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An Approximate Approach to the Economic Design of $\bar{x}$ Charts By Considering the Cost of Quality

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The selection of three parameters $\{h, k, n\}$ is necessary to design a $\bar{x}$ control chart. A cost model employing a Burr distribution is examined. Previously employed methods are refined and extended. A series of approximations are proposed that enable a rapid parameter selection. It is hoped that reducing the computational complexity of previous approaches will lead to wider utilization of $\bar{x}$ control charts.

Key words: Burr distribution, control charts, cost of quality, generalized charts.

Introduction

When designing a $\bar{x}$ control chart to monitor a process three parameters must be selected, sample size (n), interval between successive subgroups (h) and control limits (k). A cost model is employed to assist in the selection of these parameters (Duncan 1956, Alexander, et al. 1995, Chou, et al. 2000). (A summary of notations adopted, which closely follows that of Chou, et al. (2000), is listed in the Appendix.)

Chou, et al. (2000) did not fully describe their numerical approach: They used tables presented in Burr (1942) to estimate the parameters of the eponymous distribution given the first four central moments for a data set; the mean and standard deviation of this distribution corresponding to the parameters from tables presented in Burr (1942) were then introduced. Finally a computer program, which employs a grid search to minimize the cost model, is mentioned and finally key parameters $\{h, k, n\}$ are estimated. No technical details were provided of this procedure, for example, which interval was scanned and what step length was employed.

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In this study, neither Burr's (1942) tables nor a grid search is employed. Instead, numerical procedures are employed to address estimation of the parameters for the Burr distribution. The key estimation step is divided into three sub-problems, one for each parameter $\{h, k, n\}$, to make estimation more efficient and accurate. Chou, et als. (2000) original work presented two key equations and a related table (see equations (9) and (12) and Table 1). Although the equations are correct, their range of application is limited, as noted when the density function is defined in equation 5 of the original work; thus the equations are not strictly consistent with the results presented in their table. This shortcoming will be addressed, which is essentially notational. In addition a four-step procedure for parameter selection is developed and is further refined as a series of approximate methods to allow more rapid calculation of these parameters.

The Key Equations

The Burr (1942) distribution density function (1) is defined as

$$f(y) = \frac{qcy^{c-1}}{\left(1+y^c\right)^{q+1}} \tag{1}$$

for $y \geq 0$,

with cumulative density

$$F(y)=1-\frac{1}{\left(1+y^{c}\right)^{q}}, \text{ for } y \geq 0. \quad (2)$$

This is particularly useful in modeling data that does not follow a normal distribution; note that the argument (*y*) is non-negative. The Burr distribution density function is frequently used to model data that has a slight positive skewness, such as for the size of insurance claims. To avoid the problem of the argument straying outside the region of interest the Heaviside step function (3) is employed.

$$H(y)=1 \text{ for } y \geq 0, \text{ or } 0 \text{ otherwise.} \quad (3)$$

The expected cost (*E*, equation 4) is from equation 4 in Chou, et al. (2000). All parameters are defined in the Appendix and the elements of the equation (B, υ_1, υ_2, L_1 and L_2) are presented in (5), (6) and (7).

$$E=\frac{a_1+a_2 n}{h}+\frac{a_3\lambda+\frac{a_5\alpha}{h}+L_1 P+L_2 P\lambda B}{1+\lambda B}, \quad (4)$$

$$B=\left(\frac{1}{1-\beta}-\frac{1}{2}+\frac{\lambda h}{12}\right)h+D+gn, \quad (5)$$

$$\left.\begin{aligned} v_1&=\sigma \\ v_2&=\sigma\sqrt{1+\delta^2} \end{aligned}\right\} \quad (6)$$

$$L_i=\frac{A}{\Delta^2}v_i^2 \text{ for } i=1, 2. \quad (7)$$

The probabilities associated with Type I errors (α) and Type II errors (β) are of key interest. The Type I error, or producers risk, is defined in terms of the probability of deviation of the sample mean outside the upper and lower confidence limits (UCL, LCL). It is essential, at this stage, that the Heaviside function be associated with the density to correctly restrict its range. This was omitted, although tacitly assumed, in the original work.

$$\alpha=Pr\left(\bar{x}>UCL\right)+Pr\left(\bar{x}<LCL\right)$$

$$\alpha=Pr\left(y>M+kS\right)+Pr\left(y<M-kS\right)$$

$$\alpha=\begin{pmatrix}\left(1-F\left(M+kS\right)\right)H\left(M+kS\right)\\ +F\left(M-kS\right)H\left(M-kS\right)\end{pmatrix} \quad (8)$$

