Journal of Modern Applied Statistical Methods

Volume 11 | Issue 2 Article 20

11-1-2012

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Recommended Citation
Rao, B. Srinivasa; Reddy, J. Pratapa; and Babu, G. Sarath (2012) "Extreme Value Charts and Analysis of Means (ANOM) Based on the Log Logistic Distribution," *Journal of Modern Applied Statistical Methods*: Vol. 11 : Iss. 2 , Article 20.
DOI: 10.22237/jmasm/1351743540

Journal of Modern Applied Statistical Methods
November 2012, Vol. 11, No. 2, 493-505

1538 – 9472/12/$95.00

# Extreme Value Charts and Analysis of Means (ANOM) Based on the Log Logistic Distribution

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A probability model of a quality characteristic is assumed to follow a log logistic distribution. This article proposes variable control charts, termed extreme value charts, based on the extreme values of each subgroup. The control chart constants depend on the probability model of the extreme order statistics and the size of each subgroup. The analysis of means (ANOM) technique for a skewed population is applied with respect to log logistic distribution. Results are illustrated using examples based on real data.

Key words: ANOM, LLD, in control, equi-tailed, Q-Q plot.

## Introduction

The probability density function (PDF) of a log logistic distribution (LLD) with shape parameter b and scale parameter σ is given by

$$f(x, b, o) = \frac{b\left(\frac{x}{\sigma}\right)^{b-1}}{\sigma\left(1+\left(\frac{x}{\sigma}\right)^{b}\right)^{2}},$$

$$x > 0,\ \sigma > 0,\ \mathrm{b} > 1 \qquad (1.0.1)$$

and its cumulative distribution function (CDF) is

$$F(x, b, \sigma) = \frac{\left(\frac{x}{\sigma}\right)^{b}}{1+\left(\frac{x}{\sigma}\right)^{b}},$$

$$x > 0,\ \sigma > 0,\ \mathrm{b} > 1. \qquad (1.0.2)$$

When σ = 1 and b > 1 these equations are termed standard PDF and CDF. In order to construct a control chart using extreme observations of a subgroup drawn from a production process with the quality variate following a LLD, the percentiles of extreme order statistics from LLD samples are needed. Specifically, the test statistic on the extreme value control chart is the original sample vector X = ($x_1$, $x_2$, ...,$x_n$) from ongoing production. In this chart all individual sample observations are plotted into the control chart without calculating any statistics. A corrective action is taken after one, or either, of the extreme values – namely $x_{(1)}$ (sample minimum) and $x_{(n)}$ (sample maximum) – of the sample respectively fall above or below specified lines (limits); this is why the chart is called an extreme value controlled chart.

The Shewart (1986) controlled chart is a common quality control statistical tool: When a Shewart chart indicate the presence of an assignable cause, a process adjustment can be made if the remedy is known; otherwise the

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suspected presence of assignable cause is regarded as an indication of heterogeneity of the subgroup statistic for which the control chart was developed. For example, if the statistic is the sample mean, this leads to heterogeneity of the process mean and indicates departures from the target mean. Such an analysis is generally carried out by dividing a collection of a given number of subgroup means into categories, such that means within a category are homogenous and those between categories are heterogeneous. This procedure, developed by Ott (1967) is called analysis of means (ANOM).

When using the ANOM technique the concept of a control chart for means is viewed differently, grouping of plotted means that fall within or outside control limits. For the homogeneity of the means it is necessary that all means fall within the control limits. If $(1 - \alpha)$ is the confidence coefficient, then the probability that all subgroup means will fall within the control limits is $(1 - \alpha)$. Assuming independence of the subgroup, the probability statement becomes $n^{th}$ power of the probability that a subgroup mean will fall within the limits. This means that, in the sampling distribution of $\overline{x}$, the confidence interval for $\overline{x}$ to lie between two specified limits should be equal to $(1 - \alpha)^{1/n}$. This same principle is adapted through log logistic distribution in this study. This article explores ANOM using control limits of extreme value statistics considering only control chart aspects. (See Rao (2005) for a detailed description of ANOM; other related works include: Ramig, 1983; Bakir, 1994; Bernard & Wludyka, 2001; Wludyka, et al., 2001; Montgomery, 2001; Nelson & Dudewicz, 2002; Rao & Prankumar, 2002; Farnum, 2004; Guirguis & Tobias, 2004; Srinivasa Rao & Kantam, 2012.)

## Extreme Value Charts

The given sample observations are assumed to follow log logistic mode. The controlled lines are determined by the theory of extreme order statistics based on a half logistic model. The controlled lines are determined in such a way that an arbitrarily chosen $x_i$ of $X = (x_1, x_2, \ldots, x_n)$ lies with the probability $(1 - \alpha)^{1/n}$ within the limits. This can be formulated as a probability inequality as: $P(x_1 \leq L) = \alpha/2$ and $P(x_n \geq U) = \alpha/2$. The theory of order statistics states that the cumulative distribution function of the least and highest order statistics in a sample of size $n$ from any continuous population are $[F(x)]^n$ and $1-[1-F(x)]^n$, respectively, where $F(x)$ is a cumulative distribution function (CDF) of the population. If $1-\alpha$ is desired at 0.9973, then $\alpha$ would equal 0.0027. Taking $F(x)$ as the CDF of a standard log logistic model results in solutions of the equations $1-[1-F(x)]^n= 0.00135$ and $[F(x)]^{n} = 0.99865$ which, in turn, can be used to develop the controlled limits of an extreme value chart. The solutions for the two equations for n = 2 (1) 10 with b = 2, 3, 4 and 5 are shown in table 2.1 and denoted as $Z_* = Z_{(1)0.00135}$ and $Z_{**} = Z_{(n)0.99865}$.

