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Combining Independent Tests of Conditional Shifted Exponential Distribution

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The problem of combining n independent tests as $n \to \infty$ for testing that variables are uniformly distributed over the interval (0, 1) compared to their having a conditional shifted exponential distribution with probability density function $f(x|\theta) = e^{-(x-\gamma\theta)}, \quad x \geq \gamma\theta, \quad \theta \in [a,\infty), a \geq 0$ was studied. This was examined for the case where $\theta_1, \theta_2, \ldots$ are distributed according to the distribution function (DF) F and when the DF is Gamma (1, 2). Six omnibus methods were compared via the Bahadur efficiency. It is shown that, as $\gamma \to 0$ and $\gamma \to \infty$, the inverse normal method is the best among the methods studied.

Key words: Conditional shifted exponential, combining independent tests, omnibus methods, Bahadur efficiency.

Introduction

The combination of independent tests of hypothesis is an important and a popular statistical practice. Many methods are available to use to combine independent tests; these methods are compared by using different criteria including Exact Bahadur Slope (EBS), Approximate Bahadur Slope (ABS), Pitman Efficiency, Local Power, Admissibility and others.

If H_0 is a simple hypothesis, Birnbaum (1955) showed that, for given any non-parametric combination method with a monotone increasing acceptance region, there exists a problem for which this method is most powerful against some alternative. Littell and Folks (1971) studied four methods of combining a finite number of independent tests. They found that the Fisher method is better than the inverse normal method, the minimum of p-value method and maximum of p-vales via Bahadur efficiency. Later, Littell and Folks (1973) showed under mild conditions that the Fisher's method is optimal among all methods for combining a finite number of independent tests. Brown, Cohen and Strawderman (1976) have shown that such tests form a complete class.

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The Specific Problem

Consider n hypotheses of the form:

$$H_o^{(i)} : \eta_i = \eta_o^i \quad \text{vs} \quad H_1^{(i)} : \eta_i \in \Omega_i - \{\eta_o^i\} \tag{1}$$

such that $H_0^{(i)}$ is rejected for large values, i = 1, 2, …, n of some continuous random variable $T^{(i)}$. The n hypotheses are combined into one as

$$H_0 : (\eta_1,\ldots,\eta_n) = (\eta_o^1,\ldots,\eta_o^n) \quad \text{vs} \quad H_1^{(i)} : (\eta_1,\ldots,\eta_n) \in \left\{ \prod_{i=1}^{n} \Omega_i - \{(\eta_o^1,\ldots,\eta_o^n)\} \right\}. \tag{2}$$

For i = 1, 2, …, n the p-value of the i^{th} test is given by

$$P_i(t) = P_{H_o^{(i)}}\left(T^{(i)} > t\right) = 1 - F^i(t) \tag{3}$$

where F^i is the DF of $T^{(i)}$ under $H_o^{(i)}$. Note that under $H_o^{(i)}$ the random variable $P_i \sim U(0, 1)$ under $H_1^{(i)}$ has some distribution that is not U(0, 1).

If considering the special case where $\eta_i = \theta$ and $\eta_o^i = \theta_o$ for i = 1, …, n, and also assume that $T^{(1)}, \ldots, T^{(n)}$ are independent, then (1) reduces to

$$H_o : \theta = \theta_o \quad \text{vs} \quad H_1 : \theta \in \Omega - \{\theta_o\}. \tag{4}$$

It follows that the p-values $P_1, \ldots, P_n$ are also independent identically distributed random variables that have a U(0, 1) distribution under H_o, and under H_1 have a distribution whose support is a subset of the interval (0, 1) and is not a U(0, 1) distribution. Therefore, if f is the probability density function (pdf) of P, then (4) is equivalent to

$$H_o: P \sim U(0,1) \quad \text{vs} \quad H_1: P \nsim U(0,1) \tag{5}$$

where P has a pdf f with support a subset of the interval (0, 1).