The Type II error, or consumer's risk, is defined in terms of the probability of deviation of the sample mean outside the upper and lower confidence limits (UCL, LCL) when the deviation from the target mean (δ) is known.

$$\beta=Pr\left(LCL\leq\bar{x}\leq UCL\mid\mu=T+\delta\sigma\right)$$

$$\beta=Pr\left(M-kS-S\delta\sqrt{n}\leq y\leq M+kS-S\delta\sqrt{n}\right)$$

$$\begin{aligned}\beta=&F\left(M+kS-S\delta\sqrt{n}\right)H\left(M+kS-S\delta\sqrt{n}\right)\\ &-F\left(M-kS-S\delta\sqrt{n}\right)H\left(M-kS-S\delta\sqrt{n}\right)\end{aligned} \quad (9)$$

Equations (8) and (9) may be combined with that for the cumulative density (2) into a more appropriate form than those previously presented (Chou, et al., 2000). A four-step procedure was developed to derive the parameters from equations (4), (8) and (9) and the steps described will be used consistently throughout this article; the numerical methodology will be also be discussed. Because these procedures are eliminated from the final approach, they are included as an indication of the massive computational effort associated with the problem; they are also necessary to indicate the reliability of the approximate methods employed.

Methodology

Step 1

Previous data is used to evaluate the skewness and kurtosis coefficients of the underlying process and these are then used to estimate the parameters $\{c, q\}$ of the corresponding Burr distribution. The moments of the distribution are estimated by numerical integration.

Step 2

Burr parameters $\{c, q\}$ are used to estimate the mean and standard deviation $\{M, S\}$ of the distribution by numerical integration and equation 8 is then employed to estimate k for given α.

Step 3

The available estimates $\{M, S, k\}$ and equation (9) are used to estimate n (for consumer's risk β), and absolute mean shift (δ) for an out of control process.

Step 4

The parameters, plus the estimates $\{k, n\}$ and the minimization of equation (4) are used to estimate *h*.

This completes the procedure; the three parameters $\{h, k, n\}$ for an ideal $\bar{x}$ control chart are now available. It is now appropriate to address the lengthy numerical procedures required to complete these steps.

Numerical Method

Step 1

The Powell (1970) hybrid method is used to simultaneously solve the equations for skewness and kurtosis. For each evaluation, four calls to a numerical integration procedure (Piessensl, 1983) are required to evaluate the first four moments, providing two function estimates.

Step 2

A procedure is required to find the zero of a continuous function introduced in Step 2; the method of Bus and Dekker (1975) is employed. This procedure incorporates interpolation, extrapolation plus bisection.

Step 3

To find the zero of a continuous function employed in Step 3; the method of Bus and Dekker (1975) is employed.

Step 4

Applying simple algebra this step reduces to finding the roots of a quartic equation. Prior to substituting *B* defined in equation (5), it is written as a quadratic function of *h*:

$$B = c_0 + c_1 h + c_2 h^2$$

$$\left.\begin{aligned} c_0 &= D + gn \\ c_1 &= \frac{1}{1-\beta} - \frac{1}{2} \\ c_2 &= \frac{\lambda}{12} \end{aligned}\right\} \qquad (10)$$

The expected cost (4) may also be represented as a function of *h*.

$$E = \frac{n_0 + n_1 h + n_2 h^2 + n_3 h^3}{d_0 + d_1 h + d_2 h^2} \qquad (11)$$

The numerator coefficients are

$$\left.\begin{aligned} n_0 &= (a_1 + a_2 n)(1 + \lambda c_0) + a_5 \alpha \\ n_1 &= (a_1 + a_2 n)\lambda c_1 + a_3 \lambda + L_1 P + L_2 \lambda P c_0 \\ n_2 &= (a_1 + a_2 n)\lambda c_2 + L_2 \lambda P c_1 \\ n_3 &= L_2 \lambda P c_2 \end{aligned}\right\} \qquad (12)$$

and the denominator coefficients are

$$\left.\begin{aligned} d_0 &= 1 + \lambda c_0 \\ d_1 &= \lambda c_1 \\ d_2 &= \lambda c_2 \end{aligned}\right\} \qquad (13)$$

To minimize the cost, a turning point is required. This corresponds to a zero in the numerator of the derivative of E (11) with respect to h. This derivative polynomial is denoted by dE.

$$dE = dE_0 + dE_1 h + dE_2 h^2 + dE_3 h^3 + dE_4 h^4 \quad (14)$$

with coefficients

$$\left.\begin{aligned} dE_0 &= -n_0 d_1 \\ dE_1 &= -2n_0 d_2 \\ dE_2 &= -n_1 d_2 - 3n_0 d_3 + n_1 d_2 \\ dE_3 &= -2n_1 d_3 + 2n_1 d_3 \\ dE_4 &= -n_2 d_3 + n_3 d_2 \end{aligned}\right\} \quad (15)$$

A variant of Laguerre's method may be employed (Smith, 1967) to solve this quartic equation. It should be noted that, as h increases from the minimum, the gradient is relatively small. Thus, a large increase in h will lead to only a small increase in E. This completes the outline of the numerical approach. For given $\{\alpha, \beta\}$ the procedure may be repeated and Table 1 from the original work (Chou, et al., 2000) reproduced.

