459,\ 1.803]$ and is given by 1.631. This numerical example was performed assuming that the failure data follows the $TSL(1,1)$ model and it has been observed that to optimize the cost, maintenance should be scheduled at 1.631 units of time.

Block Replacement Policy and TSL Probability Distribution

Consider the case of the Block-Replacement Policy, or the constant interval policy. In this policy preventive maintenance is performed on the system after it has been operating a total of t^* units of time, regardless of the number of intervening failures. In the case where the system has failed prior to the time t^*, minimal repairs are be performed. Assume that the minimal repair will not change the failure rate of the system and that preventive maintenance renews the system to its original new state. Thus, the goal is to find the time t^* that minimizes the expected repair and preventive maintenance costs. The total expected cost per unit time for preventive replacement at time t^*, denoted by ECU (t^*) is given by

$$ECU(t^*) = \frac{Total\ expected\ cost\ in\ the\ interval (0,t^*)}{Length\ of\ the\ interval} \tag{4.2.1}$$

The total expected cost in the interval (0, t^*) equals the cost of preventative maintenance plus the cost of failure maintenance, which is given by $c_1 + c_2 M(t^*)$, where M (t^*) is the expected number of failures in the interval (0, t^*). Thus,

$$ECU(t^*) = \frac{c_1 + c_2 M(t^*)}{t^*}. \tag{4.2.2}$$

It is known that the expected number of failures in the interval (0, t^*) is the integral of the failure rate function, that is

$$M(t^*) = E\big(N(t^*)\big) = H(t^*) = \int_0^{t^*} h(t)dt.$$

Therefore, if the failure of the system follows the TSL distribution it may be observed that

$$M(t^*) = \int_0^{t^*} h(t)dt = \frac{(1+\lambda)t^*}{\varphi} - \log\big((2+2\lambda)\exp(\lambda t^*/\varphi) - 1\big) + \log(2\lambda+1)$$

Therefore,

$$ECU(t^*) = \frac{c_1 + c_2\left[\frac{(1+\lambda)t^*}{\varphi} - \log\big((2+2\lambda)\exp(\lambda t^*/\varphi) - 1\big) + \log(2\lambda+1)\right]}{t^*}. \tag{4.2.3}$$

Example

To minimize the total expected cost subject to the conditions $c_1 = 1$ and $c_2 = 10$, the Golden Section Method (as described above) is used to obtain the value of t^*

Iteration 1:
$[a_1, b_1] = [0,10], \alpha = 0.618$, $1-\alpha = 0.382$,

$$\lambda_1 = a_1 + (1-\alpha)(b_1 - a_1) = 3.82,$$
$$\mu_1 = a_1 + \alpha(b_1 - a_1) = 6.18,$$

$$ECU(\lambda_1) = 9.523 \text{ and } ECU(\mu_1) = 9.697.$$

Because $ECU(\lambda_1) < ECU(\mu_1)$ the next interval where the optimal solution lies is $[0,\ 6.18]$.

Iteration 2:
$[a_{2,} b_2] = [0,6.18]$, $\lambda_2 = 2.36$, $\mu_2 = 3.82$, $ECU(\lambda_2) = 9.30$, and $ECU(\mu_2) = 9.523$.

Because $ECU(\lambda_2) < ECU(\mu_2)$ the next interval where the optimal solution lies is $[0,\ 3.82]$.

Iteration 3:
$[a_{3,} b_3] = [0,3.82]$, $\lambda_3 = 1.459$, $\mu_3 = 2.36$, $ECU(\lambda_3) = 9.124$ and $ECU(\mu_3) = 9.30$.

Because $ECU(\lambda_3) < ECU(\mu_3)$ the next interval where the optimal solution lies is $[0,\ 2.36]$.

Iteration 4:
$[a_{4,} b_4] = [0,2.36]$, $\lambda_4 = 0.901$, $\mu_4 = 1.459$, $ECU(\lambda_4) = 9.102$ and $ECU(\mu_4) = 9.124$.

Because $ECU(\lambda_4) < ECU(\mu_4)$ the next interval where the optimal solution lies is $[0,\ 1.459]$.

Iteration 5:
$[a_{5,} b_5] = [0,1.459]$, $\lambda_5 = 0.557$, $\mu_5 = 0.901$, $ECU(\lambda_5) = 9.405$ and $ECU(\mu_5) = 9.124$.

Because $ECU(\lambda_5) > ECU(\mu_5)$ the next interval where the optimal solution lies is $[0.557,\ 1.459]$.

Iteration 6:
$[a_{6,} b_6] = [0.557,1.459]$, $\lambda_6 = 0.9015$, $\mu_6 = 1.114$, $ECU(\lambda_6) = 9.102$, and $ECU(\mu_6) = 9.08$.

Because $ECU(\lambda_6) > ECU(\mu_6)$ the next interval where the optimal solution lies is $[0.901,\ 1.459]$.

Iteration 7:
$[a_{7,} b_7] = [0.901,1.459]$, $\lambda_7 = 1.114$, $\mu_7 = 1.245$, $ECU(\lambda_7) = 9.08$, and $ECU(\mu_7) = 9.09$.

Because $ECU(\lambda_7) < ECU(\mu_7)$ the next interval where the optimal solution lies is $[0.901,\ 1.245]$.

