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On Bayesian Shrinkage Setup for Item Failure Data Under a Family of Life Testing Distribution

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Properties of the Bayes shrinkage estimator for the parameter are studied of a family of probability density function when item failure data are available. The symmetric and asymmetric loss functions are considered for two different prior distributions. In addition, the Bayes estimates of reliability function and hazard rate are obtained and their properties are studied.

Key words: Bayes shrinkage estimator; squared error loss function (SELF); LINEX loss function (LLF); reliability function; hazard rate.

Introduction

The probability density function (pdf) of a random variable x with parameter θ and two known positive constants w and v for the proposed family of life testing distribution is given as

$$f(x;\theta,w,v)=\frac{v}{\Gamma w}\left(\frac{x^{wv-1}}{\theta^{w}}\right)\exp\left(-\frac{x^{v}}{\theta}\right);\quad x>0,\ \theta>0,\ w,v>0. \tag{1.1}$$

For the different values of w and v, the distributions are given as:

w	v	Distribution
1	1	Exponential
	1	Two parameter Gamma
+ve Integer	1	Erlang
1		Two parameter Weibull
1	2	Rayleigh
3/2	2	Maxwell

The use of SELF in the Bayes estimation may not be appropriate when positive and negative errors have different consequences. To overcome this difficulty, an asymmetric loss function (LLF) was proposed by Varian (1975) and its invariant form for any parameter θ is given by (see Singh, et al., 2007)

$$L(\Delta)=e^{a\Delta}-a\Delta-1;\ a\neq 0$$

and

$$\Delta=\left(\hat{\theta}-\theta\right)/\theta. \tag{1.2}$$

where $\hat{\theta}$ is any estimate of the parameter θ. The sign and magnitude of 'a' represents the direction and degree of asymmetry respectively. The positive (negative) value of 'a' is used when overestimation is more (less) serious than underestimation. For small value of $|a|$, LLF is not far from SELF.

In many situations, the experimenter has some prior information about parameter in the form of a point guess value. Thompson (1968), Pandey and Singh (1977), Prakash and Singh (2006), Prakash and Singh (2008, 09) and others have suggested shrinkage estimators utilizing the point guess value of the parameter and have shown that they performed better when the guess value is in the vicinity of the true value. The shrinkage procedure has been applied in numerous problems, including mean survival time in epidemiological studies, forecasting of the money supply, estimating mortality rates and improving estimation in sample surveys.

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The shrinkage estimator of the parameter θ when a prior point guess value θ_0 is available, is given by

$$S = k\hat{\theta} + (1-k)\theta_0 . \qquad (1.3)$$

Here $k \in [0,1]$ is the shrinkage factor and the experimenter according to his belief in the guess value specifies the values of the shrinkage factor.

In the item censored situations where n items are put to test without replacement and the test terminates as soon as the r^{th} item fails $(r \le n)$. Let $x_1, x_2, ..., x_r$ be the observed failure items for the first r components, then the likelihood function of r failure items $\underline{x} = (x_1, x_2, ..., x_r)$ is

$$L(\underline{x}; \theta) = h(\underline{x}) \frac{1}{\theta^{rw}} \exp\left(-\frac{T_r}{\theta} \right), \qquad (1.4)$$

where $T_r = \sum_{i=1}^{r} x_i^v + (n-r) x_{(r)}^v$ and the function $h(\underline{x})$ is independent with the parameter θ. The statistic T_r is sufficient for θ and the UMVU estimator is $U_r = T_r / rw$. The risk of U_r under the SELF and LLF are obtained as

$$R_{(S)}(U_r) = \frac{\theta^2}{r}$$

and

$$R_{(L)}(U_r) = e^{-a}\left(1 - \frac{a}{r}\right)^{-r} - 1 .$$

Here the suffixes S and L respectively show the risk taken under the SELF and LLF.

