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Shrinkage Estimation in the Inverse Rayleigh Distribution

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The properties of the shrinkage test–estimators of the parameter were studied for an inverse Rayleigh model under the asymmetric loss function. Both the single and double–stage shrinkage test–estimators are considered.

Key words: Shrinkage factor; Shrinkage test–estimator; Level of significance; Relative bias; Relative efficiency.

Introduction

If x is a random variable that follows the inverse Rayleigh distribution with the parameter θ, then it has the distribution function

$$F(x;\theta) = \exp\left(-\frac{\theta}{x^2}\right); x > 0, \theta > 0. \quad (1.1)$$

If $x_1, x_2, ... x_n$ is the n random observations drawn from model (1.1), then the maximum likelihood estimator (MLE) and the unbiased estimator of θ are given respectively as

$$\hat{\theta}_{ML} = \frac{n}{T} \text{ and } \hat{\theta}_U = \frac{n-1}{T}. \quad (1.2)$$

Here, $T = \sum_{i=1}^{n} \frac{1}{x_i^2}$ is a sufficient statistic for the parameter θ.

In the estimation problem when positive and negative errors have different consequences, the use of SELF (Squared error loss function) is not appropriate. Varian (1975) discussed an asymmetric loss function known as the LINEX loss function (LLF). This loss function is convex and its shape is determined by the value of its shape parameter. The positive (negative) values of the shape parameter, gives more weight to underestimation (overestimation). Further, the magnitude of the shape parameter reflects the degree of asymmetry. The invariant form of the LLF is defined as

$$L(\Delta) = \left\{e^{a\Delta} - a\Delta - 1\right\}; \ a \neq 0$$

and

$$\Delta = \left(\frac{\hat{\theta}}{\theta} - 1\right). \quad (1.3)$$

Here $'a'$ is the shape parameter of the LLF and $\hat{\theta}$ is any estimate of the parameter θ. When $a > 0$, the loss function increases almost exponentially for positive Δ and almost linearly otherwise and overestimation is more heavily penalized than underestimation. When $a < 0$ the linear exponential rises are interchanged and underestimation is considered more costly than overestimation. The LINEX loss function may be considered a natural extension of SELF (for small values of $'a'$ (near to zero) the LINEX loss function tends to SELF). Srivastava and Tanna (2001), Xu and Shi (2004), Prakash and Singh (2006), Singh, et al. (2007), Prakash and Singh (2009) and others have discussed estimation procedures under LLF.

In many situations, the experimenter has some prior information about the parameter in the form of a point or guess value and it is recognized that a shrinkage estimator performs better if a guess value of the parameter is approximately the true value and the sample size

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is small. Thompson (1968), Mehta and Srinivasan (1971), Pandey and Singh (1977), Pandey (1979), Singh, et al. (1996), Singh, et al. (2007) and others have suggested shrinkage estimators utilizing the point guess value of the parameter.

The study is presented for the single and double stage shrinkage test–estimators for the parameter θ under the LLF.

Methodology

Proposed Class of Estimator for the Parameter θ

The proposed class of estimator for the parameter θ is defined as

$$\theta_C = C\,\hat{\theta}_U = C\,\frac{n-1}{T};\, C \in R^+. \quad (2.1)$$

The value of constant $C = \hat{C}$ (for example), which minimizes the risk of θ_C under the LLF, is obtained by solving the given equality numerically

$$\frac{e^a}{n-1} = I\left(0, \infty, \frac{1}{z}\left(a\,C\,\frac{n-1}{z}\right)\right), \quad (2.2)$$

where

$$I\,(p, q, \omega) = \frac{1}{\Gamma n}\int_p^q (\omega)\, e^{-z} z^{n-1}\, dz$$

and ω is the function of z.

Thus, the improved class of estimator of θ in the class (2.1) is

$$\hat{\theta}_C = \hat{C}\,\hat{\theta}_U \quad (2.3)$$

with the risk under the LLF

$$R\left(\hat{\theta}_C\right) = e^{-a}\, I\left(0, \infty, e^{a\Delta_0}\right) + a\left(1-\hat{C}\right) - 1;$$
$$\Delta_0 = \hat{C}\,\frac{n-1}{z}. \quad (2.4)$$

Proposed Shrinkage Estimator and its Properties

Following Thompson (1968), a shrinkage estimator for the parameter θ when θ_0, a guess value of θ is available, is defined as

$$\theta_{SH} = k_1\left(\hat{\theta}_U - \theta_0\right) + \theta_0;\; k_1 \in [0,1]. \quad (3.1)$$

Depending on the guessed value θ_0 used, a shrinkage factor k_1 is specified. The shrinkage procedure has been applied to a number of different problems, a few examples include: mean survival time in epidemiological studies (Harries & Shakarki, 1979), forecasting money supplies (Tso, 1990), estimating mortality rates (Marshall, 1991) and improving estimation in sample surveys (Wooff, 1985).

