Journal of Modern Applied Statistical Methods

Volume 8 | Issue 2 Article 21

11-1-2009

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Journal of Modern Applied Statistical Methods
November 2009, Vol. 8, No. 2, 566-570

1538 – 9472/09/$95.00

Test for the Equality of the Number of Signals

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A likelihood ratio test for testing the equality of the ranks of two non-negative definite covariance matrices arising in the area of signal processing is derived. The asymptotic distribution of the test statistic follows a Chi-square distribution from the general theory of likelihood ratio test.

Key words: Likelihood ratio test; signal processing; white noise.

Introduction

In the area of signal processing, signals are observed at different sensors from different sources at different time points. Wax, Shan and Kailath (1984) and Whalen (1971) discussed models and varieties of problems in signal processing. In general, the signal processing model is as follows:

$$\mathbf{X}(t) = A\mathbf{S}(t) + \mathbf{n}(t) \tag{1}$$

where, $\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_p(t))'$ is the px1 observation vector at time t, $\mathbf{S}(t) = (S_1(t), S_2(t), \ldots, S_q(t))'$ is the qx1 vector of unknown random signals at time t, $\mathbf{n}(t) = (n_1(t), n_2(t), \ldots, n_p(t))'$ is the px1 random noise vector at time t, $A = (A(\Phi_1), A(\Phi_2), \ldots, A(\Phi_q))$ is the pxq matrix of unknown coefficients, and $A(\Phi_r)$ is the px1 vector of functions of the elements of unknown vector $\mathbf{\Phi}_r$ associated with the r^th^ signal and q < p.

In model (1), $\mathbf{X}(t)$ is assumed to be distributed as p-variate normal with mean vector zero and dispersion matrix $A\Psi A' + \sigma^2 I_p = \Gamma + \sigma^2 I_p$, where $\Gamma = A\Psi A'$ is unknown n.n.d. matrix of rank q(<p) and Ψ = covariance matrix of $\mathbf{S}(t)$, σ^2 (>0) is unknown and $\sigma^2 I_p$ is the covariance matrix of the noise vector $\mathbf{n}(t)$. In this case, the model is called white noise model.

One important problem that arises in the area of signal processing is to estimate q, the number of signals transmitted. This problem is equivalent to estimating the multiplicity of the smallest eigenvalue of the covariance matrix of the observation vector. Anderson (1963), Krishnaiah (1976) and Rao (1983), among others, considered this problem. Wax and Kailath (1984) and Zhao, et al. (1986a, b) used information theoretic criteria proposed by Akaike (1972), Rissanen (1978) and Schwartz (1978) to estimate the number of signals.

More recently, Chen, et al. (2001), Chen (2002) and Kundu (2000) developed procedures for estimating the number of signals. This article considers the two sample problem of testing the equality of the number of signals between two sets of data from two populations. This problem is relevant in practice in the area of signal processing because it is important to know whether the total numbers of signals received are the same or not for two different days, which can be separated by a lengthy time. This problem is equivalent to testing the equality of multiplicity of the smallest eigenvalue of the covariance matrices of observation vectors of the two sets of data. Consider the following model:

$$X_i(t) = A_i S_i(t) + N_i(t); i = 1, 2$$

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where, $X_i(t)$ is a px1 observation vector for the i[th] population, $A_i = (A_i(\Phi_1^i),...,A_i(\Phi_{q_i}^i))$, $S_i(t) = (S_1^i(t),...,S_{q_i}^i(t))$, $i = 1,2$ and $N_i(t)$ is a px1 random noise vector for the i[th] population. The following are assumed about $N_i(t)$ and $S_i(t)$:

$$N_i(t) \sim N_p(0, \Sigma_i),$$
$$S_i(t) \sim N_p(0, \Psi_i),$$

and $N_i(t)$ and $S_i(t)$ are independently distributed. The null hypothesis to test is $H_{0k}: q_1 = q_2 = k$, and the alternative hypothesis is $H_1: q_1 \neq q_2, k = 0,1,..., p-1$. At this point, the likelihood ratio test is derived next and asymptotic distribution of the test statistic is used to obtain the critical value.