The inverted Gamma distribution with parameters α and β have been considered as the conjugate prior density for the parameter θ with pdf is

$$g_1(\theta) = \frac{\beta^{\alpha}}{\Gamma\alpha}\left(\frac{1}{\theta}\right)^{\alpha+1} \exp\left(-\frac{\beta}{\theta}\right) ; \theta > 0, \alpha, \beta > 0 \qquad (1.5)$$

having the prior mean is

$$\frac{\beta}{\alpha - 1}; \alpha > 1$$

and the prior variance is

$$\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}; \alpha > 2 .$$

For the situation where life researchers have no prior information about the parameter θ, the uniform, quasi or improper prior may be used. This study considered a class of quasi prior defined as

$$g_2(\theta) = \frac{1}{\theta^d} \exp\left(-\frac{pd}{\theta} \right) ; \theta > 0, p, d > 0 . \qquad (1.6)$$

If $d = 0$ then a diffuse prior results, and if $d = 1, p = 0$ then a non–informative prior results. For a set of values of d and p, that satisfies the equality $\Gamma(d-1) = (pd)^{d-1}$ makes $g_2(\theta)$ an proper prior. The prior mean and prior variance are given as

$$\frac{\Gamma(d-2)}{(pd)^{d-2}}; d \ge 2$$

and

$$\frac{\Gamma(d-3)}{(pd)^{2d-4}}\left((pd)^{d-1} - (d-3)\Gamma(d-2) \right); d > 3.$$

Some Bayes estimators and Bayes Shrinkage estimators are suggested for the parameter θ when other parameters are known. The properties of these estimators are studied in terms of relative efficiencies empirically and by numerical example. The Bayes estimator of reliability function and hazard rate are obtained and their properties are studied.

Methodology

Bayes Shrinkage Estimators and their Properties

The posterior density $Z_1(\theta)$ for the parameter θ corresponding to prior $g_1(\theta)$ is obtained as

$$Z_1(\theta) = \frac{(r\,U_r+\beta)^{r\,w+\alpha}}{\Gamma(r\,w+\alpha)}\,\frac{e^{-(r\,U_r+\beta)/\theta}}{\theta^{r\,w+\alpha+1}};\ \theta>0. \tag{2.1}$$

which is an inverted Gamma distribution with parameters $(r\,w+\alpha)$ and $(r\,U_r+\beta)$. The Bayes estimator of θ under SELF is obtained as

$$\hat{\theta}_1 = E_p(\theta) = \varphi_1(r\,U_r+\beta), \tag{2.2}$$

where $\varphi_1 = (r\,w+\alpha-1)^{-1}$ and suffix p indicates, the expectation is considered under the posterior density.

To utilize the prior information about the parameter θ in the form of a point guess value θ_0, the values of prior parameter β are chosen (Shirke & Nalawade, 2003) such as

$$E\left(\hat{\theta}_1\right)=\theta_0 \Rightarrow \varphi_1\beta = (1-\varphi_1\,r)\,\theta_0. \tag{2.3}$$

Using (2.3) in (2.2), the shrinkage estimator takes the form (1.3) and is named the Bayes shrinkage estimator:

$$\bar{\theta}_1 = \lambda_1 U_r + (1-\lambda_1)\theta_0\ ;\ \lambda_1 = r\,\varphi_1. \tag{2.4}$$

The Bayes estimator θ under the LLF is obtained by simplifying the given equality

$$E_p\left(\frac{1}{\theta}e^{a\hat{\theta}_2/\theta}\right) = e^a\,E_p\left(\frac{1}{\theta}\right) \Rightarrow \hat{\theta}_2 = \varphi_2(r\,U_r+\beta) \tag{2.5}$$

Similarly, the Bayes shrinkage estimator under the LLF is

$$\bar{\theta}_2 = \lambda_2 U_r + (1-\lambda_2)\,\theta_0, \tag{2.6}$$

where

$$\lambda_2 = r\,\varphi_2,\ \varphi_2 = \frac{1}{a}\left(1-\exp\left(\frac{-a}{r\,w+\alpha+1}\right)\right).$$

The risks under the SELF and LLF of these estimators are obtained as

$$R_{(S)}\left(\hat{\theta}_i\right) = \theta^2\left(r(r+1)\varphi_i^2 - 2r\,\varphi_i + 1\right) + 2\theta\beta\,\varphi_i\left(r\,\varphi_i - 1\right) + \beta^2\varphi_i^2,$$

$$R_{(L)}\left(\hat{\theta}_i\right) = \frac{e^{C_i}}{(1-a\,\varphi_i)^r} - (a\,\varphi_i\,r + C_i + 1),$$

$$C_i = a\left(\frac{\varphi_i\beta}{\theta} - 1\right),$$

$$R_{(S)}\left(\bar{\theta}_i\right) = \theta^2\left\{\lambda_i^2\left(\frac{r+1}{r} + \delta(\delta-2)\right) + (1-\delta)^2(1-2\lambda_i)\right\}$$