The risk under the LLF (1.3) for the shrinkage estimator θ_{SH} is given by

$$R\left(\hat{\theta}_{SH}\right) = e^{a(\delta-1)} I\left(0, \infty, e^{a\Delta_1}\right) + a\left(1-\delta\right)\left(1-k_1\right) - 1, \quad (3.2)$$

where

$$\Delta_1 = k_1\left(\frac{n-1}{z} - \delta\right) \text{ and } \delta = \frac{\theta_0}{\theta}.$$

The value of $k_1 = k_2$ (for example) that minimizes $R\left(\hat{\theta}_{SH}\right)$, is also obtained by solving the given equality numerically:

$$(1-\delta)\, e^{a(1-\delta)} = I\left(0, \infty, \left(\frac{\Delta_1}{k_1} e^{a\Delta_1}\right)\right). \quad (3.3)$$

Therefore, the improved shrinkage estimator for θ in the class (3.1) is

$$\hat{\theta}_{SH} = k_2\left(\hat{\theta}_U - \theta_0\right) + \theta_0. \quad (3.4)$$

The expressions of the relative bias and the risk under the LLF are obtained as

$$RB\left(\hat{\theta}_{SH}\right) = \frac{1}{\theta} E\left(\hat{\theta}_{SH}\right) - 1 = \left(1-\delta\right)\left(k_1 - 1\right) \quad (3.5)$$

and

$$R\left(\hat{\theta}_{SH}\right) = e^{a(\delta-1)} I\left(0, \infty, e^{a\Delta_2}\right) + a\left(1-\delta\right)\left(1-k_2\right) - 1 , \tag{3.6}$$

where $\Delta_2 = k_2\left(\frac{n-1}{z} - \delta\right)$.

The expression of relative bias of $\hat{\theta}_{SH}$ clearly shows that the relative bias is zero at $\delta = 1.00$ and has a tendency of being negative for $0 < \delta < 1.00$ and positive otherwise.

The relative efficiency for the shrinkage estimator $\hat{\theta}_{SH}$ with respect to the improved estimator $\hat{\theta}_C$ is defined as

$$RE\left(\hat{\theta}_{SH}, \hat{\theta}_C\right) = R\left(\hat{\theta}_C\right) / R\left(\hat{\theta}_{SH}\right).$$

The expression $RE\left(\hat{\theta}_{SH}, \hat{\theta}_C\right)$ involves δ, a and n. For the selected set of values of $\delta = 0.40\,(0.20)\,1.60$; $a = 0.25, 0.50, 1.00$ and $n = 04, 08, 12, 15$, the numerical findings of the relative efficiency are presented in Table 1 for $a = 0.25, 0.50$.

Based on the values in the table, it may be concluded that the shrinkage estimator $\hat{\theta}_{SH}$ performs better than the estimator $\hat{\theta}_C$ for the considered set of the parametric space and attains maximum efficiency at the point $\delta = 1.00$. Also, the efficiencies increase (decrease) for $a\,(n)$ increases when other parametric values are fixed (except $\delta = 1.00$).

The Shrinkage Test–Estimators and their Properties

It has been shown that the shrinkage estimator $\hat{\theta}_{SH}$ has a lower risk than the improved estimator $\hat{\theta}_C$ when a guess value θ_0 of θ is near to the true value of the parameter θ.

Thus, the shrinkage test–estimator is proposed for testing the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ based on a given set of data. The test statistic $2\theta_0 T \sim \chi^2_{(2n)}$ is used for testing H_0. If α is the level of significance then the null hypothesis H_0 is not rejected if $1 - \alpha = P[\, m_1 \le 2\theta_0 T \le m_2 \,]$.