The values shown in table 2.1 indicate the following probability statements:

$$P\begin{pmatrix} Z_{(1)0.00135} < Z_i < Z_{(n)0.99865}, \\ \forall i, = 1, 2, \ldots, n \end{pmatrix} = 0.9973 \tag{2.0.3}$$

and

$$P\begin{pmatrix} \sigma Z_{(1)0.00135} < x_i < \sigma Z_{(n)0.99865}, \\ \forall i, = 1, 2, \ldots, n \end{pmatrix} = 0.9973 \tag{2.0.4}$$

Taking $\dfrac{\overline{x}}{1.5708}, \dfrac{\overline{x}}{1.0472}, \dfrac{\overline{x}}{0.7854}$ and $\dfrac{\overline{x}}{0.6283}$ as an unbiased estimates of $\sigma$ when b = 2, b = 3, b = 4 and b = 5, respectively, the equation becomes

$$P\begin{pmatrix} D_3^* \overline{x} < x_i < D_4^* Z_{(n)0.99865}, \\ \forall i, = 1, 2, \ldots, n \end{pmatrix} = 0.9973 \tag{2.0.5}$$

where, for b = 2:

$$D_3^* = \frac{Z(1)(0.00135)}{1.5708}$$

and

$$D_4^* = \frac{Z(n)(0.99865)}{1.5708}.$$

For b = 3:

$$D_3^* = \frac{Z(1)(0.00135)}{1.0472}$$

and

$$D_4^* = \frac{Z(n)(0.99865)}{1.0472}.$$

For b = 4:

$$D_3^* = \frac{Z(1)(0.00135)}{0.7854}$$

and

$$D_4^* = \frac{Z(n)(0.99865)}{0.7854}.$$

For b = 5:

$$D_3^* = \frac{Z(1)(0.00135)}{0.6283}$$

and

$$D_4^* = \frac{Z(n)(0.99865)}{0.6283}.$$

Thus, $D_3^*$ and $D_4^*$ constitute the control chart constants for the extreme value charts (see Table 2.2 for n = 2(1)10).

Table 2.1: Control Chart Limits of Extreme Value Charts

| n | b=2 | | b=3 | | b=4 | | b=5 | |
|---|---|---|---|---|---|---|---|---|
| | $Z_*$ | $Z_{**}$ | $Z_*$ | $Z_{**}$ | $Z_*$ | $Z_{**}$ | $Z_*$ | $Z_{**}$ |
| 2 | 0.0259 | 38.4705 | 0.0877 | 11.3959 | 0.1612 | 6.2024 | 0.2322 | 4.3057 |
| 3 | 0.0212 | 47.1192 | 0.0766 | 13.0456 | 0.1456 | 6.8643 | 0.2141 | 4.6695 |
| 4 | 0.0183 | 54.4101 | 0.0696 | 14.3588 | 0.1355 | 7.3763 | 0.2021 | 4.9462 |
| 5 | 0.0164 | 60.8334 | 0.0646 | 15.4677 | 0.1282 | 7.7995 | 0.1933 | 5.1719 |
| 6 | 0.0150 | 66.6404 | 0.0608 | 16.4371 | 0.1225 | 8.1633 | 0.1864 | 5.3640 |
| 7 | 0.0138 | 71.9804 | 0.0577 | 17.3038 | 0.1178 | 8.4841 | 0.1807 | 5.5320 |
| 8 | 0.0129 | 76.9508 | 0.0552 | 18.0915 | 0.1139 | 8.7721 | 0.1760 | 5.6817 |
| 9 | 0.0122 | 81.6190 | 0.0531 | 18.8160 | 0.1106 | 9.0343 | 0.1719 | 5.8172 |
| 10 | 0.0116 | 86.0343 | 0.0513 | 19.4886 | 0.1078 | 9.2754 | 0.1683 | 5.9411 |

Analysis of Means (ANOM): Log Logistic Distribution

When the data variate follows log logistic distribution, suppose $\bar{x}_1$, $\bar{x}_2, \ldots,$ $\bar{x}_k$ are arithmetic means of $k$ subgroups of size $n$ drawn from a log logistic model. The subgroups means are used to develop control charts to assess whether the population from which these subgroups are drawn is operating with admissible quality variations. Depending on the basic population model, control chart constants may be used. In general, the process may be said to be in control if all subgroup means are within the control limits; otherwise the process is said to lack control. If α is the level of significance of this decision, the following probability statements apply:

$$P\left(LCL < x_i, \forall i, = 1, 2, \ldots, k < UCL\right) = 1 - \alpha \tag{3.0.6}$$

using the notion of independent subgroups, (3.0.6) becomes

$$P\left(LCL < x_i < UCL\right) = \left(1 - \alpha\right)^{1/k} \tag{3.0.7}$$

With equi-tailed probability for each subgroup mean, two constants, for example $L^*$ and $U^*$, may be found such that

$$P\left(x_i < L^*\right) = P\left(\bar{x}_i > U^*\right) = \frac{1 - \left(1 - \alpha\right)^{1/k}}{2} \tag{3.0.8}$$

For skewed populations, such as the LLD, it is necessary to calculate $L^*$, $U^*$ separately from the sampling distribution of $\bar{x}_i$. Accordingly, these depend on the subgroup size $n$ and number of subgroups $k$. The percentiles of the sampling distribution of $\bar{x}$ in samples from a log logistic distribution for b = 2, b = 3, b = 4 and b = 5 with σ = 1 were calculated using Monte-Carlo simulations (see Tables 3.1, 3.2, 3.3 and 3.4).