and

$$R_{(L)}\left(\bar{\theta}_i\right) = \frac{r^r e^{a(\delta(1-\lambda_i)-1)}}{(r-a\lambda_i)^r} - 1 + a(1-\delta)(1-\lambda_i);$$

$$\delta = \frac{\theta_0}{\theta},\ i=1,2.$$

Similarly, the Bayes risks of these estimators are

$$R_{(BS)}\left(\hat{\theta}_i\right) = \beta^2\left(\frac{r(r+1)\varphi_i^2 - 2r\,\varphi_i + 1}{(\alpha-1)(\alpha-2)} + \frac{2\varphi_i(r\,\varphi_i-1)}{(\alpha-1)} + \varphi_i^2\right),$$

$$R_{(BL)}\left(\hat{\theta}_i\right) = \frac{e^{-a}}{(1-a\,\varphi_i)^{\alpha+r}} - a\,\varphi_i(\alpha+r) + a - 1,$$

$$R_{(BS)}\left(\bar{\theta}_i\right) = (1-\lambda_i)^2\left(\theta_0^2 - \frac{2\beta\theta_0}{\alpha-1} + \frac{\beta^2}{(\alpha-1)(\alpha-2)}\right) + \frac{\beta^2\lambda_i^2}{r(\alpha-1)(\alpha-2)}$$

and

$$R_{(BL)}\left(\overline{\theta}_i\right)=\frac{r^r e^{-a}}{\left(r-a\lambda_i\right)^r}\left(1-\frac{a\left(1-\lambda_i\right)\theta_0}{\beta}\right)^{-\alpha}$$
$$-1+\frac{a\left(\beta-\alpha\theta_0\right)}{\beta\left(1-\lambda_i\right)^{-1}}.$$
$$-1+\frac{a\left(\beta-\alpha\theta_0\right)}{\beta\left(1-\lambda_i\right)^{-1}}.$$

The relative efficiency of $\overline{\theta}_i$ with respect to U_r under the SELF and LLF loss criterions are defined as

$$RE_{(S)}\left(\overline{\theta}_i, U_r\right)=\frac{R_{(S)}\left(U_r\right)}{R_{(S)}\left(\overline{\theta}_i\right)}$$

and

$$RE_{(L)}\left(\overline{\theta}_i, U_r\right)=\frac{R_{(L)}\left(U_r\right)}{R_{(L)}\left(\overline{\theta}_i\right)};i=1,2.$$

The expressions of relative efficiencies are the functions of r, a, v, δ and α. For the selected values of $r=04\,(02)\,10$; $a=0.25, 0.50, 1.00, 1.50$; $v=1.00, 1.50$; $\delta=0.40, 0.50\,(0.25)\,1.50, 1.60$ and $\alpha=1.50, 03, 05, 07, 10, 15$. The relative efficiencies have been calculated and are presented in Tables 1–4 for selected parametric values.

The positive values of 'a' are considered because overestimation in mean life is more serious than underestimation. To guard against the large risk, the large values of 'a' may be ignored and the smaller values of $a\,(\leq 2)$ are considered (see Singh, et al., 2007).

Results

Tables 1 and 2 show that the estimator $\overline{\theta}_1$ is more efficient than the UMVU estimator U_r under the SELF and LLF for all selected parametric set of values when $r\leq 06$. For large $r\geq 08$, the effective interval decreases with large $\alpha\geq 10$. The efficiency attains maximum at the point $\delta=1.00$ (except $\alpha=1.50$ when the loss criterion is LLF) and the efficiency decreases as r increases for all considered values of δ.

In addition, under LLF loss criterion the efficiency increases with 'a' increases for all considered values of δ for small $r\leq 06$, and for large r in the interval $\delta\leq 1.00$. The estimator $\overline{\theta}_2$ performs uniformly well with respect to U_r for all considered values of the parametric space when r is small and in the interval $0.50\leq\delta\leq 1.25$ when $r\geq 06$ (under the SELF–criterion).

In addition, the effective interval decreases as r or α increases (Table 3). The increasing trend in efficiency is also observed when 'a' increases in the interval $\delta\in[0.75, 1.25]$ for other fixed parametric values.