Likelihood Ratio Test: Case 1

Consider $\Sigma_i = \sigma^2 I_p, i = 1,2$. The test hypotheses are:

$$H_{0k}: q_1 = q_2 = k$$

and

$$H_1: q_1 \neq q_2, k = 0,1,..., p-1.$$

The observations from the two populations are as follows: $X_1(t_1),..., X_1(t_{n_1})$ are i.i.d. $\sim N_p(0, A_1\Psi_1A_1' + \sigma^2 I_p)$ and $X_2(t_{n_1+1}),..., X_2(t_{n_1+n_2})$ are i.i.d. $\sim N_p(0, A_2\Psi_2A_2' + \sigma^2 I_p)$.

Let $R_i = A_i\Psi_iA_i' + \sigma^2 I_p, i = 1,2$. It may be stated that testing H_{0k} is equivalent to testing the rank of $A_i\Psi_iA_i'$ = k, i = 1, 2.

Let $R_i = R_i^{(k)}$ under H_{0k}, i = 1,2. Using spectral decomposition of $R_1^{(k)}$ and $R_2^{(k)}$, it can be written that

$$R_1^{(k)} = (\lambda_1 - \sigma^2)U_1U_1' + ... + (\lambda_k - \sigma^2)U_kU_k' + \sigma^2 I_p$$

and

$$R_2^{(k)} = (\mu_1 - \sigma^2)V_1V_1' + ... + (\mu_k - \sigma^2)V_kV_k' + \sigma^2 I_p$$

where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k > \sigma^2$ are the eigenvalues of $R_1^{(k)}$ and $U_1,...,U_k$ are the corresponding orthonormal eigenvectors of $A_1\Psi_1A_1'$ and similarly, $\mu_1 \geq \mu_2 \geq ... \geq \mu_k > \sigma^2$ are the eigenvalues of $R_2^{(k)}$ and $V_1,...,V_k$ are the corresponding orthonormal eigen vectors of $A_2\Psi_2A_2'$.

Thus, under H_{0k} the log-likelihood (apart from a constant term) is

$$\begin{aligned}\log L &= -\frac{n_1}{2}\log\left|R_1^{(k)}\right| - \frac{n_1}{2}tr.(\hat{R}_1(R_1^{(k)})^{-1}) \\ &\quad -\frac{n_2}{2}\log\left|R_2^{(k)}\right| - \frac{n_2}{2}tr.(\hat{R}_2(R_2^{(k)})^{-1}) \\ &= -\frac{n_1}{2}tr.(\hat{R}_1(R_1^{(k)})^{-1}) - \frac{n_2}{2}tr.(\hat{R}_2(R_2^{(k)})^{-1}) \\ &\quad -\frac{n_1}{2}\sum_{i=1}^{k}\log\lambda_i - \frac{n_2}{2}\sum_{i=1}^{k}\log\mu_i \\ &\quad -\frac{n}{2}(p-k)\log\sigma^2\end{aligned} \tag{2}$$

where

$$\hat{R}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} X_1(t_i)X_1'(t_i),$$

$$\hat{R}_2 = \frac{1}{n_2}\sum_{i=n_1+1}^{n_1+n_2} X_2(t_i)X_2'(t_i)$$

and

$$n = n_1 + n_2.$$

Rather than maximizing (2), equivalently minimize

$$\begin{aligned}\log L^* &= n_1 tr.(\hat{R}_1(R_1^{(k)})^{-1}) + n_2 tr.(\hat{R}_2(R_2^{(k)})^{-1}) \\ &\quad + n_1\sum_{i=1}^{k}\log\lambda_i + n_2\sum_{i=1}^{k}\log\mu_i \\ &\quad + n(p-k)\log\sigma^2\end{aligned} \tag{3}$$

