Journal of Modern Applied Statistical Methods

Volume 9 | Issue 2 Article 21

11-1-2010

Maximum Downside Semi Deviation Stochastic Programming for Portfolio Optimization Problem

Anton Abdulbasah Kamil
Universiti Sains Malaysia, Penang, Malaysia, anton@usm.my

Khlipah Ibrahim
Universiti Teknologi Mara, Dungun, Terengganu, Malaysia, adli@usm.my

Journal of Modern Applied Statistical Methods
November 2010, Vol. 9, No. 2, 536-546

1538 – 9472/10/$95.00

Maximum Downside Semi Deviation Stochastic Programming for Portfolio Optimization Problem

Anton Abdulbasah Kamil Adli Mustafa
Universiti Sains Malaysia,
Penang, Malaysia

Khlipah Ibrahim
Universiti Teknologi Mara, Dungun,
Terengganu, Malaysia

Portfolio optimization is an important research field in financial decision making. The chief character within optimization problems is the uncertainty of future returns. Probabilistic methods are used alongside optimization techniques. Markowitz (1952, 1959) introduced the concept of risk into the problem and used a mean-variance model to identify risk with the volatility (variance) of the random objective. The mean-risk optimization paradigm has since been expanded extensively both theoretically and computationally. A single stage and two stage stochastic programming model with recourse are presented for risk averse investors with the objective of minimizing the maximum downside semi-deviation. The models employ the here-and-now approach, where a decision-maker makes a decision before observing the actual outcome for a stochastic parameter. The optimal portfolios from the two models are compared with the incorporation of the deviation measure. The models are applied to the optimal selection of stocks listed in Bursa Malaysia and the return of the optimal portfolio is compared between the two stochastic models. Results show that the two stage model outperforms the single stage model for the optimal and in-sample analysis.

Key words: Portfolio optimization, maximum semi-deviation measure, downside risk, stochastic linear programming.

Introduction

Portfolio optimization is an important research field in financial decision making. The most important character within optimization problems is the uncertainty of future returns. To handle such problems, probabilistic methods are utilized alongside optimization techniques. Stochastic programming is the approach employed in this study to deal with uncertainty. Stochastic programming is a branch of mathematical programming where the parameters are random, the objective of which is to find the optimum solution to problems with uncertain data. This approach can simultaneously deal with both the management of portfolio risk and the identification of the optimal portfolio. Stochastic programming models explicitly consider uncertainty in the model parameters and they provide optimal decisions which are hedged against such uncertainty.

In the deterministic framework, a typical mathematical programming problem could be stated as

$$\begin{aligned} &\min_x \quad f(x) \\ &s.t \quad g_i(x) \le 0, \quad i = 1,\ldots m, \end{aligned} \tag{1.1}$$

where x is from R^n or Z^n. Uncertainty, which is usually described by a random element, $\xi(\omega)$, where ω is a random outcome from a space Ω, leads to situation where one has to deal with $f(x,\xi(\omega))$ and $g_i(x,\xi(\omega))$, as

Anton Abdulbasah Kamil is an Associate Professor in the School of Distance Education, Universiti Sains Malaysia, Malaysia. Email: anton@usm.my. Adli Mustafa is a Senior lecturer in the School of Mathematical Sciences, Universiti Sains Malaysia, Malaysia. Email: adli@usm.my. Khlipah Ibrahim is an Associate Professor in the Universiti Teknologi Mara, Malaysia.

opposed to just $f(x)$ and $g_i(x)$. Traditionally, the probability distribution of ξ is assumed to be known (or can be estimated) and is unaffected by the decision vector x. The problem becomes decision making under uncertainty where decision vector x must be chosen before the outcome from the distribution of $\xi(\omega)$ can be observed.

Markowitz (1952, 1959) incorporated the concept of risk into the problem and introduced the mean-risk approach, which identifies risk with the volatility (variance) of the random objective. Since 1952, the mean-risk optimization paradigm has been extensively developed both theoretically and computationally. Konno and Yamazaki (1991) proposed mean absolute deviation (MAD) from the mean as the risk measure to estimate the nonlinear variance-covariance of the stocks in the mean-variance (MV) model. It transforms the portfolio selection problem from a quadratic programming problem into a linear problem. The popularity of downside risk among investors is growing and mean-return-downside risk portfolio selection models seem to oppress the familiar mean-variance approach.

The reason mean-variance models are successful is because they separate return fluctuations into downside risk and upside potential. This is relevant for asymmetrical return distributions, for which the mean-variance model punishes the upside potential in the same fashion as the downside risk. Thus, Markowitz (1959) proposed downside risk measures, such as semi variance, to replace variance as the risk measure. Subsequently, downside risk models for portfolio selection have grown in popularity (Sortino & Forsey, 1996).

Young (1998) introduced another linear programming model to maximize the minimum return or minimize the maximum loss (minimax) over time periods and he applied it to stock indices of eight countries from January 1991 until December 1995. The analysis showed that the model performs similarly with the classical mean-variance model. In addition, Young argued that - when data is log-normally distributed or skewed - the minimax formulation might be a more appropriate method compared to the classical mean-variance formulation, which is optimal for normally distributed data. Ogryczak (2000) also considered the minimax model but analyzed it with the maximum semi deviation.

Dantzig (1955) and Beale (1955) independently suggested an approach to stochastic programming termed stochastic programming with recourse; recourse is the ability to take corrective action after a random event has taken place. Their innovation was to amend the problem to allow a decision maker the opportunity to make corrective actions after a random event has taken place. In the first stage, a decision maker makes a here and now decision. In the second stage the decision maker sees a realization of the stochastic elements of the problem but is allowed to make further decisions to avoid the constraints of the problem becoming infeasible.