The estimator $\overline{\theta}_2$ performs uniformly well with respect to U_r under LLF loss criterion for all considered values of the parameter space (Table 4). The increasing trend in efficiency is observed when 'a' increases for all δ when r is small and in the range $\delta\leq 1.25$ when $r\geq 08$.

Using Tables 3 and 4, it may be concluded that the efficiency reaches its maximum at the point $\delta=1.00$. The efficiency decreases as r increases for all considered values of parametric space.

Further, as v increases, the gain in efficiency is recorded only in close vicinity of the guess value and true value of the parameter but the effective interval becomes smaller for both the Bayes shrinkage estimators.

Remark

Note that the posterior density with respect to the quasi prior $g_2(\theta)$ is

$$Z_2(\theta)=\frac{\left(r\,U_r+p\,d\right)^{r\,w+d-1}}{\Gamma\left(r\,w+d-1\right)}\frac{e^{-\left(r\,U_r+p\,d\right)/\theta}}{\theta^{(r\,w+d)}}.$$

The only changes in the posterior (2.1) are replacement α and β by $d-1$ and $p\,d$ respectively. Hence, all the results are valid by substitution of these two.

Table 1: Relative Efficiency for the Bayes Shrinkage Estimator $\overline{\theta}_1$ with Respect to U_r Under the SELF

r	δ	α					
		1.50	3.00	5.00	7.00	10.00	15.00
04	0.40	1.2378	1.6544	1.6393	1.4741	1.2741	1.0864
	0.50	1.2462	1.8000	2.0000	1.9231	1.7423	1.5283
	0.75	1.2607	2.1176	3.2000	4.0000	4.6621	4.9846
	1.00	1.2656	2.2500	4.0000	6.2500	10.562	20.250
	1.25	1.2607	2.1176	3.2000	4.0000	4.6621	4.9846
	1.50	1.2462	1.8000	2.0000	1.9231	1.7423	1.5283
	1.60	1.2378	1.6544	1.6393	1.4741	1.2741	1.0864
06	0.40	1.1563	1.4337	1.4172	1.2658	1.0866	1.0708
	0.50	1.1615	1.5238	1.6667	1.6000	1.4286	1.2121
	0.75	1.1706	1.7067	2.3810	2.9091	3.3898	3.6530
	1.00	1.1736	1.7778	2.7778	4.0000	6.2500	11.111
	1.25	1.1706	1.7067	2.3810	2.9091	3.3898	3.6530
	1.50	1.1615	1.5238	1.6667	1.6000	1.4286	1.2121
	1.60	1.1563	1.4337	1.4172	1.2658	1.0866	1.0708
10	0.40	1.1163	1.3242	1.3081	1.1689	1.0721	0.7701
	0.50	1.1202	1.3889	1.5000	1.4412	1.2788	1.0614
	0.75	1.1267	1.5152	2.0000	2.3902	2.7656	2.9877
	1.00	1.1289	1.5625	2.2500	3.0625	4.5156	7.5625
	1.25	1.1267	1.5152	2.0000	2.3902	2.7656	2.9877
	1.50	1.1202	1.3889	1.5000	1.4412	1.2788	1.0614
	1.60	1.1163	1.3242	1.3081	1.1689	1.0721	0.7701
15	0.40	1.0927	1.2587	1.2437	1.1150	1.0219	0.7150
	0.50	1.0957	1.3091	1.4000	1.3474	1.1934	0.9763
	0.75	1.1008	1.4049	1.7818	2.0898	2.3967	2.5888
	1.00	1.1025	1.4400	1.9600	2.5600	3.6100	5.7600
	1.25	1.1008	1.4049	1.7818	2.0898	2.3967	2.5888
	1.50	1.0957	1.3091	1.4000	1.3474	1.1934	0.9763
	1.60	1.0927	1.2587	1.2437	1.1150	1.0219	0.7150