Table 2.2: Control Chart Limits of Extreme Value Charts

| n | b = 2 | | b = 3 | | b = 4 | | b = 5 | |
|---|---|---|---|---|---|---|---|---|
| | $D_3^*$ | $D_4^*$ | $D_3^*$ | $D_4^*$ | $D_3^*$ | $D_4^*$ | $D_3^*$ | $D_4^*$ |
| 2 | 0.0165 | 24.4910 | 0.0837 | 10.8823 | 0.2052 | 7.8971 | 0.3695 | 6.8529 |
| 3 | 0.0135 | 29.9969 | 0.0731 | 12.4576 | 0.1854 | 8.7398 | 0.3423 | 7.4319 |
| 4 | 0.0116 | 34.6384 | 0.0664 | 13.7116 | 0.1726 | 9.3917 | 0.3216 | 7.8723 |
| 5 | 0.0104 | 38.7276 | 0.0617 | 14.7705 | 0.1632 | 9.9306 | 0.3076 | 8.2315 |
| 6 | 0.0095 | 42.4246 | 0.0580 | 15.6962 | 0.1559 | 10.3938 | 0.2966 | 8.5373 |
| 7 | 0.0088 | 45.8240 | 0.0551 | 16.5238 | 0.1500 | 10.8022 | 0.2876 | 8.8047 |
| 8 | 0.0082 | 48.9882 | 0.0527 | 17.2760 | 0.1450 | 11.1690 | 0.2801 | 9.0429 |
| 9 | 0.0078 | 51.9601 | 0.0507 | 17.9679 | 0.1408 | 11.5028 | 0.2735 | 9.2586 |
| 10 | 0.0074 | 54.7710 | 0.0489 | 18.6102 | 0.1372 | 11.8098 | 0.2678 | 9.4558 |

Table 3.1: Percentiles of Sample Mean in LLD with b = 2

| n | 0.99865 | 0.99 | 0.975 | 0.95 | 0.05 | 0.025 | 0.01 | 0.00135 |
|---|---|---|---|---|---|---|---|---|
| 2 | 12.9560 | 4.8648 | 3.2368 | 2.363 | 0.2646 | 0.2144 | 0.1624 | 0.1002 |
| 3 | 11.3630 | 4.4994 | 3.0711 | 2.3002 | 0.3390 | 0.2887 | 0.2368 | 0.1772 |
| 4 | 10.5971 | 3.8004 | 2.7757 | 2.0875 | 0.3848 | 0.3313 | 0.2767 | 0.2002 |
| 5 | 9.8124 | 3.5276 | 2.5244 | 1.9696 | 0.4228 | 0.3709 | 0.3180 | 0.2237 |
| 6 | 9.6424 | 3.47156 | 2.4932 | 1.9646 | 0.4518 | 0.4008 | 0.3446 | 0.2658 |
| 7 | 7.0472 | 3.2050 | 2.3660 | 1.9001 | 0.4692 | 0.4176 | 0.3662 | 0.2896 |
| 8 | 6.5408 | 2.9775 | 2.2500 | 1.8469 | 0.4965 | 0.4435 | 0.3914 | 0.3202 |
| 9 | 5.8094 | 2.8794 | 2.2181 | 1.8242 | 0.5144 | 0.4681 | 0.4201 | 0.3434 |
| 10 | 6.1672 | 2.7692 | 2.2176 | 1.8063 | 0.5284 | 0.4827 | 0.4331 | 0.3635 |

Table 3.2: Percentiles of Sample Mean in LLD with b = 3

| n | 0.99865 | 0.99 | 0.975 | 0.95 | 0.05 | 0.025 | 0.01 | 0.00135 |
|---|---|---|---|---|---|---|---|---|
| 2 | 4.8927 | 3.1605 | 2.6322 | 2.2986 | 0.7921 | 0.7115 | 0.6210 | 0.4927 |
| 3 | 4.3935 | 2.7927 | 2.4511 | 2.1816 | 0.8913 | 0.8211 | 0.7476 | 0.6406 |
| 4 | 3.3256 | 2.6008 | 2.2806 | 2.0491 | 0.9423 | 0.8750 | 0.8007 | 0.6739 |
| 5 | 3.1616 | 2.4128 | 2.1578 | 1.9784 | 0.9857 | 0.9256 | 0.8598 | 0.7290 |
| 6 | 3.2322 | 2.3737 | 2.1150 | 1.9448 | 1.0203 | 0.9629 | 0.8933 | 0.7799 |
| 7 | 2.9030 | 2.2476 | 2.0506 | 1.9071 | 1.0350 | 0.9802 | 0.9162 | 0.8115 |
| 8 | 2.7292 | 2.1760 | 2.0073 | 1.8728 | 1.0608 | 1.0041 | 0.9447 | 0.8522 |
| 9 | 2.5742 | 2.1361 | 1.9732 | 1.8443 | 1.0779 | 1.0272 | 0.9693 | 0.8833 |
| 10 | 2.5429 | 2.1235 | 1.9520 | 1.8268 | 1.0961 | 1.0490 | 0.9930 | 0.9017 |

Table 3.3: Percentiles of Sample Mean in LLD with b = 4

| n | 0.99865 | 0.99 | 0.975 | 0.95 | 0.05 | 0.025 | 0.01 | 0.00135 |
|---|---|---|---|---|---|---|---|---|
| 2 | 6.2330 | 3.4078 | 2.6332 | 2.1740 | 0.5154 | 0.4459 | 0.3739 | 0.2719 |
| 3 | 5.8539 | 2.9913 | 2.4142 | 2.0571 | 0.6056 | 0.5447 | 0.4763 | 0.3970 |
| 4 | 4.0852 | 2.6589 | 1.2396 | 1.8961 | 0.6549 | 0.5949 | 0.5293 | 0.4205 |
| 5 | 4.0895 | 2.4914 | 2.0797 | 1.8209 | 0.6973 | 0.6402 | 0.5764 | 0.4649 |
| 6 | 3.9497 | 2.4311 | 2.0150 | 1.7868 | 0.7278 | 0.6728 | 0.6120 | 0.5129 |
| 7 | 3.3344 | 2.2616 | 1.9481 | 1.7396 | 0.7435 | 0.6919 | 0.6317 | 0.5374 |
| 8 | 3.1361 | 2.1661 | 1.8742 | 1.7040 | 0.7702 | 0.7172 | 0.6603 | 0.5799 |
| 9 | 2.8196 | 2.0899 | 1.8578 | 1.6766 | 0.7869 | 0.7402 | 0.6875 | 0.6204 |
| 10 | 2.9277 | 2.1209 | 1.8353 | 1.6523 | 0.8039 | 0.7567 | 0.7036 | 0.6275 |