Orthogonal matrices P_1, P_2, U_1, U_2 exist such that

$$\hat{R}_1 = P_1 D_1 P_1', \qquad R_1^{(k)} = U_1 \Lambda_1 U_1'$$

and

$$\hat{R}_2 = P_2 D_2 P_2', \qquad R_2^{(k)} = U_2 \Lambda_2 U_2'$$

where, $D_1 = Diag.(l_1, ..., l_p)$, $l_1 \geq l_2 \geq ... \geq l_p$ are the eigenvalues of $\hat{R}_1$ and similarly, $D_2 = Diag.(\xi_1, ..., \xi_p)$, $\xi_1 \geq \xi_2 \geq ... \geq \xi_p$ are the eigenvalues of $\hat{R}_2$ $\Lambda_1 = Diag.(\lambda_1, ..., \lambda_k, \sigma^2, ..., \sigma^2)$, and $\Lambda_2 = Diag.(\mu_1, ..., \mu_k, \sigma^2, ..., \sigma^2)$. Thus, (3) can be rewritten as follows:

$$\begin{aligned}\log L^* &= n_1 tr.(P_1 D_1 P_1' U_1 \Lambda_1^{-1} U_1') \\ &\quad + n_2 tr.(P_2 D_2 P_2' U_2 \Lambda_2^{-1} U_2') \\ &\quad + n_1 \sum_{i=1}^{k} \log \lambda_i + n_2 \sum_{i=1}^{k} \log \mu_i \\ &\quad + n(p-k) \log \sigma^2 \\ &= n_1 tr.(D_1 V_1 \Lambda_1^{-1} V_1') + n_2 tr.(D_2 V_2 \Lambda_2^{-1} V_2') \\ &\quad + \text{term independent of } V_1 \text{and} V_2 \end{aligned} \tag{4}$$

where, $V_1 = P_1' U_1$ and $V_2 = P_2' U_2$ and hence V_1 and V_2 are orthogonal.

Differentiating (4) with respect to V_1 subject to $V_1 V_1' = I_p$ and equating it to 0, results in

$$D_1 \left[\Lambda_1^{-1} V_1' - V_1 \Lambda_1^{-1} V_1^{-2} \right] = 0$$

$$i.e., V_1 = I_p$$

Similarly, $V_2 = I_p$ is obtained. Hence, given λ_i's, μ_i's and σ^2,

$$\begin{aligned}\operatorname*{Inf}_{H_{0k}} \log L^* &= \\ & n_1 \left(\sum_{i=1}^{k} \frac{l_i}{\lambda_i} + \frac{\sum_{i=k+1}^{p} l_i}{\sigma^2} \right) + n_2 \left(\sum_{i=1}^{k} \frac{\xi_i}{\mu_i} + \frac{\sum_{i=k+1}^{p} \xi_i}{\sigma^2} \right) \\ & + n_1 \sum_{i=1}^{k} \log \lambda_i + n_2 \sum_{i=1}^{k} \log \mu_i + n(p-k) \log \sigma^2 \end{aligned} \tag{5}$$

Differentiating (5) with respect to λ_i's and equating it to 0, results in

$$-n_1 \frac{l_i}{\lambda_i^2} + \frac{n_1}{\lambda_i} = 0$$

$$i.e., \hat{\lambda}_i = l_i$$

Similarly, $\hat{\mu}_i = \xi_i ; i = 1, ..., k$.

Differentiating (5) with respect to σ^2 and equating it to 0, results in

$$\hat{\sigma}^2 = \frac{n_1 \sum_{i=k+1}^{p} l_i + n_2 \sum_{i=k+1}^{p} \xi_i}{n(p-k)},$$

hence,

$$\begin{aligned}\operatorname*{Sup}_{H_{0k}} \log L &= -nk - n_1 \frac{\sum_{i=k+1}^{p} l_i}{\sigma^2} - n_2 \frac{\sum_{i=k+1}^{p} \xi_i}{\sigma^2} \\ &\quad - n_1 \sum_{i=1}^{k} \log l_i - n_2 \sum_{i=1}^{k} \log \xi_i - n(p-k) \log \hat{\sigma}^2 \\ &= -np - n_1 \sum_{i=1}^{k} \log l_i - n_2 \sum_{i=1}^{k} \log \xi_i \\ &\quad - n(p-k) \log \left(\frac{n_1 \sum_{i=k+1}^{p} l_i + n_2 \sum_{i=k+1}^{p} \xi_i}{n(p-k)} \right) \\ &= L_1 \text{ (say)} \end{aligned}$$