Table 2: Relative Efficiency for the Bayes Shrinkage Estimator $\overline{\theta}_1$ with respect to U_r under the LLF

r = 04		α					
a	δ	1.50	3.00	5.00	7.00	10.00	15.00
0.25	0.40	1.2666	1.7672	1.7814	1.6073	1.3911	1.1874
	0.50	1.2715	1.9093	2.1609	2.0867	1.8925	1.6607
	0.75	1.2770	2.2010	3.3852	4.2651	4.9906	5.3386
	1.00	1.2729	2.2882	4.1003	6.4364	10.923	21.021
	1.25	1.2594	2.1145	3.2063	4.0280	4.7259	5.0860
	1.50	1.2307	1.7608	1.9872	1.9241	1.7524	1.5412
	1.60	1.2257	1.6068	1.6256	1.4709	1.2763	1.0894
0.50	0.40	1.2997	1.9024	1.9552	1.7709	1.5347	1.3115
	0.50	1.3007	2.0404	2.3580	2.2880	2.0777	1.8240
	0.75	1.2968	2.3032	3.6144	4.5947	5.4006	5.7815
	1.00	1.2835	2.3423	4.2400	6.6931	11.414	22.060
	1.25	1.2613	2.1256	3.2413	4.0963	4.8409	5.2460
	1.50	1.2311	1.7640	1.9921	1.9437	1.7805	1.5706
	1.60	1.2310	1.6108	1.6262	1.4818	1.2912	1.1036
1.00	0.40	1.3830	2.2709	2.4450	2.2347	1.9422	1.6636
	0.50	1.3757	2.3977	2.9135	2.8603	2.6054	2.2894
	0.75	1.3510	2.5174	4.2662	5.5404	6.5833	7.0633
	1.00	1.3182	2.5891	4.6862	7.5059	12.955	25.292
	1.25	1.2782	2.2041	3.4259	4.3950	5.2801	5.8092
	1.50	1.2321	1.7870	2.0697	2.0563	1.9090	1.6956
	1.60	1.2323	1.6214	1.6820	1.5583	1.3718	1.1766
1.50	0.40	1.5036	2.8418	3.2760	3.0281	2.6406	2.2668
	0.50	1.4861	2.8655	3.8543	3.8407	3.5120	3.0895
	0.75	1.4372	2.9731	5.3708	7.1652	8.6319	9.2915
	1.00	1.3822	3.0604	5.5108	9.0029	15.782	31.192
	1.25	1.3225	2.4017	3.8550	5.0453	6.1813	6.9176
	1.50	1.2594	1.9032	2.2881	2.3249	2.1938	1.9654
	1.60	1.2335	1.7157	1.8501	1.7503	1.5608	1.3455

Table 3: Relative Efficiency for the Bayes Shrinkage Estimator $\overline{\theta}_2$ with Respect to U_r Under the SELF

r = 04		α					
a	δ	1.50	3.00	5.00	7.00	10.00	15.00
	0.40	1.6932	1.6299	1.4640	1.3237	1.1778	1.0374
	0.50	1.9175	1.9995	1.9156	1.7925	1.6368	1.4674
	0.75	2.4770	3.2597	4.0396	4.5231	4.8735	5.0000
0.25	1.00	2.7438	4.1266	6.4079	9.1891	14.298	25.314
	1.25	2.4770	3.2597	4.0396	4.5231	4.8735	5.0000
	1.50	1.9175	1.9995	1.9156	1.7925	1.6368	1.4674
	1.60	1.6932	1.6299	1.4640	1.3237	1.1778	1.0374
	0.40	1.6944	1.6199	1.4540	1.3161	1.1730	1.0348
	0.50	1.9341	1.9981	1.9079	1.7850	1.6312	1.4642
	0.75	2.5485	3.3188	4.0782	4.5455	4.8817	4.9998
0.50	1.00	2.8504	4.2566	6.5691	9.3816	14.537	25.631
	1.25	2.5485	3.3188	4.0782	4.5455	4.8817	4.9998
	1.50	1.9341	1.9981	1.9079	1.7850	1.6312	1.4642
	1.60	1.6944	1.6199	1.4540	1.3161	1.1730	1.0348
	0.40	1.6917	1.5990	1.4342	1.3013	1.1636	1.0298
	0.50	1.9611	1.9928	1.8922	1.7701	1.6203	1.4578
	0.75	2.6919	3.4349	4.1529	4.5883	4.8971	4.9992
1.00	1.00	3.0737	4.5267	6.9016	9.7765	15.026	26.276
	1.25	2.6919	3.4349	4.1529	4.5883	4.8971	4.9992
	1.50	1.9611	1.9928	1.8922	1.7701	1.6203	1.4578
	1.60	1.6917	1.5990	1.4342	1.3013	1.1636	1.0298
	0.40	1.6830	1.5770	1.4147	1.2868	1.1544	1.0249
	0.50	1.9806	1.9846	1.8760	1.7554	1.6096	1.4516
	0.75	2.8351	3.5477	4.2240	4.6284	4.9113	4.9983
1.50	1.00	3.3113	4.8108	7.2479	10.185	15.528	26.934
	1.25	2.8351	3.5477	4.2240	4.6284	4.9113	4.9983
	1.50	1.9806	1.9846	1.8760	1.7554	1.6096	1.4516
	1.60	1.6830	1.5770	1.4147	1.2868	1.1544	1.0249