This study considers the case: $\eta_i = \gamma\, \theta_i$, i = 1, …, n, where $\theta_1,\ldots,\theta_n$ are independent identically distributed with DF F with support $[a, \infty)$, $a \geq 0$ and the following hypothesis is tested:

$$H_o : \gamma = 0 \quad \text{vs} \quad H_1 : \gamma > 0 \tag{6}$$

where the i^{th} problem is based on $T_1, \ldots, T_n$, which are independent random variables from a conditional shifted exponential with pdf $f(x|\theta) = e^{-(x-\gamma\theta)},\ x \geq \gamma\theta$ and $\theta_1, \ldots, \theta_n$ are independent identically distributed with DF F with support $[a, \infty)$, $a \geq 0$. Six methods are compared, namely: maximum of p-values method, Tippett's method, Fisher method, logistic method, inverse normal method, and the sum of p-values method. These methods reject H_o for large values of

$$-\max_{1\leq i\leq n}(p_i) \text{ (Maximum of p-values)},$$

$$-\min_{1\leq i\leq n}(p_i) \text{ (Tippett's)},$$

$$-2\sum_{i=1}^{n}\frac{\ln P_i}{\sqrt{n}} \text{ (Fisher)},$$

$$-\sum_{i=1}^{n}\ln\left(\frac{P_i}{1-P_i}\right)\Big/\sqrt{n} \text{ (Logistic)},$$

$$-\sum_{i=1}^{n}\frac{\Phi^{-1}(P_i)}{\sqrt{n}} \text{ (Inverse normal)}$$

and

$$-\sum_{i=1}^{n}\frac{P_i}{\sqrt{n}} \text{ (Sum of p-values)}.$$

Derivation of EBS

Let $X_1, \ldots, X_n$ be an independent identically distributed pdf with $f(x, \theta)$, the hypotheses test hypotheses are $H_o: \theta = \theta_o$ vs. $H_1: \theta \in \Omega\text{-}\{\theta_o\}$, $\{T_n^1\}$ and $\{T_n^2\}$ are two sequences of test statistics for testing H_o, and the p-value of T_n^i is $P_n^i = 1 - F_i(T_n^i)$ where $F_i(t) = P_{H_o}(T_n^i \leq t_i)$ for i = 1, 2.

Under these assumptions, there usually exists a positive valued function $C_i(\theta)$, which is termed the EBS of the sequence $\{T_n^i\}$ at θ. This EBS sequence has the property that $C_i(\theta) = \lim_{n\to\infty} -\frac{2}{n}\ln P_n^i$ w. p. 1 under θ, and the Bahadur efficiency of $\{T_n^1\}$ relative to $\{T_n^2\}$ which is given by $\frac{C_1(\theta)}{C_2(\theta)}$. Therefore, comparing two tests via the Bahadur efficiency is equivalent to comparing their corresponding EBS's. The following three theorems provide the EBS for tests based on sums of independent identically distributed random variables.

Theorem 1

Let $X_1, \ldots, X_n$ be independent identically distributed random variables with DF F and $S_n = \sum_{i=1}^{n} X_i$. Assume that the moment generating function $M(z) = E_F e^{zX}$ exists and is near zero. If $m(t) = \inf_z e^{-zt} M(z)$, then $\lim_{n\to\infty} -\frac{2}{n} \ln P_F[S_n \geq nt] = -2\ln m(t)$.

Theorem 2

Let $\{T_n\}$ be a sequence of test statistics satisfying the following conditions:

1. Under H_1: $\frac{T_n}{\sqrt{n}} \to b(\theta)$ w.p.1 under θ where b(θ) is a real function.
2. An open interval I containing $\{b(\theta): \theta \in \Omega\}$ exists and a function g continuous on I such that $\lim_{n\to\infty} -\frac{2}{n} \ln[1 - F_n(t\sqrt{n})] = g(t)$ where F_n is the DF of T_n under H_o.

Thus the EBS of $\{T_n\}$ is C(θ) = g(b(θ)).