Table 3.4: Percentiles of Sample Mean in LLD with b = 5

| n | 0.99865 | 0.99 | 0.975 | 0.95 | 0.05 | 0.025 | 0.01 | 0.00135 |
|---|---|---|---|---|---|---|---|---|
| 2 | 4.5322 | 3.2326 | 2.8121 | 2.5223 | 1.0811 | 0.9880 | 0.8909 | 0.7344 |
| 3 | 4.1151 | 2.9163 | 2.6339 | 2.4091 | 1.1882 | 1.1093 | 1.0315 | 0.9027 |
| 4 | 3.3010 | 2.7435 | 2.4671 | 2.2876 | 1.2398 | 1.1642 | 1.0868 | 0.9514 |
| 5 | 3.0996 | 2.5609 | 2.3640 | 2.2206 | 1.2812 | 1.2195 | 1.1485 | 1.0053 |
| 6 | 2.9154 | 2.4838 | 2.3201 | 2.1834 | 1.3056 | 1.2545 | 1.1927 | 1.0291 |
| 7 | 2.8736 | 2.4256 | 2.2716 | 2.1508 | 1.3344 | 1.2759 | 1.2103 | 1.0960 |
| 8 | 2.7778 | 2.3577 | 2.2310 | 2.1206 | 1.3587 | 1.3036 | 1.2420 | 1.1449 |
| 9 | 2.6438 | 2.3260 | 2.1990 | 2.0882 | 1.3776 | 1.3254 | 1.2603 | 1.1749 |
| 10 | 2.6299 | 2.3118 | 2.1803 | 2.0749 | 1.3973 | 1.3463 | 1.2899 | 1.1963 |

The percentiles shown in Tables 3.1 – 3.4 are used in equation (3.0.8) for specified *n* and *k* to determine $L^*$ and $U^*$ for $\alpha = 0.05$ (see Tables 3.5, 3.6, 3.7 and 3.8). A control chart for averages showing in control conclusions indicates that all subgroups means, though varying among themselves, are homogenous in some cells. This is the null hypothesis in an analysis of variance technique, hence, the constants shown in tables 3.5 - 3.8 can be used as an alternative to analysis of variance techniques. For a normal population Ott's (1967) tables can be used, and for a LLD the tables shown herein can be used.

Table 3.5: LLD Constants for Analysis of Means for b = 2, (1−α) = 0.95

| *k* | *n* = 2 | | 3 | | 4 | | 5 | | 6 | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.3303 | 5.1477 | 0.4539 | 4.7591 | 0.5212 | 4.3511 | 0.5768 | 4.0660 | 0.6266 | 3.9113 |
| 2 | 0.2639 | 7.0248 | 0.9864 | 6.6701 | 0.4599 | 5.8876 | 0.5161 | 5.4263 | 0.5798 | 4.9528 |
| 3 | 0.2365 | 8.9046 | 0.3569 | 7.7521 | 0.4292 | 6.9287 | 0.4819 | 6.2444 | 0.5517 | 5.7732 |
| 4 | 0.2192 | 10.4238 | 0.3355 | 8.9742 | 0.4080 | 7.7136 | 0.4591 | 7.0343 | 0.5307 | 6.3341 |
| 5 | 0.2068 | 11.5178 | 0.3169 | 9.5190 | 0.3742 | 8.4092 | 0.4451 | 7.4928 | 0.5140 | 7.3627 |
| 6 | 0.2015 | 12.4982 | 0.3054 | 10.1839 | 0.3697 | 9.1787 | 0.4303 | 8.0430 | 0.4153 | 7.5537 |
| 7 | 0.1972 | 12.9499 | 0.3007 | 10.8149 | 0.3592 | 9.9402 | 0.4199 | 8.7234 | 0.4779 | 7.8955 |
| 8 | 0.1890 | 13.4191 | 0.2950 | 11.0937 | 0.3544 | 10.4101 | 0.4131 | 8.8267 | 0.4708 | 8.3140 |
| 9 | 0.1866 | 13.9979 | 0.2931 | 12.1805 | 0.3532 | 10.8970 | 0.4056 | 9.1813 | 0.4665 | 8.8593 |
| 10 | 0.1716 | 14.6854 | 0.2626 | 12.4166 | 0.3374 | 11.0593 | 0.3878 | 9.6395 | 0.4592 | 9.0119 |
| 20 | 0.1331 | 17.9113 | 0.2247 | 16.9479 | 0.3130 | 14.8274 | 0.3423 | 12.3236 | 0.4244 | 11.9848 |
| 30 | 0.1028 | 23.7632 | 0.2097 | 20.7346 | 0.3016 | 18.0296 | 0.3300 | 14.8560 | 0.3885 | 13.4620 |
| 40 | 0.0946 | 33.6935 | 0.1780 | 24.0424 | 0.2751 | 19.7351 | 0.3118 | 16.4754 | 0.3687 | 16.1631 |
| 50 | 0.0929 | 34.7754 | 0.1686 | 25.1053 | 0.2745 | 22.5189 | 0.3088 | 18.1307 | 0.3637 | 16.4374 |