Table 4: Relative Efficiency for the Bayes Shrinkage Estimator $\overline{\theta}_2$ with respect to U_r under the LLF

r = 04		α					
a	δ	1.50	3.00	5.00	7.00	10.00	15.00
0.25	0.40	1.8240	1.7718	1.5965	1.4448	1.2866	1.1344
	0.50	2.0515	2.1615	2.0788	1.9469	1.7782	1.5948
	0.75	2.5926	3.4508	4.3086	4.8389	5.2200	5.3538
	1.00	2.7988	4.2316	6.6005	9.4934	14.815	26.304
	1.25	2.4751	3.2671	4.0691	4.5772	4.9570	5.1097
	1.50	1.8947	1.9878	1.9172	1.8010	1.6490	1.4802
	1.60	1.6636	1.6171	1.4613	1.3250	1.1808	1.0401
0.50	0.40	1.9892	1.9353	1.7475	1.5846	1.4144	1.2504
	0.50	2.2395	2.3606	2.2710	2.1283	1.9461	1.7481
	0.75	2.8215	3.7598	4.6905	5.2598	5.6626	5.7964
	1.00	2.9917	4.5192	7.0418	10.119	15.778	27.986
	1.25	2.5656	3.3663	4.1820	4.7057	5.1063	5.2799
	1.50	1.9049	1.9946	1.9310	1.8208	1.6731	1.5056
	1.60	1.6644	1.6175	1.4633	1.3322	1.1910	1.0510
1.00	0.40	2.4607	2.4012	2.1776	1.9825	1.7780	1.5802
	0.50	2.7745	2.9280	2.8197	2.6462	2.4254	2.1857
	0.75	3.4681	4.6392	5.7837	6.4691	6.9391	7.0775
	1.00	3.5344	5.3440	8.3263	11.959	18.629	32.999
	1.25	2.8427	3.7029	4.5902	5.1748	5.6420	5.8720
	1.50	1.9870	2.0828	2.0351	1.9345	1.7908	1.6201
	1.60	1.6975	1.6564	1.5234	1.3986	1.2587	1.1151
1.50	0.40	3.2712	3.2019	2.9164	2.6657	2.4019	2.1455
	0.50	3.6926	3.9030	3.7633	3.5375	3.2503	2.9387
	0.75	4.4534	6.1474	7.6678	8.5593	9.1522	9.3049
	1.00	4.5695	6.7637	10.564	15.186	23.661	41.896
	1.25	3.3496	4.3562	5.4076	6.1173	6.7074	7.0303
	1.50	2.2055	2.3275	2.3004	2.2058	2.0581	1.8726
	1.60	1.8476	1.8227	1.6995	1.5742	1.4268	1.2692

Example: Exponential Failure Model

Two hundred electronic tubes were tested under the exponential failure model with the parameter $\theta = 04$ and the test was terminated after the first six items failed. The failure times (in hours) were recorded as follows:

83.5, 221, 356, 478, 535, 632

The relative efficiencies and Bayes risks of the proposed estimators were obtained for $\alpha = 5.00, 8.50, 10$; $\beta = 20, 32, 50$; a = 0.50 1.00,1.50 and are presented in Table 5 for $\alpha = 8.50, \beta = 32$ and $a = 0.50$.

It may be concluded that the relative efficiency attains maximum at point $\theta = \theta_0$ for all considered values. Under the LLF criterion the gain in efficiency is larger than the SELF–criterion when $\delta \le 1.25$. Further, the Bayes risks are nominal when the loss criterion is LLF. The risks have the tendency to be smaller when $\theta > \theta_0$ and attains minimum when $\theta = \theta_0$ and then increases. Further, both the risk and Bayes risk decreases (increases) when $'a'(\alpha)$ increases under both loss criterions when other parametric values are fixed.