A Numerical Example for an Exact Solution

To illustrate the numerical methods procedure, consider an example employed by Chou, et al. (2000), which corresponds to the first row of Table 1 (Alexander, et al., 1995).

Step 1

The first four moments of a given data set, from the process for which an $\bar{x}$ chart is required, are calculated. The required measures are: $\text{skewness} = 0.4836$ and $\text{kurtosis} = 3.3801$. Resulting in $c = 3.0003$ and $q = 5.9989$. The exact measures for $c = 3$ and $q = 6$ correspond to a skewness of 0.48364038 and a kurtosis of 3.38009234.

Step 2

Calculating the moments corresponding to $\{c, q\}$ results in: $M = 0.5109$ and $S = 0.2022$. Selecting the Type I error probability as $\alpha = 0.005$ gives $k = 3.0299$.

Step 3

The Type II error probability and the magnitude of the mean shift for an out of control process are now selected as $\beta = 0.08114$ and $\delta = 1$, where $\beta = 1 - \text{power}$. These result in $n = 18.9998$. Because n is the sample size it is adjusted to 19; in general estimates of n are rounded up to the nearest integer.

Step 4

All additional parameters (see Table 1), reflecting details of the cost and frequency of the measurement process, are now required. These result in numerical values (see Table 2) where the estimates are obtained from the equations described.

Table 1: Input Coefficients for the Numerical Example

Coefficient	Description
$a_1 = 1$	fixed cost of taking a sample
$a_2 = 0.1$	variable cost of sampling
$a_3 = 50$	cost of eliminating an assignable cause
$a_5 = 50$	cost of investigating a false alarm
$A = 5$	cost to rework or scrap a faulty item
$D = 2$	time required to investigate an out-of-control signal
$g = 0.01$	time to measure and record the quantity of interest
$\Delta = 0.3$	tolerance
$\lambda = 0.25$	frequency of process shifts
$\sigma = 0.1$	process standard deviation, from past data
$P = 100$	production rate

Table 2: Calculated Coefficients for the Numerical Example

Source Equation					
10	6	7	12	13	15
$c_0 = 2.19$	$v_1 = 0.01$	$L_1 = 0.5555$	$n_0 = 4.7378$	$d_1 = 1.5475$	$dE_0 = 7.3317$
$c_1 = 0.5883$	$v_2 = 0.1414$	$L_2 = 1.1111$	$n_1 = 129.3569$	$d_2 = 0.1471$	$dE_1 = 1.3936$
$c_2 = 0.0208$			$n_2 = 16.3569$	$d_3 = 0.0052$	$dE_2 = -6.2191$
			$n_3 = 0.5787$		$dE_3 = -0.4441$
					$dE_4 = 0.00008$

For the final parameter (Step 4) the root of interest (14) is $h = 1.1523$. These estimates agree with those previously reported (Chou, et al., 2000) and result in an expected cost: $E = 88.7779$.

A great deal of computer code was required to reach this point. To encourage the adoption of this approach approximate methods for each step of the procedure are proposed. These eliminate the computational complexity; however this results in the introduction of additional notation and the consideration of certain special cases.

Approximate Methods

The computational complexity of the numerical methods described may discourage some users. In view of this, a series of approximations are proposed. Although care must be taken in creating the various terms required it is not necessary to employ the complex numerical procedures utilized previously.

Step 1

The proposed method is entirely different from that previously employed. The tail areas $\{\beta_1, \beta_2\}$ associated with the upper/lower cut-off $\{y_U, y_L\}$ for the raw data are selected. The values adopted would be $\{0.75, 0.25\}$ for quartiles. Employing (2) with some manipulation results in a function that should vanish for c.

$$Z(c) = \left(1 + y_U^c\right) - \left(1 + y_L^c\right)^{\frac{ln(1-\beta_1)}{ln(1-\beta_2)}} \tag{16}$$

To obtain the zero the Newton-Raphson method, which requires evaluation of the first derivative of Z, may be employed.

$$Z'(c) = \ln(y_U)y_U^c - \frac{\ln(1-\beta_1)}{\ln(1-\beta_2)}\ln(y_L)y_L^c\left(1 + y_L^c\right)^{\frac{\ln(1-\beta_1)}{\ln(1-\beta_2)}-1} \tag{17}$$

Then successive estimates $\{c_0, c_1, ..., c_n\}$ for c are then obtained

$$c_{i+1} = c_i - \frac{Z(c_i)}{Z'(c_i)} \tag{18}$$

based on an initial estimate for c_0, for example, 2.