| *k* | *n* = 7 | | 8 | | 9 | | 10 | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.6592 | 3.6837 | 0.6945 | 3.6131 | 0.7253 | 3.5156 | 0.7536 | 3.4029 |
| 2 | 0.5999 | 4.6640 | 0.6423 | 4.5937 | 0.6657 | 4.4740 | 0.6873 | 4.2973 |
| 3 | 0.5712 | 5.4824 | 0.6103 | 5.2898 | 0.6424 | 5.1286 | 0.6513 | 4.8740 |
| 4 | 0.5561 | 5.9729 | 0.5949 | 5.7324 | 0.6134 | 5.7535 | 0.6309 | 5.0412 |
| 5 | 0.5378 | 6.7590 | 0.5838 | 6.2880 | 0.6025 | 6.2461 | 0.6199 | 5.8142 |
| 6 | 0.5280 | 7.1752 | 0.5682 | 6.9044 | 0.5905 | 6.6190 | 0.6140 | 6.0589 |
| 7 | 0.5243 | 7.7579 | 0.5596 | 7.2190 | 0.5833 | 6.9793 | 0.5962 | 6.3483 |
| 8 | 0.5224 | 7.9220 | 0.5499 | 7.6895 | 0.5821 | 7.2779 | 0.5916 | 6.6536 |
| 9 | 0.5213 | 8.6406 | 0.5493 | 7.8990 | 0.5800 | 7.6447 | 0.5863 | 7.0156 |
| 10 | 0.5022 | 8.6938 | 0.5437 | 8.0013 | 0.5672 | 7.7415 | 0.5752 | 7.0463 |
| 20 | 0.4771 | 11.6854 | 0.5106 | 10.6656 | 0.5192 | 9.8981 | 0.5427 | 9.9847 |
| 30 | 0.4510 | 13.1287 | 0.4994 | 12.2854 | 0.5059 | 11.3192 | 0.5237 | 10.3004 |
| 40 | 0.4249 | 14.5202 | 0.4815 | 14.1756 | 0.4751 | 13.1701 | 0.4990 | 12.7299 |
| 50 | 0.4242 | 16.1104 | 0.4777 | 14.5343 | 0.4677 | 13.2737 | 0.4972 | 13.4114 |

Table 3.6: LLD Constants for Analysis of Means for b = 3, (1−α) = 0.95

| *k* | *n* | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | | 3 | | 4 | | 5 | | 6 | |
| 1 | 0.4695 | 2.7584 | 0.5683 | 2.5524 | 0.6248 | 2.3464 | 0.6666 | 2.2172 | 0.6988 | 2.1238 |
| 2 | 0.4059 | 3.3751 | 0.5101 | 3.0785 | 0.5720 | 2.7479 | 0.6177 | 2.5241 | 0.6660 | 2.3903 |
| 3 | 0.3733 | 3.8157 | 0.4875 | 3.3458 | 0.5469 | 2.9638 | 0.5929 | 2.8291 | 0.6448 | 2.5548 |
| 4 | 0.3593 | 4.1241 | 0.4584 | 3.6189 | 0.5203 | 3.0344 | 0.5695 | 2.9484 | 0.6255 | 2.6804 |
| 5 | 0.3427 | 4.5531 | 0.4451 | 3.8694 | 0.5063 | 3.4598 | 0.5526 | 0.0452 | 0.6095 | 2.8544 |
| 6 | 0.3388 | 4.7973 | 0.4414 | 3.9541 | 0.4956 | 3.5578 | 0.5437 | 3.1117 | 0.5999 | 2.9205 |
| 7 | 0.3368 | 4.9647 | 0.4295 | 4.0572 | 0.4881 | 3.6225 | 0.5349 | 3.2163 | 0.5868 | 3.0714 |
| 8 | 0.3145 | 5.2882 | 0.4167 | 4.1558 | 0.4787 | 3.6740 | 0.5208 | 3.3446 | 0.5772 | 3.1013 |
| 9 | 0.3136 | 5.3570 | 0.4015 | 4.2656 | 0.4700 | 3.7688 | 0.5152 | 3.3985 | 0.5730 | 3.2142 |
| 10 | 0.3099 | 5.4002 | 0.3986 | 4.3335 | 0.4645 | 3.8646 | 0.5135 | 3.4171 | 0.5712 | 3.2412 |
| 20 | 0.2624 | 6.1439 | 0.3477 | 5.6020 | 0.4415 | 4.6598 | 0.4769 | 4.1357 | 0.5375 | 3.6726 |
| 30 | 0.2455 | 8.6207 | 0.3273 | 6.1971 | 0.4184 | 5.0747 | 0.4486 | 4.4602 | 0.5233 | 4.0896 |
| 40 | 0.2144 | 8.7280 | 0.2993 | 6.3545 | 0.4108 | 5.4735 | 0.4358 | 4.5999 | 0.5046 | 4.3879 |
| 50 | 0.2044 | 9.0887 | 0.2835 | 6.6497 | 0.3955 | 5.7045 | 0.4300 | 4.6674 | 0.5240 | 4.4110 |