Table 1: Relative efficiency for the Shrinkage Estimator $\hat{\theta}_{SH}$ with respect to $\hat{\theta}_C$ for $a = 0.25$ and 0.50

n	a	δ						
		0.40	0.60	0.80	1.00	1.20	1.40	1.60
04	0.25	1.7810	3.1662	10.358	15.774	6.9258	2.1400	1.5999
	0.50	1.9967	3.6602	11.779	16.036	9.9987	2.9404	1.6161
08	0.25	1.2857	1.8153	4.5882	19.534	4.4207	1.6963	1.1901
	0.50	1.5537	2.2628	5.7189	23.606	5.3097	1.9740	1.3167
12	0.25	1.1796	1.5122	3.2651	20.664	3.1455	1.4269	1.1138
	0.50	1.2762	1.6553	3.5781	22.116	3.3206	1.4723	1.1313
15	0.25	1.1422	1.4045	2.7725	21.122	2.6720	1.3076	1.0604
	0.50	1.1699	1.4513	2.8794	21.419	2.6799	1.3339	1.0889

Thus, the proposed shrinkage test–estimators are

$$\hat{\theta}_{SH\,i} = \hat{\theta}_C + \left(\left(1-k_i\right)\theta_0 + \left(k_i - \hat{C}\right)\hat{\theta}_U\right) I_{(t_1 \le T \le t_2)}\ ; \tag{4.1}$$

where $I_{(A)}$ denotes the indicator of A, $t_i = \dfrac{m_i}{2\theta_0}$ and $i = 1, 2$. Here m_1 and m_2 are the values of the lower and upper $100\alpha/2\%$ points of the Chi–square distribution with $2n$ degrees of freedom.

The expression of the relative bias is obtained as:

$$RB\left(\hat{\theta}_{SHi}\right) = I\left(y_1, y_2, \Delta_i'\right) + \hat{C} - 1; \tag{4.2}$$

where $\Delta_i' = \left(\Delta_i - \Delta_0 + \delta\right)$, $y_i = \dfrac{m_i}{2\delta}$ and $i = 1, 2.$

Similarly, the expressions of the risk under the LLF for the proposed shrinkage test–estimators are

$$\begin{aligned} R\left(\hat{\theta}_{SHi}\right) = {} & e^{a(\delta-1)}\ I\left(y_1, y_2, e^{a\Delta_i}\right) \\ & - e^{-a}\ I\left(y_1, y_2, e^{a\Delta_0}\right) \\ & - aI\left(y_1, y_2, \Delta_i'\right) + e^{-a} I\left(0, \infty, e^{a\Delta_0}\right) \\ & + a\left(1-\hat{C}\right) - 1; \quad i = 1, 2. \end{aligned} \tag{4.3}$$

The value of $k_1 = k_3$ (for example) that minimizes the risk of the shrinkage test–estimator $\hat{\theta}_{SH1}$ may be obtained by solving following equality

$$e^{a(1-\delta)} I\left(y_1, y_2, \left(\frac{\Delta_1}{k_1}\right)\right) = I\left(y_1, y_2, \left(\frac{\Delta_1}{k_1} e^{a\Delta_1}\right)\right). \tag{4.4}$$

Hence, the improved shrinkage test–estimator is defined as

$$\hat{\theta}_{SH3} = \hat{\theta}_C + \left(\left(1-k_3\right)\theta_0 + \left(k_3 - \hat{C}\right)\hat{\theta}_U\right) I_{(t_1 \le T \le t_2)} \tag{4.5}$$

The expressions of the relative bias and the risk under the LLF are given as

$$RB\left(\hat{\theta}_{SH3}\right) = I\left(y_1, y_2, \Delta_3'\right) + \hat{C} - 1;$$
$$\Delta_3' = \left(\Delta_3 - \Delta_0 + \delta\right)$$

and

$$\begin{aligned} R\left(\hat{\theta}_{SH3}\right) = {} & e^{a(\delta-1)}\ I\left(y_1, y_2, e^{a\Delta_3}\right) \\ & - e^{-a}\ I\left(y_1, y_2, e^{a\Delta_0}\right) \\ & - aI\left(y_1, y_2, \Delta_3'\right) \\ & + e^{-a} I\left(0, \infty, e^{a\Delta_0}\right) + a\left(1-\hat{C}\right) - 1. \end{aligned} \tag{4.6}$$