Theorem 3

Let $U_1, U_2, \ldots$ be independent identically distributed random variables. To test H_o: $U_i \sim U(0, 1)$ vs H_1: $U_i \sim f$ on (0, 1), which is not U(0, 1), then

1. $C_{\max}(f) = -2\ln(ess.Sup_f(u))$, where $ess.Sup_f(u) = Sup\{u: f(u) > 0\}$ w.p.1 under f.
2. If $t(\ln t)^2 f(t) \to 0$ as $t \to 0$, then $C_{\min}(f)=0$.

Note that for testing problem (6), the i^{th} p-value is:

$$P_i = P(X \geq x_i) = e^{-x_i} \quad (7)$$

The next four lemmas give the EBS for Fisher (C_F), logistic (C_L), inverse normal (C_N), sum of p-values (C_S), Tippett's (C_T) and maximum of p-values (C_{max}) methods.

Lemma 1

$$C_F(\gamma) = 2\gamma E_F\theta - 2\ln(1 + \gamma E_F\theta) \quad (8)$$

$$C_S(\gamma) = -2 \ln m_S\left(\frac{1}{2} E_F\left(e^{-\gamma\theta}\right)\right),$$

where

$$m_S(t) = \inf_{z>0}\left\{e^{-tz} \frac{1-e^{-z}}{z}\right\} \quad (9)$$

$$C_L(\gamma) = -2 \ln m_L\left(b_L(\gamma)\right),$$

where

$$m_L(\gamma) = \inf_{0<z<1}\left\{e^{-b_L(\gamma)z} \pi z\, CSC(\pi z)\right\}$$

and

$$b_L(\gamma) = \gamma E_F\theta - E_F(e^{\gamma\theta} - 1)\ln(1 - e^{-\gamma\theta}) \quad (10)$$

$$C_N(\gamma) = \left(E_F\left\{e^{\gamma\theta}\varphi\left(\Phi^{-1}(e^{\gamma\theta})\right)\right\}\right)^2 \quad (11)$$

The proof for Lemma 1 follows from Theorems (1) and (2).

Lemma 2

Let $U_1, \ldots, U_n$ be independent identically distributed like U with pdf f, if the test hypotheses are:

H_o: $U_i \sim U(0, 1)$

vs

H_1: $U_i \sim f$ on (0, 1) but not U(0, 1),

then

$$C_{max}(f) = -2\ln\left(ess.\sup_f(U)\right)$$

Where $ess.Sup_f\, U = Sup\{u: f(u) > 0\}$ w.p.1 under f.

Proof: Lemma 2

Let $G_o(t)$ be the DF of $-\max_i U_i$ under H_o. Then for

$$-1 < t < 0,\ 1 - G_o(t) = (-t)^n,$$

which implies that

$$-2/n\ln[1-G_o(-\max_i U_i)]=-2\ln\max_i U_i.$$

Therefore,

$$C_{\max}(f)=\lim_{n\to\infty}[-2\ln\max_i U_i]$$
$$=-2\ln\lim_{n\to\infty}\max_i U_i$$
$$=-2\ln(ess\sup_f U).$$

Lemma 3

$$C_{\max}(\gamma)=2\gamma a \quad (12)$$

Proof: Lemma 3

Assume g(θ) is the pdf of the DF F, then the joint pdf of x and θ is

$$h(x,\theta)=g(\theta)\,f(x|\theta),$$

where $f(x|\theta)=e^{-(x-\gamma\theta)}, x>\gamma\theta$. Then the marginal pdf of x is

$$f(x)=\int_a^{x/\gamma}h(x,\theta)d\theta$$
$$=\int_a^{x/\gamma}e^{-(x-\gamma\theta)}g(\theta)d\theta,\quad x>a\gamma, a\geq 0$$
$$=e^{-x}\int_a^{x/\gamma}e^{\gamma\theta}dF(\theta),\quad x>\gamma\theta$$

The under γ the p-value is $e^{-x}\equiv$ P satisfies $0<P<e^{-\gamma a}$, then ess.sup P = $e^{-\gamma a}$, which implies $C_{\max}(\gamma)=2\gamma a$ by theorem (3).