| *k* | *n* | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 7 | | 8 | | 9 | | 10 | |
| 1 | 0.7261 | 2.0437 | 0.7474 | 1.9820 | 0.7730 | 1.9419 | 0.7902 | 1.9098 |
| 2 | 0.6783 | 2.2991 | 0.7126 | 2.2268 | 0.7322 | 2.1810 | 0.7442 | 2.1333 |
| 3 | 0.6559 | 2.4746 | 0.6947 | 2.3405 | 0.7071 | 2.3472 | 0.7205 | 2.2488 |
| 4 | 0.6404 | 2.5873 | 0.6761 | 2.4314 | 0.6957 | 2.4292 | 0.7071 | 2.3594 |
| 5 | 0.6281 | 2.6892 | 0.6663 | 2.8787 | 0.6826 | 2.5798 | 0.6962 | 2.4315 |
| 6 | 0.6225 | 2.8124 | 0.6627 | 2.6851 | 0.6744 | 2.6859 | 0.6848 | 2.5297 |
| 7 | 0.6145 | 2.8789 | 0.6419 | 2.7863 | 0.6701 | 2.7588 | 0.6824 | 2.5859 |
| 8 | 0.6110 | 3.0330 | 0.6362 | 2.8695 | 0.6638 | 2.7963 | 0.6731 | 2.6091 |
| 9 | 0.6087 | 3.1342 | 0.6344 | 2.9247 | 0.6527 | 2.8245 | 0.6703 | 2.6647 |
| 10 | 0.6048 | 3.1402 | 0.6317 | 2.9568 | 0.6522 | 2.8585 | 0.6652 | 2.7014 |
| 20 | 0.5782 | 3.6224 | 0.6163 | 3.2793 | 0.6165 | 3.1551 | 0.6323 | 2.9422 |
| 30 | 0.5664 | 3.9450 | 0.6036 | 3.6797 | 0.6092 | 3.6316 | 0.6161 | 3.3713 |
| 40 | 0.5332 | 4.0987 | 0.5951 | 3.8300 | 0.5934 | 3.5372 | 0.6112 | 3.3857 |
| 50 | 0.5277 | 4.2365 | 0.5902 | 3.9501 | 0.5824 | 3.6456 | 0.6035 | 3.4009 |

Table 3.7: LLD Constants for Analysis of Means for b = 4, (1−α) = 0.95

| *k* | *n* | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | | 3 | | 4 | | 5 | | 6 | |
| 1 | 0.5538 | 2.0708 | 0.6462 | 1.9460 | 0.6916 | 1.8023 | 0.7255 | 1.7171 | 0.7522 | 1.6570 |
| 2 | 0.4998 | 2.3840 | 0.5967 | 2.1913 | 0.6472 | 1.9817 | 0.6821 | 1.8793 | 0.7205 | 1.8782 |
| 3 | 0.4692 | 2.5861 | 0.5711 | 2.3067 | 0.6235 | 2.0966 | 0.6641 | 1.9984 | 0.7016 | 1.8626 |
| 4 | 0.4501 | 2.7621 | 0.5495 | 2.4355 | 0.6032 | 2.2188 | 0.6423 | 2.0633 | 0.6858 | 1.9325 |
| 5 | 0.4424 | 2.9285 | 0.5382 | 2.5761 | 0.5927 | 2.3001 | 0.6290 | 2.1164 | 0.6741 | 1.9779 |
| 6 | 0.4363 | 2.9917 | 0.5261 | 2.6173 | 0.5881 | 2.3841 | 0.6182 | 2.1564 | 0.6696 | 2.0239 |
| 7 | 0.4280 | 3.0573 | 0.5194 | 2.6437 | 0.5769 | 2.4234 | 0.6101 | 2.1760 | 0.6782 | 2.0083 |
| 8 | 0.4122 | 3.1024 | 0.5090 | 2.6733 | 0.5636 | 2.4560 | 0.6063 | 2.2113 | 0.6521 | 2.0791 |
| 9 | 0.4077 | 3.1713 | 0.5000 | 2.7301 | 0.5562 | 2.4843 | 0.5925 | 2.2169 | 0.6491 | 2.1297 |
| 10 | 0.4024 | 3.2275 | 0.4962 | 2.7719 | 0.5518 | 2.5069 | 0.5880 | 2.2393 | 0.6472 | 2.1553 |
| 20 | 0.3486 | 3.5220 | 0.4395 | 3.2248 | 0.5232 | 2.7649 | 0.5633 | 2.4762 | 0.6228 | 2.3591 |
| 30 | 0.3057 | 4.0758 | 0.4203 | 0.5425 | 0.5083 | .3.0566 | 0.5348 | 2.5192 | 0.5973 | 2.3924 |
| 40 | 0.2952 | 4.3930 | 0.3891 | 3.6392 | 0.4958 | 3.1190 | 0.5131 | 2.6812 | 0.5904 | 2.5104 |
| 50 | 0.2793 | 4.4353 | 0.3874 | 3.7004 | 0.4843 | 3.1396 | 0.5121 | 2.7258 | 0.5870 | 2.5539 |

| *k* | *n* | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 7 | | 8 | | 9 | | 10 | |
| 1 | 0.7698 | 1.6077 | 0.7891 | 1.5732 | 0.8069 | 1.5495 | 0.8201 | 1.5341 |
| 2 | 0.7335 | 1.7229 | 0.7605 | 1.6883 | 0.7728 | 1.6479 | 0.7835 | 1.6446 |
| 3 | 0.7123 | 1.8107 | 0.7444 | 1.7368 | 0.7550 | 1.7385 | 0.7666 | 1.6908 |
| 4 | 0.7018 | 1.8610 | 0.7310 | 1.5922 | 0.7477 | 1.7852 | 0.7558 | 0.7328 |
| 5 | 0.6192 | 1.9104 | 0.7250 | 1.8373 | 0.7378 | 1.8349 | 0.7467 | 1.7653 |
| 6 | 0.6841 | 1.5945 | 0.7181 | 1.8821 | 0.7316 | 1.8768 | 0.7398 | 1.7819 |
| 7 | 0.7104 | 1.9209 | 0.7298 | 1.9051 | 0.7338 | 1.8111 | 0.7552 | 1.5561 |
| 8 | 0.6744 | 2.0342 | 0.6980 | 1.9459 | 0.7190 | 1.9222 | 0.7306 | 1.8189 |
| 9 | 0.6711 | 2.0588 | 0.6964 | 1.9618 | 0.7152 | 1.9513 | 0.7281 | 1.8303 |
| 10 | 0.6693 | 2.0766 | 0.6943 | 1.9997 | 0.7093 | 1.9538 | 0.7224 | 1.8422 |
| 20 | 0.6504 | 2.2074 | 0.6809 | 2.1062 | 0.6810 | 2.0959 | 0.6985 | 1.9428 |
| 30 | 0.6345 | 2.3967 | 0.6672 | 2.2398 | 0.6731 | 2.1668 | 0.6848 | 2.0243 |
| 40 | 0.6109 | 2.4582 | 0.6621 | 2.3577 | 0.6608 | 2.1886 | 0.6739 | 2.0624 |
| 50 | 0.6092 | 2.5502 | 0.6585 | 2.3803 | 0.6519 | 2.2144 | 0.6729 | 2.0980 |