The relative efficiency of the shrinkage test–estimator $\hat{\theta}_{SHi}$; $i = 1, 2, 3,$ with respect to improved estimator $\hat{\theta}_C$ is defined as

$$RE\left(\hat{\theta}_{SHi}, \hat{\theta}_C\right) = R\left(\hat{\theta}_C\right) \Big/ R\left(\hat{\theta}_{SHi}\right); \; i = 1, 2, 3.$$

The relative bias $RB\left(\hat{\theta}_{SH1}\right)$ and the relative efficiency $RE\left(\hat{\theta}_{SH1}, \hat{\theta}_C\right)$ are the functions of δ, k, α, a and n. For a similar set of values as considered previously with $k = 0.25, 0.50, 0.75$ and $\alpha = 0.01, 0.05,$ the relative bias (not presented) and the relative efficiency are presented in Table 2, for $n = 08$ and 12 .

The relative biases are negligibly small and lie between −0.014 and 0.019. For small values of $\delta \le 1.00,$ the relative bias is negative but for large δ it has a tendency to be positive. The value of the absolute relative bias (ARB) decreases as n increases for $\delta \ge 1.00$ when other parametric values are fixed. The ARB increases as a (α) increases for small $\delta \le 1.00$

and decreases otherwise. In addition, the ARB decreases when k increases for the all considered values of δ when other parametric values are fixed.

The shrinkage test–estimator $\hat{\theta}_{SH1}$ has smaller risk than $\hat{\theta}_C$ for the all considered values of the parametric space. The efficiency decreases as 'a' or k increases in the region $0.40 \le \delta \le 1.20$ when other parametric values are fixed and the efficiency attains maximum at the point $\delta = 1.00$. In addition, as the level of significance α increases, the efficiency decreases for the all considered values of δ.

The expressions of the relative bias and the relative efficiency for the test–estimator $\hat{\theta}_{SHi}$; $i = 2, 3$ are the functions of δ, α, a and n. For a similar set of values as considered earlier, the relative biases (not presented here) and the relative efficiencies are shown in Tables 3 and 4.

The relative biases of $\hat{\theta}_{SH2}$ are negligibly small and lie between −0.017 and 0.029. For small values of $\delta \le 1.00$, the relative bias is negative, but for large δ it has a tendency of being positive. The ARB increases as $a(\alpha)$ increases for small $\delta \le 1.00$ and decreases otherwise. The relative biases of $\hat{\theta}_{SH3}$ are also negligibly small and lie between −0.018 and 0.031. Other properties are similar to shrinkage test–estimator $\hat{\theta}_{SH2}$.

The shrinkage test–estimator $\hat{\theta}_{SH2}$ performs well with respect to $\hat{\theta}_C$ for the all considered parametric values and attains maximum efficiency at the point $\delta = 1.00$ (Table 3). The efficiency decreases as 'a' increases when $\delta \le 1.00$ for other fixed parametric values. This decreasing trend has also been observed when α increases for all considered values of δ.

Table 4 shows that the shrinkage test–estimator $\hat{\theta}_{SH3}$ performs uniformly well with respect to $\hat{\theta}_C$ for the all considered parametric values. The efficiency decreases as n increases in the region $0.80 \le \delta \le 1.40$ for other fixed parametric values. Other properties are observed to be similar to the shrinkage test–estimator $\hat{\theta}_{SH2}$.

The Double–Stage Shrinkage Test–Estimator

A double–stage procedure using prior information in the form of an initial estimate or a guessed value has been considered by many authors (Katti, 1962; Shah, 1964; Waikar & Katti, 1971; Al–Bayyati & Arnold, 1972; Waikar, et al., 1984; Adke, et al., 1987). Arnold & Al–Bayyati (1970) considered the double–stage shrinkage estimator for the mean of a normal population when a prior guessed value of the mean is available. Pandey, et al. (1988) proposed some shrinkage estimators for the variance of a Normal distribution at double–stage under mean square error criterion.