If the δ level is fixed at 0.5 it can be concluded that the optimum value lies in the interval $[0.901,\ 1.245]$ and it is given by 1.07. As in the case of age replacement in this numerical example it was assumed that the failure data follows $TSL(1,1)$ model. Observe that, in order to optimize the cost, maintenance must be scheduled at every 1.07 units of time.

Maintenance Over a Finite Time Span

The problem concerning the preventive maintenance over a finite time span is of great importance in industry. It can be viewed in two different perspectives based on whether the total number of replacements (failure + planned) times are known or unknown. The first case is straightforward and has been addressed in the literature for a long time. Barlow *et al.* (1962) derived the expression for this case. Let T^* represent the total time span, meaning minimization of the cost due to forced replacement or planned replacement until $T = T^*$. Let $C_n(T^*, T)$ represent the expected cost in the time span 0 to T^*, $[0, T^*]$, considering only the first n replacements and following a policy of replacement at interval T. It is clear that considering the case when $T^* \le T$ is equivalent to zero planned replacements, that

$$C_1(T^*, T) = \begin{cases} c_2 F(T^*),\ if\ T^* \le T, \\ c_2 F(T) + c_1\left(1 - F(T)\right),\ if\ T^* \ge T \end{cases} \tag{5.1}$$

Thus, for $n = 1, 2, 3, \ldots$, $C_{n+1}(T^*, T) =$

$$\begin{cases} \int_0^{T^*} \left[c_2 + C_n(T^* - y, T)\right] dF(y) \ \ if\ T^* \le T \\ \int_0^{T} \left[c_2 + C_n(T^* - y, T)\right] dF(y) \\ + \left[c_1 + C_n(T^* - T, T)\right] \times \left[1 - F(T)\right], \ \ otherwise \end{cases} \tag{5.2}$$

A statistical model may now be developed that can be used to predict the total cost of maintenance before an item is actually used. Let T equal the predetermined replacement time, and assume that an item is always replaced exactly at the time of failure T^* or T hours after its installation, whichever occurs first. Let τ denotes the failure time then we have two cases to consider,

Case 1: $T < T^*$

In this case the preventative maintenance (PM) interval is less than the finite planning horizon. For this case if the component fails after time, say, $\tau\,(for\,\tau < T)$ then the cost due to failure is incurred and the planning horizon is reduced to $[T^* - \tau]$. But if the component works till the preventive replacement time T then the cost due to preventive maintenance is incurred and the planning horizon is reduced to $[T^*\text{-}T]$.

The total cost incurred in these two cases is

$$C(T^*, T) = \int_0^{T} \left[c_2 + C(T^* - \tau, T)\right] f(\tau)\, d\tau + \left[1 - F(T)\right] \times \left[c_1 + C(T^* - T, T)\right] \tag{5.3}$$

where c_1 is the cost for preventive maintenance and $c_2 (> c_1)$ is the cost for failure maintenance.

Case 2: $T^* < T$

In this case the PM interval is greater than the planning horizon so there is no preventive maintenance but there is a chance of failure maintenance. Hence the total cost incurred will be

$$C(T^*, T) = \int_0^{T^*} \left[c_2 + C(T^* - \boldsymbol{\tau}, T)\right] f(\boldsymbol{\tau})\, d\boldsymbol{\tau} \tag{5.4}$$

The interest here lies in finding the preventative maintenance time T that minimizes the cost of the system. Consider a numerical example to determine whether the minimum exists if the failure model is assumed to be TSL *(1, 1)*. A random sample of size 100 was generated from *TSL (1, 1)* and a time T to perform preventive maintenance was fixed. The preventive maintenance cost $c_1 = 1$ was set along with the failure replacement cost $c_2 = 1$, 2 and 10. The process was repeated several times and the total cost for first 40 failures was computed. All necessary calculations were performed using the statistical language R. In the table 3, C_i, for $i = 1, 2, \& 10$ represents the total cost due to preventive maintenance cost $c_1 = 1$ and the failure replacement cost $c_2 = i$, $i = 1, 2\ \& 10$. It can be observed from table 3 that the minimum C_i exists around at T = 1.1 units of time. A preliminary study of the application of the TSL distribution in such environment can be found in Aryal, et al. (2008).

Table 3: Expected Costs at Different Maintenance Times

T	C_{10}	C_2	C_1
1.00	340.55	88.95	57.50
1.01	347.25	89.65	57.45
1.02	336.95	87.75	56.60
1.03	342.95	88.15	56.30
1.04	339.15	87.15	55.65
1.05	341.25	87.25	55.50
1.06	334.40	86.40	55.40
1.07	343.75	87.35	55.30
1.08	332.15	84.95	54.05
1.09	338.55	85.81	54.22
1.10	318.48	82.67	53.19
1.11	327.68	84.04	52.59
1.12	344.76	86.48	54.19
1.13	333.70	84.50	53.35
1.14	340.40	85.20	53.30
1.15	338.86	84.68	53.90
1.16	331.28	82.90	53.86
1.17	338.27	84.09	54.31
1.18	335.24	83.05	53.52
1.19	341.90	84.00	54.76
1.20	363.90	87.50	56.95

Conclusion

This study presented a comparison of the truncated skew Laplace probability distribution with the two parameter gamma probability distribution and hypoexponential probability distribution. A detailed procedure was also provided to apply the truncated skew Laplace probability distribution in the maintenance system.

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