Table 3.8: LLD Constants for Analysis of Means for b = 5, (1−α) = 0.95

| *k* | *n* = 2 | | 3 | | 4 | | 5 | | 6 | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.6183 | 1.7596 | 0.7003 | 1.6595 | 0.7380 | 1.5590 | 0.7661 | 1.5036 | 0.7871 | 1.4560 |
| 2 | 0.5683 | 1.9464 | 0.6561 | 1.8041 | 0.6999 | 1.6716 | 0.7287 | 1.6030 | 0.7602 | 1.5422 |
| 3 | 0.5418 | 2.0700 | 0.6306 | 1.8956 | 0.6762 | 1.7528 | 0.7123 | 1.6647 | 0.7428 | 1.5898 |
| 4 | 0.5224 | 2.1907 | 0.6141 | 1.9736 | 0.6657 | 1.8008 | 0.6928 | 1.7119 | 0.7313 | 1.6292 |
| 5 | 0.5175 | 2.2863 | 0.6031 | 2.0374 | 0.6534 | 1.8694 | 0.6816 | 1.7496 | 0.7201 | 1.6586 |
| 6 | 0.5076 | 2.3282 | 0.5957 | 2.0855 | 0.6480 | 1.9026 | 0.6732 | 1.7694 | 0.7145 | 1.6720 |
| 7 | 0.5044 | 2.3661 | 0.5802 | 2.1047 | 0.6395 | 1.9335 | 0.6691 | 1.7829 | 0.7093 | 1.6910 |
| 8 | 0.4913 | 2.4061 | 0.5747 | 2.1234 | 0.6248 | 1.9751 | 0.6587 | 1.7873 | 0.7057 | 1.7134 |
| 9 | 0.4858 | 2.4285 | 0.5708 | 2.1327 | 0.6195 | 2.0120 | 0.6550 | 1.8075 | 0.7002 | 1.7293 |
| 10 | 0.4778 | 2.4406 | 0.5634 | 2.1710 | 0.6175 | 2.0183 | 0.6414 | 1.8256 | 0.6942 | 1.7310 |
| 20 | 0.4292 | 2.6741 | 0.5022 | 2.3900 | 0.5846 | 2.1100 | 0.6218 | 1.9631 | 0.6724 | 1.8581 |
| 30 | 0.3870 | 2.9332 | 0.4963 | 2.6166 | 0.5771 | 2.2720 | 0.5977 | 2.0171 | 0.6607 | 1.9060 |
| 40 | 0.3741 | 2.9979 | 0.4683 | 2.6941 | 0.5573 | 2.3353 | 0.5760 | 2.0587 | 0.6469 | 1.9297 |
| 50 | 0.3563 | 3.0159 | 0.4598 | 2.7000 | 0.5521 | 2.3364 | 0.5729 | 2.0806 | 0.6426 | 1.9329 |

| *k* | *n* = 7 | | 8 | | 9 | | 10 | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.8032 | 1.4224 | 0.8184 | 1.3976 | 0.8342 | 1.3850 | 0.8433 | 1.3707 |
| 2 | 0.7721 | 1.4942 | 0.7943 | 1.4671 | 0.8050 | 1.4502 | 0.8148 | 1.4294 |
| 3 | 0.7546 | 1.5472 | 0.7805 | 1.5109 | 0.7900 | 1.4973 | 0.8000 | 1.4701 |
| 4 | 0.7436 | 1.5846 | 0.7699 | 1.5274 | 0.7838 | 1.5264 | 0.7906 | 1.4873 |
| 5 | 0.7366 | 1.6084 | 0.7647 | 1.5548 | 0.7760 | 1.5550 | 0.7837 | 1.5172 |
| 6 | 0.7297 | 1.6367 | 0.7592 | 1.5695 | 0.7694 | 1.5742 | 0.7753 | 1.5216 |
| 7 | 0.7256 | 1.6484 | 0.7497 | 1.5892 | 0.7669 | 1.5986 | 0.7737 | 1.5360 |
| 8 | 0.7213 | 1.6674 | 0.7437 | 1.6120 | 0.7580 | 1.6170 | 0.7684 | 1.5413 |
| 9 | 0.7154 | 1.6722 | 0.7402 | 1.6340 | 0.7507 | 1.6787 | 0.7655 | 1.5481 |
| 10 | 0.7131 | 1.6751 | 0.7396 | 1.6457 | 0.7500 | 1.6457 | 0.7616 | 1.5503 |
| 20 | 0.6973 | 1.7975 | 0.7269 | 1.7162 | 0.7264 | 1.6930 | 0.7450 | 1.6237 |
| 30 | 0.6866 | 1.8671 | 0.7179 | 1.8126 | 0.7159 | 1.7138 | 0.7351 | 1.6402 |
| 40 | 0.6674 | 1.9378 | 0.7092 | 1.8794 | 0.7115 | 1.7430 | 0.7326 | 1.6644 |
| 50 | 0.6673 | 1.9386 | 0.7085 | 1.8884 | 0.7012 | 1.7599 | 0.7163 | 1.6892 |

Example 1 (Wadsworth, 1986)

The following 25 observations are from a metal product manufacturing site. Variations in iron content were suspected in raw material supplied by 5 different suppliers. Five ingots were randomly selected from each of the suppliers. The following table contains the iron determinations for each ingot by weight from each of the 5 suppliers.