Stochastic programming is becoming more popular in finance as computing power increases and there have been numerous applications of stochastic programming methodology to real life problems over the last two decades. The applicability of stochastic programs to financial planning problems was first recognized by Crane (1971). More recently Worzel, et al. (1994) and Zenios, et al. (1998) have developed multistage stochastic programs with recourse to address portfolio management problems with fixed-income securities under uncertainty in interest rates. Their models integrate stochastic programming for the selection of portfolios using Monte Carlo simulation models of the term structure of interest rates.

Hiller and Eckstein (1994), Zenios (1995) and Consiglo and Zenios (2001) also applied stochastic programs to fixed-income portfolio management problems. Chang, et al. (2002) modeled a portfolio selection problem with transaction costs as a two-stage stochastic programming problem and evaluated the model using historical data obtained from the Taiwan Stock Exchange; their results show that the model outperforms the market and the MV and MAD models.

In this article, a single stage and two stage stochastic programming model are developed with recourse for portfolio selection. The objective is to minimize the maximum downside deviation measure of portfolio returns from the expected return. The so-called here-and-now approach is utilized: a decision-maker makes a decision (now) before observing the actual outcome for the stochastic parameter. The portfolio optimization problem considered follows the original Markowitz (1959) formulation and is based on a single period model of investment. At the beginning of a period, an investor allocates capital among various securities assuming that each security is represented by a variable; this is equivalent to assigning a nonnegative weight to each variable. During the investment period, a security generates a random rate of return. The change of invested capital observed at the end of the period is measured by the weighted average of the individual rates of return.

The objective of this study is to compare the optimal portfolio selected using two different stochastic programming models. The optimal portfolios are compared between the single stage and two stage models with the incorporation of deviation measure. This method is applied to the optimal selection of stocks listed in Bursa Malaysia and the return of the optimal portfolio from the two models is compared.

Methodology

Consider a set of securities $I=\{i : i=1,2,...,n\}$ for an investment; at the end of a certain holding period the assets generate returns, $\tilde{r}=(\tilde{r}_1,\tilde{r}_2,...,\tilde{r}_n)^T$. The returns are unknown at the beginning of the holding period, that is at the time of the portfolio selection, and are treated as random variables; their mean value is denoted by, $\bar{r}=E(\tilde{r})=(\bar{r}_1,\bar{r}_2,...,\bar{r}_n)^T$. At the beginning of a holding period an investor wishes to apportion his budget to these assets by deciding on a specific allocation $\boldsymbol{x}=(x_1,x_2,...,x_n)^T$ such that $x_i \geq 0$ (i.e., short sales are not allowed) and $\sum_{i\in I} x_i = 1$ (budget constraint). In this article, boldface characters are used to denote vectors, and the symbol ~ denotes random variables.

The uncertain return of a portfolio at the end of a holding period is $\tilde{R} = R(\boldsymbol{x},\tilde{\boldsymbol{r}}) = \boldsymbol{x}^{\boldsymbol{T}}\tilde{\boldsymbol{r}}$. This is a random variable with a distribution function F, that is, $F(x,\mu) = P\{R(x,\tilde{r}) \leq \mu\}$. It is assumed that F does not depend on the portfolio composition $\boldsymbol{x}$. The expected return of the portfolio is

$$\bar{R} = E[\tilde{R}] = E[R(\boldsymbol{x},\tilde{\boldsymbol{r}})] = \bar{R}(\boldsymbol{x},\tilde{\boldsymbol{r}}).$$

Suppose the uncertain returns of the assets, $\tilde{\boldsymbol{r}}$, are represented by a finite set of discrete scenarios $\boldsymbol{\Omega} = \{\omega : \omega = 1,2,...,S\}$, whereby the returns under a particular scenario $\omega \in \boldsymbol{\Omega}$ take the values $\boldsymbol{r}_\omega = (r_{1\omega}, r_{2\omega},...,r_{n\omega})^{\boldsymbol{T}}$ with associated probability $p_\omega > 0$, $\sum_{\omega\in\Omega} p_\omega = 1$. The mean return of the assets is $\bar{\boldsymbol{r}} = \sum_{\omega\in\Omega} p_\omega \boldsymbol{r}_\omega$. The portfolio return under a particular realization of asset return $\boldsymbol{r}_\omega$ is denoted by $R_\omega = R(\boldsymbol{x},\boldsymbol{r}_\omega)$. The expected portfolio return is expressed as:

$$\begin{aligned}\bar{\mathrm{R}} &= \bar{R}(\mathrm{x},\mathrm{r}_\omega)\\ &= E[R(\mathrm{x},\mathrm{r}_\omega)]\\ &= \sum_{\omega\in\Omega} p_\omega R(\mathrm{x},\mathrm{r}_\omega).\end{aligned}$$

Let $M[R(\boldsymbol{x},\boldsymbol{r}_\omega)]$ be the minimum of the portfolio return. The maximum (downside) semideviation measure is defined as

$$\begin{aligned}\kappa(\mathrm{x}) &= MM[R(\mathrm{x},\mathrm{r}_\omega)]\\ &= [\mathrm{E}[R(\mathrm{x},\mathrm{r}_\omega)] - Min\,[R(\mathrm{x},\mathrm{r}_\omega)]\end{aligned} \tag{2.1}$$

Maximum downside deviation risk $MM[R(\boldsymbol{x},\boldsymbol{r}_\omega)]$ is a very pessimistic risk measure related to the worst case analysis. It does not take into account any distribution of outcomes other than the worst one.

Properties of the $MM[R(\boldsymbol{x},\tilde{\boldsymbol{r}})]$ Measures

Artzner, et al. (1999) introduced the axiomatic approach to construction of risk measures. This approach has since been repeatedly employed by many authors for the development of other types of risk measures tailored to specific preferences and applications (see Rockafellar, et al., 2002, 2004; Acerbi, 2002; Ruszcynski & Shapiro, 2004).