Let $x_{ji}(i = 1, 2, \ldots, n_j)$; $j = 1, 2$ be two random samples of size n_1 and n_2 respectively, drawn independently from the model (1.1) with the parameter θ. The pooled unbiased estimate of θ based on two samples of size n_1 and n_2 is

$$\theta_P = \frac{(n_1 - 1)T_2 + (n_2 - 1)T_1}{2T_1T_2};.$$

$$T_j = \sum_{i=1}^{n_j} \frac{1}{x_{ji}^2}, \ j = 1, 2 \qquad (5.1)$$

The proposed class of estimators for the pooled estimate of θ is given by

$$\theta_{PC} = l\theta_P ; \ l \in R^+. \qquad (5.2)$$

The value of $l = \hat{l}$ (for example), for which $R(\theta_{PC})$ is minimum is obtained by simplifying the given equality numerically

$$2e^a = G\left(0, \infty, 0, \infty, \left(\frac{\Delta'_D}{l} e^{a\Delta'_D}\right)\right), \qquad (5.3)$$

where

Table 2: Relative Efficiency for the Shrinkage Test-Estimator $\hat{\theta}_{SH1}$ with respect to $\hat{\theta}_C$ for $n = 08$ and 12

$n = 08$			δ						
α	a	k	0.40	0.60	0.80	1.00	1.20	1.40	1.60
0.01	0.25	0.25	1.2151	2.1922	2.3196	4.6215	3.8156	1.7756	1.6830
		0.50	1.1481	1.8933	2.0205	2.9787	2.7729	2.1138	1.3801
		0.75	1.1479	1.4051	1.4303	1.8448	1.7469	1.6798	1.5326
	0.50	0.25	1.1945	1.4547	1.9428	3.8979	3.5790	1.9019	1.8913
		0.50	1.1381	1.4508	1.7480	2.7789	2.7129	2.1823	1.4858
		0.75	1.1375	1.1574	1.3149	1.8276	1.7426	1.7329	1.5948
0.05	0.25	0.25	1.1347	2.1089	2.1210	2.8377	2.3843	1.5066	1.2878
		0.50	1.1345	1.7516	1.8392	2.1867	1.9566	1.7256	1.2700
		0.75	1.1323	1.2456	1.3262	1.5639	1.4307	1.4857	1.4403
	0.50	0.25	1.1332	1.4234	1.5926	2.3318	2.2350	1.6070	1.1905
		0.50	1.1315	1.4197	1.4455	1.9222	1.8855	1.7907	1.3849
		0.75	1.1132	1.1282	1.1312	1.4561	1.4220	1.3367	1.3282
$n = 12$									
0.01	0.25	0.25	1.6316	2.1191	2.3444	3.3353	2.5675	1.1708	1.1617
		0.50	1.6011	2.1061	2.0147	2.5170	2.2819	1.6603	1.0200
		0.75	1.4517	1.5849	1.4843	1.7736	1.6626	1.6518	1.4559
	0.50	0.25	1.5354	1.6908	1.7535	2.8634	2.4524	1.1878	1.1736
		0.50	1.5087	1.6781	1.6401	2.2561	2.1817	1.6498	1.0498
		0.75	1.3727	1.1940	1.2765	1.6437	1.6092	1.6217	1.4366
0.05	0.25	0.25	1.4620	1.9624	2.2676	2.5836	1.8173	1.0414	1.1474
		0.50	1.4437	1.8656	2.0132	2.0955	1.6523	1.3561	1.0191
		0.75	1.3858	1.5181	1.4754	1.5833	1.3628	1.3994	1.3366
	0.50	0.25	1.4319	1.6854	1.7453	2.0929	1.6666	1.0416	1.0582
		0.50	1.4145	1.6152	1.5948	1.7697	1.5237	1.3307	1.0478
		0.75	1.3557	1.1478	1.2402	1.3909	1.2701	1.3559	1.3211

Table 3: Relative efficiency for the Shrinkage Test-Estimator $\hat{\theta}_{SH2}$ with respect to $\hat{\theta}_C$