The simplest choice for the exponent in equation 16 is 2. If $\beta_1 + \beta_2 = 1$ is set – which is a reasonable choice for the tail areas – then the corresponding estimates are

$$\beta_1 = \frac{-1+\sqrt{5}}{2} = 0.6180$$

and

$$\beta_2 = 1 - \frac{-1+\sqrt{5}}{2} = 0.3820 .$$

This leads to associated simplifications in (17). Recognizing that the expected value of y^{Nc} takes the form

$$\varepsilon\left(y^{Nc}\right)=\frac{N!}{\prod_{i=1}^{N}(q-i)} \text{ for } N<q .$$

(Note that the above expression may be proved inductively employing the following result: $\varepsilon\left(\left(1+y^{c}\right)^{N}\right)=\frac{q}{q-N}$ for $N<q$.) In particular for $N=1$, $\varepsilon\left(y^{c}\right)=\frac{1}{q-1}$ $\left(q=\frac{1}{\varepsilon\left(y^{c}\right)}+1\right)$, thus using c and the raw data $\left(\varepsilon\left(y^{c}\right)\right)$ the parameter q may be immediately estimated. Alternately employing the median (y_M) for the sample

$$q=\frac{\ln(2)}{\ln\left(1+y_{M}^{c}\right)}$$

the parameter q may again be immediately estimated.

Step 2

The program employed for Step 1 was used to generate 5,041 examples of the mean and standard deviation for given c and q $(c: 2.5, 2.75, \ldots, 20,\ q: 2.5, 2.75, \ldots, 20)$. This data was used to train a neural network (Goodman, 2001) to provide estimates $\{M, S\}$. A simple topology was employed with 2 inputs, 2 outputs and a hidden layer of width 2. The inputs are $x_1=c$ and $x_2=q$ and the transfer functions are $\phi_0(x)=x$ and $\phi_1(x)=\frac{1-e^{-x}}{2\left(1+e^{-x}\right)}$. The activation level of the neurons in the hidden layer is $x_j=\phi_1\left(u_j+\sum_{i=1}^{2} x_i a_{ij}\right)$ $j=3,4$ and the activation of the output neurons is $x_j=\phi_0\left(u_j+\sum_{i=3}^{4} x_i a_{ij}\right)$ $j=5,6$. Which result in the required outputs: $M=x_5$ and $S=x_6$. In this case the fitted values from the software (Goodman, 2001) are: $u_3=-1.06871$, $u_4=0.00297517$, $u_5=0.344999$ and $u_6=0.254552$. The weights representing the links between neurons are: $a_{13}=0.120744$, $a_{23}=0.185339$, $a_{14}=-0.22152$, $a_{24}=-0.00139201$, $a_{35}=-0.544563$, $a_{36}=-0.157016$, $a_{45}=-1.55586$ and $a_{46}=0.252306$.

Alternately the integrals may be evaluated numerically employing Simpson's rule or other quadrature formulas. Given the mean and standard deviation two cases must be considered when estimating k.

Case 1: $M - kS > 0$

In this case both tails of equation (8) contribute, a Taylor expansion of $\ln(\alpha)$ as a function of k, because k is small, is generated.

$$\ln(\alpha)\approx-\frac{2qM^{c-1}cS}{\left(1+M^{c}\right)^{q+1}}k$$

$$k=-ln(\alpha)\frac{\left(1+M^{c}\right)^{q+1}}{2qM^{c-1}cS} .$$

It is necessary to ensure that the expected criterion, $M-kS>0$, is satisfied.

Case 2: $M - kS \le 0$

In this case only one tail of equation (8) contributes, a Maclaurin expansion of $\ln(\alpha)$ as a function of k about z is generated.

$$\ln(\alpha)\approx\ln\left(\frac{1}{\left(1+(M+zS)^{c}\right)^{q}}\right)-\frac{q(M+zS)^{c-1}cS}{1+(M+zS)^{c}}(k-z)$$

$$k=z+\frac{1+(M+zS)^{c}}{q(M+zS)^{c-1}cS}\left(ln\left(\frac{1}{\left(1+(M+zS)^{c}\right)^{q}}\right)-ln(\alpha)\right)$$

Because $k \geq \frac{M}{S}$, z is chosen to satisfy this criterion, typically $z = \left(\frac{M}{S}\right)(1.4 - 40\alpha)$. It is necessary to ensure that the expected criterion, $M - kS \leq 0$, is satisfied.