Proposition 1: $MM[R(\boldsymbol{x},\tilde{\boldsymbol{r}})]$ measure is a deviation measure.

Proof:

1. Subadditivity:
$\kappa(X_1 + X_2) \le \kappa(X_1) + \kappa(X_2)$.

$$\begin{aligned}
MM[R_1(\mathrm{x},\tilde{\mathrm{r}}) + R_2(\mathrm{x},\tilde{\mathrm{r}})] &= \max\{E[R_1(\mathrm{x},\tilde{\mathrm{r}}) + R_2(\mathrm{x},\tilde{\mathrm{r}})] \\
&\quad - [R_1(\mathrm{x},\tilde{\mathrm{r}}) + R_2(\mathrm{x},\tilde{\mathrm{r}})]\} \\
&= \max\{(E[R_1(\mathrm{x},\tilde{\mathrm{r}})] - R_1(\mathrm{x},\tilde{\mathrm{r}})) \\
&\quad + (E[R_2(\mathrm{x},\tilde{\mathrm{r}})] - R_2(\mathrm{x},\tilde{\mathrm{r}})\} \\
&\le \max\{E[R_1(\mathrm{x},\tilde{\mathrm{r}})] - R_1(\mathrm{x},\tilde{\mathrm{r}})\} \\
&\quad + \max\{E[R_2(\mathrm{x},\tilde{\mathrm{r}})] - R_2(\mathrm{x},\tilde{\mathrm{r}})\} \\
&\le MM[R_1(\mathrm{x},\tilde{\mathrm{r}})] + MM[R_2(\mathrm{x},\tilde{\mathrm{r}})]
\end{aligned}$$

2. Positive Homogeneity:
$MM[0] = max(E[0] - 0) = 0.$

$$\begin{aligned}
MM[\lambda(R(x,\tilde{r})] &= max\{E[\lambda R(x,\tilde{r})] - \lambda R(x,\tilde{r})\} \\
&= \lambda\, max\{E[R(x,\tilde{r})] - R(x,\tilde{r})\} \\
&= \lambda MM[R(x,\tilde{r})], \quad for\ all\ \lambda > 0
\end{aligned}$$

3. Translation invariance:
$\kappa(X + \alpha) = \kappa(X) - \alpha$, for all real constants α.

$$\begin{aligned}
MM[(R(x,\tilde{r}) + \alpha] &= max\{E([R(x,\tilde{r}) + \alpha] - [R(x,\tilde{r}) + \alpha])\} \\
&= max\{E[R(x,\tilde{r})] + \alpha - R(x,\tilde{r}) - \alpha\} \\
&= max\{E[R(x,\tilde{r})] - R(x,\tilde{r})\} \\
&= MM[(R(x,\tilde{r})]
\end{aligned}$$

4. Convexity:
$\kappa[\lambda X_1 + (1-\lambda)X_2] \le \lambda\kappa(X_1) + (1-\lambda)\kappa(X_2)$
for all $\lambda \in [0,1]$.

$$\begin{aligned}
MM[\lambda R_1(\mathrm{x},\tilde{\mathrm{r}}) + (1-\lambda)R_2(\mathrm{x},\tilde{\mathrm{r}})] &= \max\{E[\lambda R_1(\mathrm{x},\tilde{\mathrm{r}}) + (1-\lambda)R_2(\mathrm{x},\tilde{\mathrm{r}})] \\
&\quad - [\lambda R_1(\mathrm{x},\tilde{\mathrm{r}}) + (1-\lambda)R_2(\mathrm{x},\tilde{\mathrm{r}})]\} \\
&= \max\{(E[\lambda R_1(\mathrm{x},\tilde{\mathrm{r}})] + E[(1-\lambda)R_2(\mathrm{x},\tilde{\mathrm{r}}))] \\
&\quad - \lambda R_1(\mathrm{x},\tilde{\mathrm{r}}) + (1-\lambda)R_2(\mathrm{x},\tilde{\mathrm{r}})\} \\
&= \max\{\lambda(E[R_1(\mathrm{x},\tilde{\mathrm{r}})] - R_1(\mathrm{x},\tilde{\mathrm{r}})) \\
&\quad + (1-\lambda)(E[R_2(\mathrm{x},\tilde{\mathrm{r}})] - R_2(\mathrm{x},\tilde{\mathrm{r}}))\} \\
&\le \lambda \max\{(E[R_1(\mathrm{x},\tilde{\mathrm{r}})] - R_1(\mathrm{x},\tilde{\mathrm{r}}))\} + \\
&\quad (1-\lambda)\max\{E[R_2(\mathrm{x},\tilde{\mathrm{r}})] - R_2(\mathrm{x},\tilde{\mathrm{r}}))\} \\
&\le \lambda MM[R_1(\mathrm{x},\tilde{\mathrm{r}})] + (1-\lambda)MM[R_2(\mathrm{x},\tilde{\mathrm{r}})]
\end{aligned}$$

Single Stage Stochastic Programming Portfolio Optimization Model with MM Deviation Measure

The portfolio selection optimization model is formulated as a single stage stochastic programming model as follows.