Lemma 4

$$C_T(\gamma)=0 \quad (13)$$

Proof: Lemma 4

$$g(p)=\int_a^{-\ln p/\gamma}e^{\gamma\theta}g(\theta)d\theta$$

$$\lim_{p\to 0}p(\ln p)^2 g(p)$$
$$=\lim_{p\to 0}\frac{(\ln p)^2}{1/p}\int_a^{-\ln p/\gamma}e^{\gamma\theta}g(\theta)d\theta$$
$$=\lim_{p\to 0}-p^2\left[\frac{2\ln p}{p}\int_a^{-\ln p/\gamma}e^{\gamma\theta}g(\theta)d\theta+\frac{(\ln p)^2}{p}g\left(\frac{-\ln p}{\gamma}\right)\right]$$
$$=\lim_{p\to 0}-2p\ln p\int_a^{-\ln p/\gamma}e^{\gamma\theta}g(\theta)d\theta$$
$$=0$$

using L'hopital rule because g(∞) =0 and $\lim_{p\to 0}p(\ln p)^2=0$.

Results

First, it the limits of the ratio of any two methods under study were found as γ→0 and γ→∞. This gives the following results.

Corollary 1

$$\lim_{\gamma\to 0}\frac{C_{\max}(\gamma)}{C_a(\gamma)}=0,$$

where

$$C_a(\gamma)=\{C_S(\gamma),C_N(\gamma),C_L(\gamma),C_F(\gamma)\},$$

and

$$\lim_{\gamma\to 0}\frac{C_F(\gamma)}{C_S(\gamma)}=\frac{1}{3},$$

$$\lim_{\gamma\to 0}\frac{C_S(\gamma)}{C_L(\gamma)}=\lim_{\gamma\to 0}\frac{C_S(\gamma)}{C_N(\gamma)}$$
$$=\lim_{\gamma\to 0}\frac{C_F(\gamma)}{C_L(\gamma)}$$
$$=\lim_{\gamma\to 0}\frac{C_F(\gamma)}{C_N(\gamma)}$$
$$=\lim_{\gamma\to 0}\frac{C_L(\gamma)}{C_N(\gamma)}$$
$$=0.$$

Corollary 2

$$\lim_{\gamma\to\infty}\frac{C_L(\gamma)}{C_F(\gamma)}=1,\ \lim_{\gamma\to\infty}\frac{C_F(\gamma)}{C_N(\gamma)}=\lim_{\gamma\to\infty}\frac{C_L(\gamma)}{C_N(\gamma)}=0.$$

Proof 1

By (8) and (10)

$$\frac{C_L(\gamma)}{C_F(\gamma)}\le\frac{-2\ln 2-2\ln b_L(\gamma)+2b_L(\gamma)}{2\gamma E_F\theta-2\ln(1+2\gamma E_F\theta)},$$

and

$$\lim_{\gamma\to\infty}\frac{C_L(\gamma)}{C_F(\gamma)}\le\lim_{\gamma\to\infty}\frac{-2\ln 2-2\ln b_L(\gamma)+2b_L(\gamma)}{2\gamma E_F\theta-2\ln(1+2\gamma E_F\theta)}$$

$$=\lim_{\gamma\to\infty}\frac{-2\frac{b_L'(\gamma)}{b_L(\gamma)}+2b_L'(\gamma)}{2E_F\theta-2\frac{E_F\theta}{1+2\gamma E_F\theta}}$$

$$=1$$

by using the L'Hopital rule.