| | Supplier | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| Ingot Iron Content (g) | 3.46 | 3.59 | 3.51 | 3.38 | 3.29 |
| | 3.48 | 3.46 | 3.64 | 3.4 | 3.46 |
| | 3.56 | 3.42 | 3.46 | 3.37 | 3.37 |
| | 3.39 | 3.49 | 3.52 | 3.46 | 3.32 |
| | 3.4 | 3.5 | 3.49 | 3.39 | 3.38 |

Example 2

In a study of 3 brands of batteries, it was suspected that the life (in weeks) of the three brands was different. Five of each brand of battery were tested with the following results:

| Battery Life (weeks) | | |
|---|---|---|
| Brand 1 | Brand 2 | Brand 3 |
| 100 | 76 | 108 |
| 96 | 80 | 100 |
| 92 | 75 | 96 |
| 96 | 84 | 98 |
| 92 | 82 | 100 |

Example 3

Four catalysts that may affect the concentration of a component in a three component liquid mixture were investigated. The following concentrations were obtained:

| Catalyst | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 58.2 | 56.3 | 50.1 | 52.9 |
| 57.2 | 54.5 | 54.2 | 49.9 |
| 58.4 | 57.0 | 55.4 | 50.0 |
| 55.8 | 55.3 | 54.9 | 51.7 |

The goodness of fit of data, as revealed by a Q-Q plot (correlation coefficient), for the 3 examples are summarized Table 3.9, which shows that the log logistic distribution is a better model than the normal because it exhibits a significant linear relation between sample and population quantiles.

Table 3.9: Goodness of Fit Data from Q-Q Plot

| Example | b | LLD | Normal |
|---|---|---|---|
| 1 | 2 | 0.9306 | 0.2067 |
| | 3 | 0.9673 | |
| | 4 | 0.9801 | |
| | 5 | 0.9854 | |
| 2 | 2 | 0.8484 | 0.4149 |
| | 3 | 0.8986 | |
| | 4 | 0.9206 | |
| | 5 | 0.9324 | |
| 3 | 2 | 0.8424 | 0.2067 |
| | 3 | 0.8981 | |
| | 4 | 0.9223 | |
| | 5 | 0.9351 | |

Treating the observations in the data as single samples, the decision limits for the normal and the LLD populations were calculated and are shown in Tables 3.10 and 3.11 respectively.

Table 3.10: Normal Distribution

| Examples | [LDL, UDL] (Ott, 1967) | Number of Counts | | | |
|---|---|---|---|---|---|
| | | In | P = in/k | Out | Out/k |
| 1: n=5, k=5, α=0.05 | [3.517, 3.879] | 3 | 0.6 | 2 | 0.4 |
| 2: n=5, k=3, α=0.05 | [87.82, 95.52] | 2 | 0.7 | 1 | 0.3 |
| 3: n=4, k=4, α=0.05 | [26.14, 82.84] | 2 | 0.5 | 2 | 0.5 |

Table 3.11: Log Logistic Distribution

| Examples | [LDL, UDL] | Number of Counts | | | |
|---|---|---|---|---|---|
| | | In | P=in/k | Out | Out/k |
| b = 2 | | | | | |
| 1<br>n=5, k=5, α=0.05 | [1.5345, 25.8322] | 5 | 1 | 0 | 0 |
| 2<br>n=5, k=3, α=0.05 | [44.1805, 572.4044] | 3 | 1 | 0 | 0 |
| 3<br>n=4, k=4, α=0.05 | [22.2330, 420.2953] | 4 | 1 | 0 | 0 |
| b = 2 | | | | | |
| 1<br>n=5, k=5, α=0.05 | [1.9053, 10.4989] | 5 | 1 | 0 | 0 |
| 2<br>n=5, k=3, α=0.05 | [54.3510, 259.3351] | 3 | 1 | 0 | 0 |
| 3<br>n=4, k=4, α=0.05 | [28.3547, 180.0506] | 4 | 1 | 0 | 0 |
| b = 3 | | | | | |
| 1<br>n=5, k=5, α=0.05 | [2.1685, 7.2966] | 5 | 1 | 0 | 0 |
| 2<br>n=5, k=3, α=0.05 | [60.8822, 183.1867] | 3 | 1 | 0 | 0 |
| 3<br>n=4, k=4, α=0.05 | [32.8685, 120.9012] | 4 | 1 | 0 | 0 |
| b = 5 | | | | | |
| 1<br>n=5, k=5, α=0.05 | [2.3499, 6.0319] | 5 | 1 | 0 | 0 |
| 2<br>n=5, k=3, α=0.05 | [65.2987, 152.5993] | 3 | 1 | 0 | 0 |
| 3<br>n=4, k=4, α=0.05 | [36.2766, 98.1243] | 4 | 1 | 0 | 0 |

## Conclusion

Ott's (1967) ANOM tables are designed for normal distributions, the number of homogenous means for each data set was 3, 2, 2, respectively, and those not homogeneous are 2, 1 and 2, respectively. When the ANOM tables of the proposed model (LLD) are used for the same data sets, the number of homogenous means are 5, 3 and 4, respectively, and they do not exhibit deviation from homogeneity for values of b = 2, b = 3,b = 4 and b = 5. Use of the normal model resulted in homogeneity for some means and deviation for other means, thus indicating possible rejection of those means. The rejection decision is valid if a normal distribution is a good fit for the data. However, by comparison, results show that the LLD is a better model than the normal. Results are supported by the Q-Q plot correlation coefficient for each data set with the normal and with the LLD. It is therefore assumed that more error is likely to be associated with the decision process when data are from a normal distribution, thus, making all the means homogenous using LLD (see Table 3.11) is recommended over using the normal-ANOM procedure.

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