The Bayes Estimator of Reliability Function and Hazard Rate

The Reliability function $\Psi(t)$ at time $t\,(>0)$ is defined as

$$\Psi(t) = \frac{1}{\Gamma(w)} \int_{t^v/\theta}^{\infty} e^{-S} S^{w-1}\, dS. \qquad (3.1)$$

Similarly, the Hazard rate at time $t\,(>0)$ is given by

$$\rho(t) = \frac{v\, t^{wv-1}}{\theta^w e^{t^v/\theta}} \left\{ \int_{t^v/\theta}^{\infty} e^{-S} S^{w-1}\, dS \right\}^{-1}. \qquad (3.2)$$

In particular, for the exponential distribution $(w = v = 1)$ the Reliability function and Hazard rate are given as

$$\Psi(t) = \exp(-t/\theta) \text{ and } \rho(t) = 1/\theta. \qquad (3.3)$$

The Bayes estimator of the reliability function and hazard rates under the SELF, corresponding to the posterior density $Z_1(\theta)$ are obtained as

$$\Psi_1 = G\left(0, \infty, J_1\right) \text{ and } \rho_1 = G\left(0, \infty, J_2\right); \qquad (3.4)$$

where

$$G(u_1, u_2, \xi) = \frac{(r\,U_r + \beta)^{r_1}}{\Gamma(r_1)} \int_{u_1}^{u_2} e^{-(r U_r + \beta) z} z^{r_1 - 1} (\xi)\, dz,$$

ξ be the function of z, $r_1 = r\,w + \alpha$,

$$J_1 = \int_{t^v z}^{\infty} \frac{e^{-S} S^{w-1}}{\Gamma(w)} dS \text{ and } J_2 = \frac{v\, z^w t^{wv-1}}{e^{t^v z}} \left\{ \int_{t^v z}^{\infty} e^{-S} S^{w-1}\, dS \right\}^{-1}.$$

Similarly, the Bayes estimate of the reliability function Ψ_2, and the Hazard rate ρ_2 under the LLF for the given posterior $Z_1(\theta)$ are obtained by solving the given equality

$$G\left(0, \infty, \left(J_1^{-1} \exp\left(a\ J_1^{-1}\, \Psi_{(L)}(t)\right)\right)\right) = e^a\, G\left(0, \infty, J_1^{-1}\right)$$

and

$$G\left(0, \infty, \left(J_2^{-1} \exp\left(a\, J_2^{-1}\, \rho_{(L)}(t)\right)\right)\right) = e^a\, G\left(0, \infty, J_2^{-1}\right) \qquad (3.5)$$

The close form of the Bayes estimators $\Psi(t)$ and $\rho(t)$ under the LLF are nonexistent, therefore, the risk and Bayes risks do not exist in the closed form. For convenience, consider Varian’s (1975) asymmetric loss function defined for any parameter θ as

$$L(\Delta') = e^{a\,\Delta'} - a\,\Delta' - 1;\ \Delta' = \hat{\theta} - \theta.$$

Hence, the Bayes estimators $\Psi(t)$ and $\rho(t)$ under the LLF are given by

$$\Psi_3 = -\frac{1}{a}\ln\left\{\frac{\left(r\,U_r+\beta\right)^{r_1}}{\Gamma\left(r_1\right)}\;G\left(0,\infty,e^{-a\,J_1}\right)\right\}$$

and

$$\rho_3 = -\frac{1}{a}\ln\left\{\frac{\left(r\,U_r+\beta\right)^{r_1}}{\Gamma\left(r_1\right)}\;G\left(0,\infty,e^{-a\,J_2}\right)\right\}. \tag{3.6}$$

The risk and Bayes risk of these Bayes estimators under the SELF and LLF do not exist in a closed form. However, the numerical findings of the risk and Bayes risk for these Bayes estimators under SELF and LLF $R_{(S)}(\Psi_i)$, $R_{(L)}(\Psi_i)$, $R_{(BS)}(\Psi_i)$, $R_{(BL)}(\Psi_i)$, $R_{(S)}(\rho_i)$, $R_{(L)}(\rho_i)$, $R_{(BS)}(\rho_i)$ and $R_{(BL)}(\rho_i)$; $i=1,3$ are obtained for a particular case when $v=w=1$.

Example: Risks and Bayes Risks

Consider the above example with $t=250h$ the Bayes estimates for the reliability function and hazard rate, risks and Bayes risks as obtained are presented in Table 6.