Definition 1: S_MM

The stochastic portfolio optimization problem where the difference between the expected portfolio return and the maximum of minimum portfolio returns is minimized and constraining the expected portfolio return is:

$$\underset{\mathrm{x}\in \mathrm{X}}{Minimize}\ \max_{\omega\in\Omega}\ [\bar{R}(\mathrm{x},\mathrm{r}_\omega) - R(\mathrm{x},\mathrm{r}_\omega)] \tag{2.2a}$$

Subject to:

$$R(x,r_\omega) = \sum_{i\in I} x_i r_{\omega i} \quad \forall\,\omega\in\Omega \tag{2.2b}$$

$$\bar{R}(x,r_\omega) = \sum_{\omega\in\Omega} p_\omega R(x,r_\omega) \tag{2.2c}$$

$$\bar{R}(x,r_\omega) \ge \alpha \tag{2.2d}$$

$$\sum_{i\in I} x_i = 1 \tag{2.2e}$$

$$L_i \le x_i \le U_i \quad \forall\, i\in I \tag{2.2f}$$

Model S_MM minimizes the maximum semi deviation of portfolio returns from the expected portfolio return at the end of the investment horizon. Equation (2.2b) defines the total portfolio return under each scenario ω. Equation (2.2c) defines the expected return of the portfolio at the end of the horizon, while equation (2.2d) constrains the expected return by the target return α. Equation (2.2e) insures that the total weights of all investments sum to one, that is, budget constraints ensuring full investment of available budget. Finally equation (2.2f) insures that the weights on assets purchased are nonnegative, disallowing short sales and placing upper bounds on the weights. Solving the parametric programs (2.2) for different values of the expected portfolio return α yields the MM-efficient frontier.

Linear Programming Formulation for S_MM

Models S_MM have a non linear objective function and a set of linear constraints, thus the models are non linear stochastic programming. However, the models can be transformed to linear models as follows.

For every scenario $\omega \in \Omega$, let an auxiliary variable,

$$\eta = \max_{\omega \in \Omega} [\bar{R}(\boldsymbol{x}, \boldsymbol{r}_\omega) - R(\boldsymbol{x}, \boldsymbol{r}_\omega)] \quad (2.3)$$

subject to

$$\eta \geq \max_{\omega \in \Omega} [\bar{R}(x, r_\omega) - R(x, r_\omega)] \text{ for } \forall \omega \in \Omega,$$

then,

$$MM[R(\boldsymbol{x}, \boldsymbol{r}_\omega)] = \eta \quad (2.4)$$

subject to

$$\eta \geq \max_{\omega \in \Omega} [\bar{R}(x, r_\omega) - R(x, r_\omega)] \text{ for } \forall \omega \in \Omega.$$

Substituting (2.4) in the portfolio optimization models (2.2) results in the following stochastic linear programming model:

$$\text{Minimize } \eta, \quad (2.5a)$$

subject to:

$$R(x, r_\omega) = \sum_{i \in I} x_i r_{\omega i} \quad (2.5b)$$

$$\bar{R}(x, r_\omega) = \sum_{\omega \in \Omega} p_\omega R(x, r_\omega) \quad (2.5c)$$

$$\bar{R}(x, r_\omega) \geq \alpha \quad (2.5d)$$

$$\bar{R}(x, r_\omega) - R(x, r_\omega) \leq \eta \quad (2.5e)$$

$$\sum_{i \in I} x_i = 1 \quad (2.5f)$$

$$L_i \leq x_i \leq U_i \quad \forall i \in I \quad (2.5g)$$

Theorem 1

If $\boldsymbol{x}^*$ is an optimal solution to (2.2), then $(\boldsymbol{x}^*, \eta^*)$ is an optimal solution to (2.5), where $\eta = \max_{\omega \in \Omega} [\bar{R}(\boldsymbol{x}, \boldsymbol{r}_\omega) - R(\boldsymbol{x}, \boldsymbol{r}_\omega)]$. Conversely, if (x^*, η^*) where $\eta = \max_{\omega \in \Omega} [\bar{R}(\boldsymbol{x}, \boldsymbol{r}_\omega) - R(\boldsymbol{x}, \boldsymbol{r}_\omega)]$ is an optimal solution to (2.5), then $\boldsymbol{x}^*$ is an optimal solution to (2.2).

Proof:

If $\boldsymbol{x}^*$ is an optimal solution to (2.2), then $(\boldsymbol{x}^*, \eta^*)$ is a feasible solution to (2.5), where $\eta = \max_{\omega \in \Omega} [\bar{R}(\boldsymbol{x}^*, \boldsymbol{r}_\omega) - R(\boldsymbol{x}^*, \boldsymbol{r}_\omega)]$. If $(\boldsymbol{x}^*, \eta^*)$ is not an optimal solution to (2.5), then a feasible solution $(\boldsymbol{x}, \eta)$ exists to (2.5) where $\eta = \max_{\omega \in \Omega} [\bar{R}(\boldsymbol{x}, \boldsymbol{r}_\omega) - R(\boldsymbol{x}, \boldsymbol{r}_\omega)]$ such that $\eta \leq \eta^*$.

If $\max_{\omega \in \Omega} [\bar{R}(\boldsymbol{x}, \boldsymbol{r}_\omega) - R(\boldsymbol{x}, \boldsymbol{r}_\omega)] \leq \eta$, then

$$\max_{\omega \in \Omega} [\bar{R}(x, r_\omega) - R(x, r_\omega)] \leq \eta < \eta^* < \max_{\omega \in \Omega} [\bar{R}(x^*, r_\omega) - R(x^*, r_\omega)]$$

which contradicts that $\boldsymbol{x}^*$ is an optimal solution to (2.2).

However, if $(\boldsymbol{x}^*, \eta^*)$ is an optimal solution to (2.5), where $\eta = \max_{\omega \in \Omega} [\bar{R}(\boldsymbol{x}, \boldsymbol{r}_\omega) - R(\boldsymbol{x}, \boldsymbol{r}_\omega)]$ then $\boldsymbol{x}^*$ is an optimal solution to (2.2). Otherwise, a feasible solution $\boldsymbol{x}$ to (2.2) exists such that

$$\max_{\omega \in \Omega} [\bar{R}(\mathrm{x}, \mathrm{r}_\omega) - R(\mathrm{x}, \mathrm{r}_\omega)] < \max_{\omega \in \Omega} [\bar{R}(\mathrm{x}^*, \mathrm{r}_\omega) - R(\mathrm{x}^*, \mathrm{r}_\omega)]$$

Denoting $\eta = \max_{\omega \in \Omega} [\bar{R}(\boldsymbol{x}, \boldsymbol{r}_\omega) - R(\boldsymbol{x}, \boldsymbol{r}_\omega)]$, leads to

$$\begin{aligned} \eta &= \max_{\omega \in \Omega} [\bar{R}(\boldsymbol{x}, \boldsymbol{r}_\omega) - R(\boldsymbol{x}, \boldsymbol{r}_\omega)] \\ &< \max_{\omega \in \Omega} [\bar{R}(\mathrm{x}^*, \mathrm{r}_\omega) - R(\mathrm{x}^*, \mathrm{r}_\omega)] \\ &< \eta^* \end{aligned}$$

which contradicts that $(\boldsymbol{x}^*, \eta^*)$ is an optimal solution to (2.5).