Similarly, it can be shown that

$$\lim_{\gamma\to\infty}\frac{C_L(\gamma)}{C_F(\gamma)}\ge1.$$

Regarding $C_S(\gamma)$, nothing can be concluded about general prior F because $\lim_{\gamma\to\infty}\frac{b_S'(\gamma)}{b_S(\gamma)}$ has an indeterminate for (0/0), thus, only a certain prior - namely, $G(\alpha,\beta)$ with $\alpha=1$ and $\beta=2$, is considered:

$$\lim_{\gamma\to\infty}\frac{C_S(\gamma)}{C_F(\gamma)}=\lim_{\gamma\to\infty}\frac{-\frac{b_S'(\gamma)}{b_S(\gamma)}}{E_F\theta}.$$

Proof 2

By (8) and (10)

$$\lim_{\gamma\to\infty}\frac{C_S(\gamma)}{C_F(\gamma)}\le\lim_{\gamma\to\infty}\frac{-2\ln 2-2\ln b_S(\gamma)}{2\gamma E_F\theta-2\ln(1+\lambda E_F\theta)}$$

$$=\lim_{\gamma\to\infty}\frac{-2\frac{b_S'(\gamma)}{b_S(\gamma)}}{2E_F\theta-2\frac{E_F\theta}{1+\gamma E_F\theta}}$$

$$=\lim_{\gamma\to\infty}\frac{-\frac{b_S'(\gamma)}{b_S(\gamma)}}{E_F\theta}$$

Similarly,

$$\lim_{\gamma\to\infty}\frac{C_S(\gamma)}{C_F(\gamma)}\ge\lim_{\gamma\to\infty}\frac{-\frac{b_S'(\gamma)}{b_S(\gamma)}}{E_F\theta},$$

hence,

$$\lim_{\gamma\to\infty}\frac{C_L(\gamma)}{C_F(\gamma)}=1. \tag{14}$$

From the above relations it may be concluded that locally as $\gamma\to0$

$$C_N(\gamma)>C_L(\gamma)>C_S(\gamma)>C_F(\gamma)>C_{max}(\gamma)>C_T(\gamma),$$

but as $\gamma\to\infty$

$$C_N(\gamma)>C_L(\gamma)>C_{max}(\gamma)>C_F(\gamma)>C_T(\gamma),$$

The dominance of one method over the other in case $\gamma>0$ can be shown mathematically only in some cases. The proof is omitted because, although it is straightforward, it is lengthy; however, numerical comparison for all methods is shown in Table 1. It appears from Table 1 that

$\gamma=0.05$: $C_N(\gamma)>C_L(\gamma)>C_S(\gamma)>C_F(\gamma)$;

$\gamma\in[0.1, 0.5]$: $C_S(\gamma)>C_N(\gamma)>C_L(\gamma)>C_F(\gamma)$;

$\gamma=1.00$: $C_N(\gamma)>C_L(\gamma)>C_F(\gamma)>C_S(\gamma)$;

$\gamma\in[2, 3]$: $C_L(\gamma)>C_F(\gamma)>C_N(\gamma)>C_S(\gamma)$;

$\gamma\in[5, 8]$: $C_F(\gamma)>C_L(\gamma)>C_N(\gamma)>C_S(\gamma)$;

$\gamma\in[10, 20]$: $C_N(\gamma)>C_F(\gamma)>C_L(\gamma)>C_S(\gamma)$.

Table 1: The Exact Bahadur Slopes for Conditional Shifted Exponential with Prior G(1, 2)

γ	$C_S(\gamma)$	$C_L(\gamma)$	$C_F(\gamma)$	$C_N(\gamma)$
0.050	0.0249	0.0298	0.0094	0.0320
0.100	0.0903	0.0818	0.0354	0.8447
0.200	0.2512	0.2168	0.1271	0.2323
0.300	0.4414	0.3796	0.2599	0.4096
0.400	0.6329	0.5633	0.4244	0.6059
0.500	0.8173	0.7644	0.6137	0.8153
1.000	1.5887	1.9598	1.8028	1.9829
2.000	2.6053	4.9002	4.7811	4.5961
3.000	3.2781	8.1865	8.1082	7.3718
5.000	4.1821	15.1632	15.2042	12.6267
8.000	5.0527	26.0621	26.3336	21.1932
10.00	5.4753	33.4909	33.9110	40.3568
20.00	6.8134	71.6401	72.5729	162.3284

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