In the above expression of L_1, the unknown k can be estimated by using Zhao, Krishnaiah and Bai's (1986a,b) information criterion as follows: Estimate k by $\hat{k}$ such that

$$I(\hat{k}, c_n) = \min_{0 \le k \le p-1} I(k, c_n),$$

where

$$I(k, c_n) = -L_1 + c_n \left\{ (2k+1) + 2\left(pk - k - \frac{k(k-1)}{2} \right) \right\}$$

and c_n is such that

$$\text{(i)} \lim_{n \to \infty} \frac{c_n}{n} = 0$$

$$\text{(ii)} \lim_{n \to \infty} \frac{c_n}{\log \log n} = \infty$$

For practical purposes, choose $c_n = \log n$ which satisfies conditions (i) and (ii). Hence,

$$L_1^* = -np - n_1 \sum_{i=1}^{\hat{k}} \log l_i - n_2 \sum_{i=1}^{\hat{k}} \log \xi_i - n(p-\hat{k}) \log \left(\frac{n_1 \sum_{i=\hat{k}+1}^{p} l_i + n_2 \sum_{i=\hat{k}+1}^{p} \xi_i}{n(p-\hat{k})} \right) \quad (6)$$

Similarly,

$$L_2^* = \underset{H_1}{Sup} \log L = -np - n_1 \sum_{i=1}^{\hat{q}_1} \log l_i - n_2 \sum_{i=1}^{\hat{q}_2} \log \xi_i - \left[n_1(p-\hat{q}_1) + n_2(p-\hat{q}_2) \right] \log \left(\frac{n_1 \sum_{i=\hat{q}_1+1}^{p} l_i + n_2 \sum_{i=\hat{q}_2+1}^{p} \xi_i}{n_1(p-\hat{q}_1) + n_2(p-\hat{q}_2)} \right) \quad (7)$$

where, $\hat{q}_1$ and $\hat{q}_2$ are obtained such that

$$I(\hat{q}_1, \hat{q}_2, c_n) = \min_{\substack{0 \le q_1 \le p-1 \\ 0 \le q_2 \le p-1}} I(q_1, q_2, c_n)$$

and

$$I(q_1, q_2, c_n) = np + n_1 \sum_{i=1}^{q_1} \log l_i + n_2 \sum_{i=1}^{q_2} \log \xi_i + \begin{bmatrix} n_1(p-q_1) \\ +n_2(p-q_2) \end{bmatrix} \log \left(\frac{n_1 \sum_{i=q_1+1}^{p} l_i + n_2 \sum_{i=q_2+1}^{p} \xi_i}{n_1(p-q_1) + n_2(p-q_2)} \right) + c_n \left\{ \begin{array}{l} (q_1+q_2)+1 \\ +(p-1)(q_1+q_2) - \frac{q_1(q_1-1)}{2} - \frac{q_2(q_2-1)}{2} \end{array} \right\}$$

and c_n is defined the same as previously.

Hence log of likelihood ratio statistic is $L_1^* - L_2^*$, where L_1^* and L_2^* are given by (6) and (7) respectively. The critical value for this test can be approximated from the fact that asymptotically, $-2(L_1^* - L_2^*) \sim \chi^2_{\gamma(\hat{q}_1, \hat{q}_2, \hat{k})}$ under H_0 where,

$$\gamma(\hat{q}_1, \hat{q}_2, \hat{k}) = (\hat{q}_1 + \hat{q}_2 - 2\hat{k}) + (p-1)(\hat{q}_1 + \hat{q}_2 - 2\hat{k}) + \hat{k}(\hat{k}-1) - \frac{\hat{q}_1(\hat{q}_1-1)}{2} - \frac{\hat{q}_2(\hat{q}_2-1)}{2}.$$

Likelihood Ratio Test: Case 2

Consider, $\Sigma_i = \sigma_i^2 I_p, i = 1, 2$. For case 2, the problem can be solved similarly and the problem is easier than that in case 1.

Likelihood Ratio Test: Case 3

Consider, σ^2 is known in case 1. Without loss of generality, σ^2 = 1 can be assumed and in that case, the log likelihood must be maximized with respect to the eigenvalues subject to the condition that the eigenvalues are greater than 1, in which case the technique presented by Zhao, Krishnaiah and Bai (1986a, b) can be used.

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