Two Stage Stochastic Programming Model with Recourse

A dynamic model where not only the uncertainty of the returns is included in the model but future changes, recourse, to the initial compositions are allowed is now introduced. The portfolio optimization is formulated by assuming an investor can make corrective action after the realization of random values by changing the composition of the optimal portfolio. This can be accomplished by formulating the single period stochastic linear programming models with the mean absolute negative deviation measure as a two-stage stochastic programming problem with recourse. The two-stage stochastic programming problem allows a recourse decision to be made after uncertainty of the returns is realized.

Consider the case when the investor is interested in a first stage decision $\boldsymbol{x}$ which hedges against the risk of the second-stage action. At the beginning of the investment period, the investor selects the initial composition of the portfolio, $\boldsymbol{x}$. The first stage decision, $\boldsymbol{x}$, is made when there is a known distribution of future returns. At the end of the planning horizon, after a particular scenario of return is realized, the investor rebalances the composition by either purchasing or selling selected stocks. In addition to the initial - or first stage - decision variables $\boldsymbol{x}$, let a set of second stage variables, $\boldsymbol{y}_{i,\omega}$ represent the composition of stock i after rebalancing is done, that is, $\boldsymbol{y}_{i,\omega} = \boldsymbol{x}_i + \boldsymbol{P}_{i,\omega}$ or $\boldsymbol{y}_{i,\omega} = \boldsymbol{x}_i - \boldsymbol{Q}_{i,\omega}$, where $\boldsymbol{P}_{i,\omega}$ and $\boldsymbol{Q}_{i,\omega}$ are the quantity purchased and sold respectively and $\boldsymbol{y}_{i,\omega}$ is selected after the uncertainty of returns is realized.

Linear Representation of MM

Before formulating the two stage stochastic programming models to minimize the second stage risk measure to address the portfolio optimization problem, the mean absolute negative deviation and maximum downside deviation of portfolio returns are formulated from the expected return in terms of the second stage variables $\boldsymbol{y}$.

$$\begin{aligned} \text{Let } \kappa(R(\boldsymbol{y}_\omega, \boldsymbol{r}_\omega)) &= MM[R(\boldsymbol{y}_\omega, \boldsymbol{r}_\omega)] \\ &= \max_{\omega \in \Xi} [\bar{R}(\boldsymbol{y}_\omega, \boldsymbol{r}_\omega) - R(\boldsymbol{y}_\omega, \boldsymbol{r}_\omega)] \end{aligned} \tag{2.6}$$

For every scenario $\omega \in \Omega$, if the auxiliary variable is

$$\eta = \max_{\omega \in \Omega} [\bar{R}(\boldsymbol{y}_\omega, \boldsymbol{r}_{,\omega}) - R(\boldsymbol{y}_\omega, \boldsymbol{r}_\omega)] \tag{2.7}$$

subject to

$$\eta \geq \max_{\omega \in \Omega} [\bar{R}(y_\omega, r_\omega) - R(y_\omega, r_\omega)] \text{ for } \forall \omega \in \Omega \tag{2.8}$$

then

$$MM[R(\boldsymbol{x}, \boldsymbol{r}_\omega)] = \eta \tag{2.9}$$

subject to

$$\eta \geq \max_{\omega \in \Omega} [\bar{R}(y_\omega, r_\omega) - R(y_\omega, r_\omega)] \text{ for } \forall \omega \in \Omega.$$

Two Stage Stochastic Linear Programming Formulation of 2S_MM

The two stage stochastic linear programming model is formulated for the portfolio optimization problem that hedges against second stage MM as follows.

Definition 2: 2S_MM

The stochastic portfolio optimization problem where the downside maximum semi-deviation of portfolio returns from the expected return is minimized and the expected portfolio return is constrained is:

$$\text{Minimize } \eta \qquad (2.10a)$$

$$\sum_{i \in I} x_i = 1 \qquad (2.10b)$$

$$\sum_{i \in I} y_{\omega i} = 1 \quad \forall \omega \in \Omega \qquad (2.10c)$$

$$\bar{R}(x, r_\omega) + R(y_\omega, r_\omega) \geq \alpha \quad \forall \omega \in \Omega \qquad (2.10d)$$

$$L_i \leq x_i \leq U_i \quad \forall i \in I \qquad (2.10e)$$

$$L_{\omega i} \leq y_{\omega i} \leq U_{\omega i} \quad \forall i \in I, \forall \omega \in \Omega \qquad (2.10f)$$

$$R(y_\omega, r_\omega) \geq \eta \quad \forall \omega \in \Omega \qquad (2.10g)$$

Model (2.10) minimizes the maximum downside semi deviation of the portfolio return from the expected portfolio return of the second stage variable, $\boldsymbol{y}$, at the end of the investment period. Equation (2.10b) insures that the total weights of all investments in the first stage sum to one, and equation (2.10c) insures that the total weights of all investments in the second stage under each scenario, ω, sum to one - that is, budget constraints ensuring full investment of available budget. Equation (2.10d) constrains the expected return by the target return, α, while equations (2.10e) and (2.10f) insure that the weights on assets purchased are nonnegative, disallowing short sales and placing an upper bound on the weights in the first stage and second stage respectively. Finally, equations (2.10g) and (2.10h) define the mean absolute negative deviation of portfolio returns from the expected portfolio return in the second stage and the auxiliary variables for the linear representation of the deviation measure.