$n = 04$		δ						
α	a	0.40	0.60	0.80	1.00	1.20	1.40	1.60
0.01	0.25	1.6548	2.8665	4.7368	11.755	6.6751	3.1841	1.9473
	0.50	1.5454	2.3551	4.0652	9.9677	2.8202	2.1233	1.7596
	1.00	1.5401	2.1247	3.8357	9.6791	5.9872	3.2399	1.8926
0.05	0.25	1.5548	1.7367	2.5544	4.9625	3.6768	2.5214	1.7863
	0.50	1.4454	1.6955	2.2922	4.1782	2.4534	2.0365	1.7452
	1.00	1.4402	1.6507	2.2569	4.0661	3.1790	3.0188	1.7381
$n = 08$								
0.01	0.25	1.6543	1.9973	2.4546	6.0587	3.8926	2.1165	1.5681
	0.50	1.4361	1.4759	2.2995	5.8657	3.6473	2.1909	1.6250
	1.00	1.3288	1.0766	1.9320	5.5925	4.4410	2.8794	2.5328
0.05	0.25	1.5502	1.9872	2.1138	3.3657	2.4321	1.7246	1.4518
	0.50	1.3626	1.4470	1.7291	3.1299	2.2758	1.7918	1.5200
	1.00	1.3191	1.0424	1.5898	3.0576	2.0576	1.5325	1.1507
$n = 12$								
0.01	0.25	1.4375	1.8921	2.3629	3.8744	2.5683	1.7466	1.4662
	0.50	1.3713	1.5575	1.8136	3.7197	2.4520	1.7204	1.4438
	1.00	1.3244	1.3924	1.7434	3.7049	3.6813	3.1246	1.1731
0.05	0.25	1.3801	1.7143	2.2637	2.7897	1.8072	1.4265	1.3546
	0.50	1.3551	1.5169	1.7135	2.7499	1.6598	1.3866	1.3322
	1.00	1.3159	1.3391	1.4702	2.3182	2.2879	2.2782	1.1685
$n = 15$								
0.01	0.25	1.3716	1.7533	2.2959	3.8034	2.1006	1.6055	1.4303
	0.50	1.3628	1.6522	2.0846	3.7610	2.0189	1.5494	1.3781
	1.00	1.3114	1.3493	1.6074	3.2557	3.1649	2.9006	1.2831
0.05	0.25	1.3486	1.5474	2.2562	2.8378	1.6021	1.3135	1.3179
	0.50	1.3466	1.5130	2.0011	2.6023	1.4944	1.2548	1.2696
	1.00	1.2355	1.3321	1.3961	2.4358	2.1663	2.0297	1.1815

Table 4: Relative efficiency for the Shrinkage Test-Estimator $\hat{\theta}_{SH3}$ with respect to $\hat{\theta}_{C}$

$n = 04$		δ						
α	a	0.40	0.60	0.80	1.00	1.20	1.40	1.60
0.01	0.25	1.6548	2.9751	4.9163	12.671	6.9280	3.3047	1.9473
	0.50	1.5454	2.4443	4.2197	10.744	3.7855	2.2037	1.7596
	1.00	1.5401	2.2052	3.9810	10.433	6.2141	3.3629	1.8926
0.05	0.25	1.5548	1.8025	2.6512	5.3498	3.8165	2.6166	1.7863
	0.50	1.4454	1.7597	2.3790	4.5038	2.5468	2.1138	1.7452
	1.00	1.4402	1.7132	2.3424	4.3828	3.2998	3.1332	1.7381
$n = 08$								
0.01	0.25	1.6543	2.073	2.5476	6.5306	4.0401	2.1967	1.5681
	0.50	1.4361	1.5318	2.3866	6.3226	2.9271	2.1739	1.6250
	1.00	1.3288	1.1174	2.0052	6.0281	4.6093	2.9885	2.5328
0.05	0.25	1.5502	2.0625	2.3939	3.6278	2.5242	1.7899	1.4518
	0.50	1.3626	1.5018	1.7946	3.3737	2.3620	1.8597	1.5200
	1.00	1.3191	1.0819	1.6500	3.2957	2.4355	2.3905	1.1507
$n = 12$								
0.01	0.25	1.4375	1.9638	2.4524	4.1762	2.6656	1.8128	1.4662
	0.50	1.3713	1.6165	1.8823	4.1094	2.5449	1.7856	1.4438
	1.00	1.3244	1.4451	1.8094	3.9931	3.8208	2.4243	1.1731
0.05	0.25	1.3801	1.7792	2.3494	3.1007	1.8756	1.4805	1.3546
	0.50	1.3551	1.5743	1.7784	2.9641	1.7227	1.4395	1.3322
	1.00	1.3159	1.3898	1.5259	2.6987	2.3746	2.3645	1.1685
$n = 15$								
0.01	0.25	1.3716	1.8197	2.3829	4.0996	2.1802	1.6663	1.4303
	0.50	1.3628	1.7148	1.7636	4.0539	2.0954	1.6081	1.3781
	1.00	1.3114	1.4004	1.6683	3.5093	3.2848	2.1105	1.2831
0.05	0.25	1.3486	1.6060	2.3417	3.0588	1.6628	1.3632	1.3179
	0.50	1.3466	1.5703	1.7069	2.8050	1.5510	1.3023	1.2696
	1.00	1.2355	1.3825	1.4490	2.6255	2.2484	2.1063	1.1815