The risk of the estimator Ψ_1 increases as β increases when $\alpha \ge 8.50$ under the LLF. A similar trend is observed for the risk and Bayes risk of Ψ_3 as β increases when $\alpha \ge 8.50$ under the LLF.

Further, the Bayes risk of Ψ_1 and Ψ_3 increases when 'a' increases under both loss criterions. The risk and Bayes risk decreases when α increases (except $\beta \ge 50.00$) when other parametric values are fixed for Ψ_1 (LLF–criterion) and Ψ_3 (SELF and LLF criterions).

The risk and Bayes risk for the estimators ρ_1 and ρ_3 increases as β increases for all the considered values of α under the SELF and LLF (except $\alpha = 5.00$) when other parametric values are fixed. The risk and Bayes risk of ρ_1 and ρ_3 also increases (decreases) under both loss criterions when 'a'(α) increases.

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Table 5: Risk and Bayes Risk of the Bayes Shrinkage Estimators

$r = 06$ :: $\theta = 04$ :: $\alpha = 8.50$ :: $\beta = 32.00$ :: $a = 0.50$							
δ	0.40	0.50	0.75	1.00	1.25	1.50	1.60
θ_0	1.60	2.00	3.00	4.00	5.00	6.00	6.40
$RE_{(S)}\left(\overline{\theta}_1, U_r\right)$	1.1571	1.5140	3.1921	5.0625	3.1921	1.5140	1.1571
$RE_{(L)}\left(\overline{\theta}_1, U_r\right)$	1.3339	1.7305	3.5411	5.2767	3.1873	1.4909	1.1333
$RE_{(S)}\left(\overline{\theta}_2, U_r\right)$	1.0278	1.3889	3.4623	6.8918	3.4623	1.3889	1.0278
$RE_{(L)}\left(\overline{\theta}_2, U_r\right)$	1.1872	1.5902	3.8596	7.2137	3.4783	1.3748	1.0101
$R_{(BS)}\left(\overline{\theta}_1\right)$	3.7507	3.1416	2.0511	1.5778	1.7219	2.4832	2.9606
$R_{(BL)}\left(\overline{\theta}_1\right)$	0.0164	0.0130	0.0103	0.0098	0.0180	0.0336	0.0421
$R_{(BS)}\left(\overline{\theta}_2\right)$	4.3068	3.5505	2.1963	1.6086	1.7875	2.7328	3.3256
$R_{(BL)}\left(\overline{\theta}_2\right)$	0.0184	0.0141	0.0183	0.0100	0.0202	0.0398	0.0505

Table 6: Risk and Bayes Risk of the Reliability Function and Hazard Rates

$r = 06$ :: $\theta = 04$ :: $t = 250$ :: $\alpha = 8.50$ :: $\beta = 32.00$							
$a \rightarrow$	0.50	1.00	1.50	$a \rightarrow$	0.50	1.00	1.50
Ψ_1	0.0097	0.0097	0.0097	Ψ_3	97.860	48.930	32.620
$R_{(S)}(\Psi_1)$	16.000	16.000	16.000	$R_{(S)}(\Psi_3)$	7.1480	9.0530	10.810
$R_{(BS)}(\Psi_1)$	35.192	35.192	35.192	$R_{(BS)}(\Psi_3)$	37.160	38.380	42.950
$R_{(L)}(\Psi_1)$	1.1353	3.0183	5.0025	$R_{(L)}(\Psi_3)$	1.1550	1.9160	3.8300
$R_{(BL)}(\Psi_1)$	1.7744	4.1825	6.6077	$R_{(BL)}(\Psi_3)$	1.9700	4.3600	6.7790
$a \rightarrow$	0.50	1.00	1.50	$a \rightarrow$	0.50	1.00	1.50
ρ_1	0.0218	0.0218	0.0218	ρ_3	53.270	26.704	17.849
$R_{(S)}(\rho_1)$	13.941	13.941	13.941	$R_{(S)}(\rho_3)$	6.7758	8.2750	9.9006
$R_{(BS)}(\rho_1)$	33.257	33.257	33.257	$R_{(BS)}(\rho_3)$	35.803	37.005	41.536
$R_{(L)}(\rho_1)$	1.0214	2.7575	4.6040	$R_{(L)}(\rho_3)$	1.1513	1.7864	3.6102
$R_{(BL)}(\rho_1)$	1.7085	4.0468	6.4038	$R_{(BL)}(\rho_3)$	1.9298	4.2714	6.6433