Numerical Analysis

Models developed herein were tested on ten common stocks listed on the main board of Bursa Malaysia. These stocks were randomly selected from a set of stocks that were listed on December 1989 and were still in the list in May 2004; closing prices were obtained from Investors Digest. At first, sixty companies were selected at random, ten stocks were then selected and the criterion used to select the ten stocks in the analysis is as follows:

i. Those companies which do not have a complete closing monthly price during the analysis period were excluded.

ii. Because the portfolios were examined on the basis of historical data, those with negative average returns over the analysis period were excluded.

Empirical distributions computed from past returns were used as equiprobable scenarios. Observations of returns over N_S overlapping periods of length Δt are considered as the N_S possible outcomes (or scenarios) of future returns and a probability of $\frac{1}{N_s}$ is assigned to each of them. Assume T historical prices, $P_t,\ t = 1,2,...,T$ of the stocks under consideration. For each point of time, the realized return vector over the previous period of 1 month is computed, which will be further considered as one of the N_S scenarios for future returns on the assets. Thus, for example, a scenario r_{is} for the return on asset i is obtained as:

$$r_{is} = \frac{P_i(t+1) - P_i(t)}{P_i(t)}. \qquad (3.1)$$

For each stock, 100 scenarios of the overlapping periods of length 1 month were obtained, that is, N_S.

To evaluate the performance of the two models, the portfolio returns resulting from applying the two stochastic optimization models were examined. A comparison is made between the S_MM and 2S_MM models by analyzing the optimal portfolio returns in-sample portfolio returns and out-of-sample portfolio returns over a 60-month period from June 1998 to May 2004. At each month, the historical data from the previous 100 monthly observations is used to solve the resulting optimization models and record the return of the optimal portfolio. The in-sample realized portfolio return is then calculated. The clock is advanced one month and the out-of-sample realized return of the portfolio is determined from the actual return of the assets. The same procedure is repeated for the next period and the average returns are computed for in-sample and out-of-sample realized portfolio return. The minimum monthly required return α is equal to one in the analysis for both the S_MM and 2S_MM models.

Results

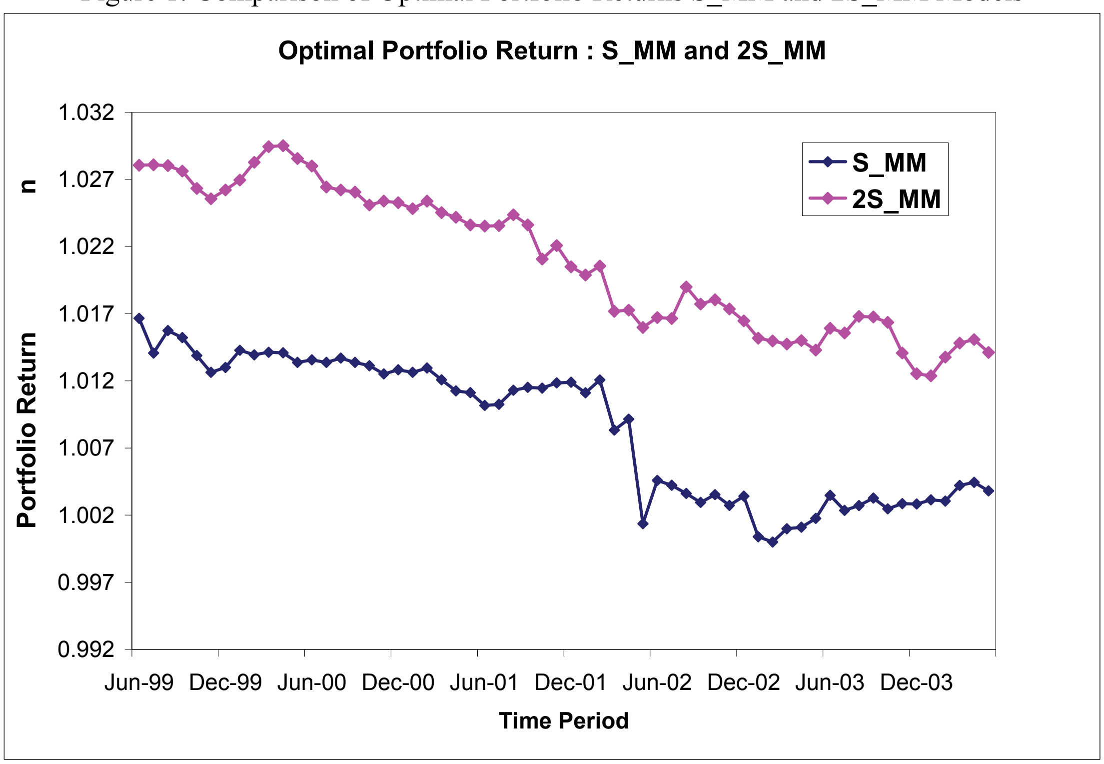
Comparison of Optimal Portfolio Returns between S_MM and 2S_MM

Figure 1 presents the graphs of optimal portfolio returns resulting from solving the two models; S_MM and 2S_MM. The optimal portfolio returns of the two models exhibit a similar pattern: a decreasing trend is observed in the optimal returns for both models. However, as illustrated in Figure 1, the optimal portfolio returns from the two stage stochastic programming with recourse model (2S_MM) are higher than the optimal portfolio returns from the single stage stochastic programming model (S_MM) in all testing periods. This shows that an investor can make a better decision regarding the selection of stocks in a portfolio when taking into consideration both making decision facing the uncertainty and the ability of making corrective actions when the uncertain returns are realized compared to considering only making decisions facing the uncertainty alone.