$$G(p_1, q_1, p_2, q_2, \omega) = \frac{1}{\Gamma n_1 \, \Gamma n_2}$$

$$\int_{z_1=p_1}^{q_1} \int_{z_2=p_2}^{q_2} (\omega) \, e^{-z_1} \, z_1^{n_1-1} \, e^{-z_1} \, z_1^{n_1-1} \, dz_1 \, dz_2 \, ,$$

$\Delta'_D = \frac{l}{2}\left(\frac{n_1-1}{z_1} - \frac{n_2-1}{z_2}\right)$ and ω may be the function of z_1 and z_2.

Thus, the improved pooled estimator among the class (5.2) is

$$\hat{\theta}_{PC} = \hat{l}\theta_P \qquad (5.4)$$

with the risk

$$R\left(\hat{\theta}_{PC}\right) = e^{-a} \, G\left(0, \infty, 0, \infty, \, e^{a\Delta_D}\right) + a\left(1 - \hat{l}\right) - 1, \qquad (5.5)$$

where

$$\Delta_D = \frac{\hat{l}}{2}\left(\frac{n_1-1}{z_1} - \frac{n_2-1}{z_2}\right).$$

The performances of the shrinkage test–estimator $\hat{\theta}_{SH3}$ are better in terms of the magnitude of efficiency when they are compared with $\hat{\theta}_{SH2}$. Hence, $\hat{\theta}_{SH3}$ has been considered in double–stage setup. The proposed double–stage shrinkage test–estimator is given as

$$\hat{\theta}_{DSH} = \hat{\theta}_{PC} + \left(\left(1 - k_3\right)\theta_0 + k_3 \, \hat{\theta}_{U1} - \hat{\theta}_{PC}\right)$$

$$I_{(t_1 \le T \le t_2)} \; ; \; \hat{\theta}_{U1} = \frac{n_1 - 1}{T_1} \, .$$

The proposed double–stage technique is to first obtain a sample size n_1 and compute $\hat{\theta}_{U1}$. If $\hat{\theta}_{U1}$ implies that the prior estimate θ_0 was reasonable, the sampling is stopped and the parameter is estimated with the help of a shrinkage estimator. Otherwise, n_2 additional observations are obtained and used to improve the estimate based on all $(n_1 + n_2)$ observations. The risk under the LLF for $\hat{\theta}_{DSH}$ is obtained as

$$R\left(\hat{\theta}_{DSH}\right) =$$

$$e^{a(\delta-1)} I\left(y_1, y_2, e^{a\Delta_3}\right)$$
$$- e^{-a} \, G\left(y_1, y_2, 0, \infty, e^{a\Delta_D}\right)$$
$$- aI\left(y_1, y_2, \Delta'_D\right)$$
$$+ e^{-a} G\left(0, \infty, 0, \infty, e^{a\Delta_D}\right)$$
$$+ a\left(1 - \hat{l}\right) - 1,$$

where

$$\Delta'_D = \left(\Delta_3 - \frac{\hat{l}}{2}\left(\frac{n_1-1}{z_1} + 1\right) + \delta\right).$$

The problem is considered as a sequential estimation problem with stopping random variable N defined as

$$N = \begin{cases} n_1 & \text{if } t_1 \le T_1 \le t_2 \\ n_1 + n_2 & \text{otherwise.} \end{cases} \qquad (5.6)$$

If a cost $d\,(>0)$ is introduced for each observation. Then the risk of $\hat{\theta}_{DSH}$ is:

$$\tilde{R}\left(\hat{\theta}_{DSH}\right) = R\left(\hat{\theta}_{DSH}\right) + d\,E\,(N)$$

Similarly the risk of $\hat{\theta}_{PC}$ is:

$$\tilde{R}\left(\hat{\theta}_{PC}\right) = R\left(\hat{\theta}_{PC}\right) + d\,(n_1 + n_2)$$

Therefore, the relative efficiency of $\hat{\theta}_{DSH}$ with respect to $\hat{\theta}_{PC}$ is given by:

$$RE\left(\hat{\theta}_{DSH}, \hat{\theta}_{PC}\right) = \frac{\tilde{R}\left(\hat{\theta}_{PC}\right)}{\tilde{R}\left(\hat{\theta}_{DSH}\right)}.$$

The function of the relative efficiency involves $n_1, n_2, \delta, a, \alpha$ and per unit cost d. For a similar set of selected values as considered previously with $n_2 = 04, 08$ and $d = 0.50, 05, 10, 50,$ calculated relative efficiencies are presented for $n_1 = 04, 08$ and $d = 0.50$ in Table 5.