Comparison of Average In-Sample Portfolio returns between S_MM and 2S_MM

The average realized returns were used

Figure 1: Comparison of Optimal Portfolio Returns S_MM and 2S_MM Models



to compare in-sample portfolio returns between the S_MM model and 2S_MM model; results are presented in Figure 2. An increasing trend is observed in the months from December 1999 until April 2000, and then a decreasing trend is noted until June 2001. From June 2001 until May 2004 both averages show an increasing trend. The average in-sample portfolio returns of 2S_MM are higher than the average in-sample portfolio returns in all testing periods.

Comparison of Out-Of-Sample Portfolio Returns between S_MM and 2S_MM Models

In a real-life environment, model comparison is usually accomplished by means of ex-post analysis. Several approaches can be used to compare models. One of the most commonly applied methods is based on the representation of the ex-post returns of selected portfolios over a given period and on comparing them against a required level of return. The comparison of out-of-sample portfolio returns between the single stage stochastic programming model S_MM and the two stage stochastic programming with recourse model 2S_MM is also accomplished using the average return. The results of the out-of-sample analysis are presented in Figure 3.

Throughout the testing periods, the average returns from the two models show similar patterns. An increasing trend is observed in the months from December 1999 until December 2000, and then a decreasing trend is observed until June 2001. Starting from June 2001, both averages show an increasing trend. The average out-of-sample of the two-stage model 2S_MM is higher than those of single stage model S_MM. The models have been applied directly to the original historical data treated as future returns scenarios, thus loosening the trend information. Possible application of forecasting procedures prior to the portfolio optimization models considered may be an interesting direction for future research. For references on scenario generation see Carino, et al., (1998).

Figure 2: Comparison of Average In-Sample Portfolio Return between S_MM and 2S_MM Models

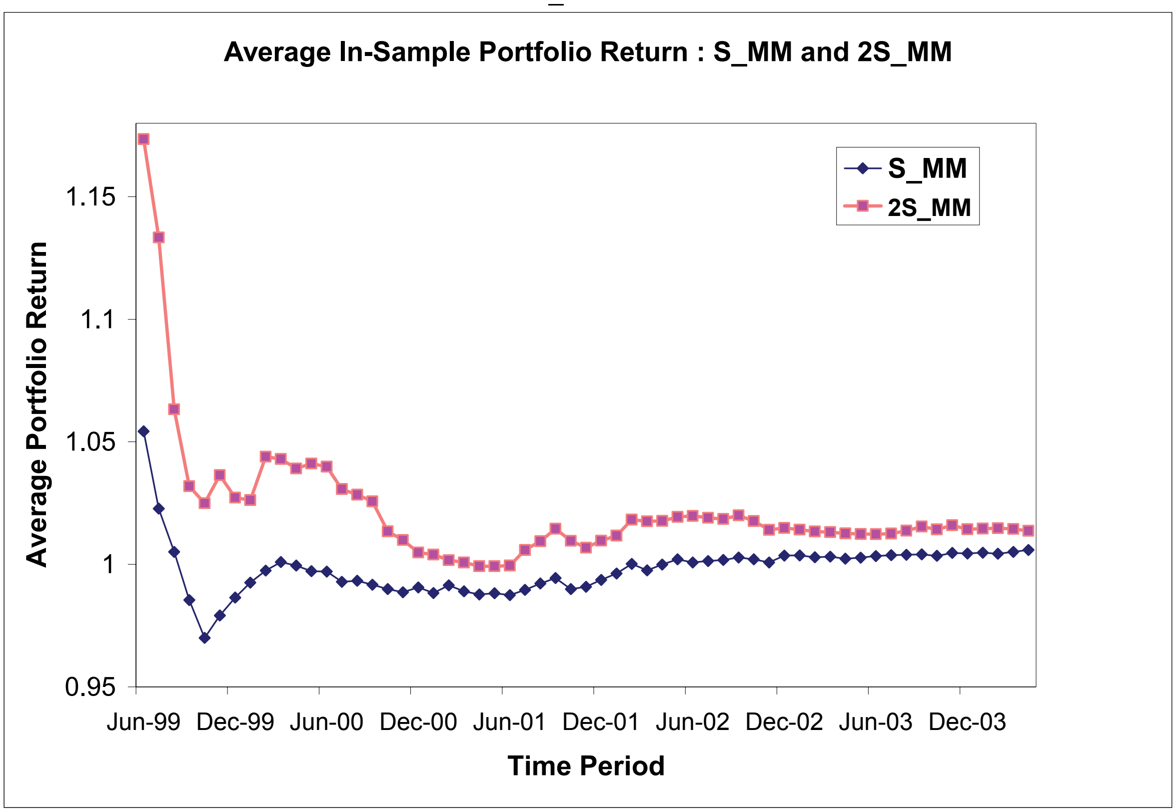
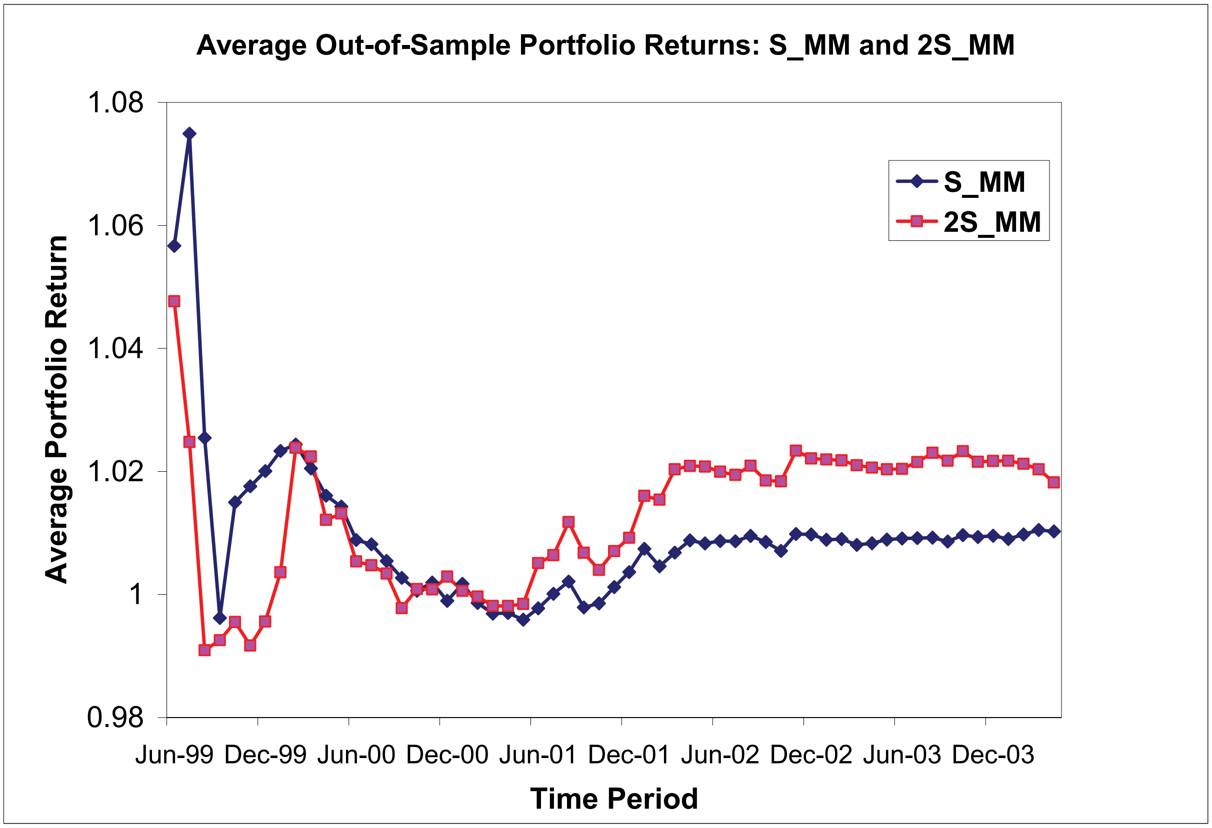