The double–stage shrinkage test–estimator $\hat{\theta}_{DSH}$ performs well with respect to improved pooled estimator $\hat{\theta}_{PC}$ for the all considered parametric set of values and attains maximum efficiency at the point $\delta = 1.00$. The efficiency decreases as $\alpha(n_1)$ and increases for all δ when other parametric values are fixed. The decreasing trend was observed when n_2 increased for all considered values of δ. The nominal loss was recorded when per unit cost increased but the effective interval did not alter.

Conclusion

Based on the data presented, the performances of both the shrinkage test–estimators are uniformly well respect to the improved estimator $\hat{\theta}_C$ for the considered parametric set of values. Based on the gain in efficiency, $\hat{\theta}_{SH3}$ may be preferred over $\hat{\theta}_{SH2}$ in the region $0.60 \le \delta \le 1.40$. The double–stage shrinkage test–estimator $\hat{\theta}_{DSH}$ performs well with respect to improved pooled estimator $\hat{\theta}_{PC}$ for the all considered parametric set of values.

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Table 5: Relative efficiency for the Shrinkage Test-Estimator $\hat{\theta}_{DSH}$ with respect to $\hat{\theta}_{PC}$ for $n_1 = 04, 08$ and $d = 0.50$

$n_1 = 04$			δ						
α	n_2	a	0.40	0.60	0.80	1.00	1.20	1.40	1.60
0.01	04	0.25	1.8017	3.0445	6.7037	20.117	9.3750	4.2732	3.1907
		0.50	1.7424	3.0036	6.6478	20.104	5.8755	3.8743	3.0036
		1.00	1.4921	2.9083	6.4982	20.089	11.093	5.3159	2.9084
	08	0.25	1.2600	1.3411	1.6286	5.0506	3.3255	3.1786	1.8908
		0.50	1.2572	1.3010	1.6276	3.5493	2.4466	2.4118	2.3561
		1.00	1.1686	1.2825	1.6619	2.9527	2.8975	2.3116	2.1039
0.05	04	0.25	1.5173	2.1061	3.4159	6.5139	3.0115	2.7075	2.4881
		0.50	1.4216	2.0964	3.4164	6.5169	3.9956	3.2718	2.1564
		1.00	1.3883	2.0692	3.3801	7.5212	6.0421	4.2974	2.0266
	08	0.25	1.1299	1.1812	1.3278	2.7313	1.9040	1.6100	1.3276
		0.50	1.0965	1.2719	1.3730	2.6014	1.7896	1.7238	1.1896
		1.00	1.0832	1.2612	1.6608	2.4602	1.3531	1.2305	1.1708
$n_1 = 08$									
0.01	04	0.25	2.7420	5.3945	6.7491	27.239	13.595	5.0900	3.6150
		0.50	3.0881	8.4469	17.943	25.154	16.671	8.4366	6.0597
		1.00	3.0841	3.7177	17.822	24.781	15.313	5.6013	4.6651
	08	0.25	1.3572	1.9032	3.4935	8.7832	7.5900	4.2296	2.2389
		0.50	1.3639	1.9013	3.4851	8.7877	7.9879	7.0423	5.5448
		1.00	1.3311	1.4116	1.7516	3.5838	3.0122	2.6166	2.2118
0.05	04	0.25	1.9372	2.5090	3.4191	7.5824	4.6038	2.7900	2.5022
		0.50	2.2009	4.6659	10.215	17.766	11.694	6.5119	4.7137
		1.00	2.2546	2.9908	10.270	18.120	11.022	4.9076	3.9610
	08	0.25	1.2981	1.5169	2.1274	7.1942	3.9665	2.2372	2.1651
		0.50	1.3015	1.5169	2.1264	6.7378	5.8711	4.7378	3.6704
		1.00	1.3114	1.3557	1.7388	2.5168	1.5108	1.3964	1.2609

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