Figure 3: Comparison of Out-Of-Sample Analysis between Single Stage S_MM and Two Stage 2S_MM Models



Conclusion

A portfolio selection of stocks with maximum downside semi deviation measure is modeled as single stage and two stage stochastic programming models in this article. The single stage model and the two stage model incorporate uncertainty and at the same consider rebalancing the portfolio composition at the end of investment period. The comparison of the optimal portfolio returns, the in-sample portfolio returns and the out-of-sample portfolio returns show that the performance of the two stage model is better than that of the single stage model. Historical data was used for scenarios of future returns. Future research should generate scenarios of future asset returns using an appropriate scenario generation method before applying models developed in this article.

Acknowledgements

The work funded by the FRGS (Fundamental Research Grant Scheme) of Ministry for Higher Education of Malaysia, Grant 203/PJJAUH/671128 Universiti Sains Malaysia.

References

Acerbi, C. (2002). Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking Finance*, *26*(*7*), 1487-1503.

Beale, E. M. L. (1955). On minimizing a convex function subject to linear inequalities. *Journal of the Royal Statistical Society*, *Series B*, *17*, 173-184.

Carino, D. R., Myers, D. H., & Ziemba, W. T. (1998). Concepts, technical issues and use of the Russel-Yasuda Kasai financial planning model. *Operations Research*, *46*, 450-463.

Chang, K. W., Chen, H. J., & Liu, C. Y. (2002). A stochastic programming model for portfolio selection. *Journal of Chinese Institute of Industrial Engineers*, *19*(*3*), 31-41.

Crane, D. B. (1971). A stochastic programming model for commercial bank bond portfolio management. *Journal of Financial and Quantitative Analysis*, *6*, 955-976.

Dantzig, G. B. (1955). Linear programming under uncertainty. *Management Science*, *1*, 197-206.

Konno, H., & Yamazaki, H. (1991). Mean-absolute deviation portfolio optimization model and its application to Tokyo stock market. *Management Science*, *7*, 519-531.

Markowitz, H. M. (1952). Portfolio selection. *Journal of Finance*, *8*, 77-91.

Markowitz,H. M. (1959). *Portfolio selection: Efficient diversification of investmen*t. New York: John Wiley& Sons.

Rockafellar, R. T., Uryasev, S., & Zabarankin, M. (2002). Deviation measures in risk analysis and optimization. *Technical Report 2002-7*, ISE Dept., University of Florida.

Rockafellar, R. T., Uryasev, S., & Zabarankin, M. (2004). Generalized deviation in risk analysis. *Technical Report 2004-4*, ISE Dept., University of Florida.

Ruszczynski, A., & Shapiro, A. (2004). Optimization of convex risk functions. *Working paper*.

Sortino, F. A., & Forsey, H. J. (1996). On the use and misuse of downside risk. *Journal of Portfolio Management*, *Winter*, 35-42.

Young, M. R. (1998). A minimax portfolio selection rule with linear programming solution. *Management Science*, *44*, 673-683

Worzel,K. J., Vassiadou-Zeniou, C., & Zenios, S. A. (1994). Integrated simulation and optimization models for tracking fixed-income securities. *Operations Research*, *42*(*2*), 223-233.

Zenios, S. A, Holmer, M. R., McKendall, R., & Vassiadou-Zeniou, C. (1998). Dynamic models for fixedincome portfolio management under uncertainty. *Journal of Economic Dynamics and Control*, *22*, 1517-1541.