Step 3

Two cases arise when evaluating n.

Case 1: $\sqrt{n} \leq \frac{M + kS}{S\delta}$

In this case only one tail of equation (9) contributes, a Maclaurin expansion of β as a function of $\sqrt{n}$, about z is generated.

$$\beta \approx 1 - \frac{1}{\left(1 + (M + kS - S\delta z)^c\right)^q} - \frac{cS\delta q(M + kS - S\delta z)^{c-1}}{\left(1 + (M + kS - S\delta z)^c\right)^{q+1}}\left(\sqrt{n} - z\right)$$

$$n = floor\left(\left(z + \frac{\left(1 + (M + kS - S\delta z)^c\right)^{q+1}}{cS\delta q(M + kS - S\delta z)^{c-1}}\left(1 - \frac{1}{\left(1 + (M + kS - S\delta z)^c\right)^q} - \beta\right)\right)^2 + 1\right)$$

Where the floor function returns the largest previous integer. Because $\sqrt{n} \leq \frac{M + kS}{S\delta}$, z is chosen to satisfy this criterion, typically $z = \left(\frac{M + kS}{S\delta}\right)(0.9 - 1.9\beta)$. It is necessary to ensure that the expected criterion, $\sqrt{n} \leq \frac{M + kS}{S\delta}$, is satisfied.

Case 2: $\sqrt{n} \leq \frac{M - kS}{S\delta}$

This case is extremely complex requiring examination of the two-tail situation for (9). It is unlikely to be satisfied because

$$n \leq \left(\frac{M - kS}{S\delta}\right)^2$$

for the parameters generally considered, as is the case here, leads to $n = 0$. Thus this case is not considered.

Step 4

It is straightforward to employ Step 4 from the numerical methods, however, if desired, a slight variant on the procedure may be adopted. Noticing that $dE_4 \approx 0$, the problem reduces to solving a cubic equation. To derive a solution the following coefficients (recall equation 14) are useful:

$$p_3 = \frac{3dE_1 dE_3 - dE_2^2}{3dE_3^2},$$

$$q_3 = \frac{2dE_2^3 - 9dE_1 dE_2 dE_3 + 27dE_0 dE_3^2}{27dE_3^3},$$

$$u_3 = \left(-\frac{q_3}{2} + \sqrt{\left(\frac{q_3}{2}\right)^2 + \left(\frac{p_3}{3}\right)^3}\right)^{\frac{1}{3}}$$

and

$$v_3 = \left(-\frac{q_3}{2} - \sqrt{\left(\frac{q_3}{2}\right)^2 + \left(\frac{p_3}{3}\right)^3}\right)^{\frac{1}{3}}$$

result in:

$$h = u_3 + v_3 - \frac{dE_2}{3dE_3}$$

A Numerical Example for the Approximate Solution

The approximation procedure outlined may be used to reproduce the results presented in Table 1 of Chou, et al. (2000); bearing in mind that in this case the parameters and moments of the Burr distribution are already available. Thus, $c = 3$, $q = 6$, $M = 0.5109$, $S = 0.2022$, $\text{skewness} = 0.4836$ and $\text{kurtosis} = 3.3768$..

The values presented in Table 3 are in exact agreement to the precision given with those earlier reported (Chou, et al., 2000). This suggests that the numerical approximations described are reliable and may safely replace the previously described complex computational approach. This leaves the accuracy of two of the numerical procedures undemonstrated. To check

Table 3: Results of the Numerical Approximation

α	Power (1−β)	*n*	*h*	*k*	Cost
0.00500	0.918860	19	1.15	3.03	88.78
0.00500	0.939008	20	1.19	3.03	88.80
0.00500	0.955365	21	1.23	3.03	88.84
0.00500	0.968362	22	1.26	3.03	88.89
0.00500	0.978435	23	1.30	3.03	88.96
0.00500	0.986008	24	1.33	3.03	89.03
0.00500	0.991489	25	1.35	3.03	89.12
0.00455	0.993538	26	1.37	3.08	89.21
0.00406	0.994851	27	1.39	3.14	89.30
0.00362	0.995924	28	1.41	3.20	89.39
0.00322	0.996797	29	1.43	3.26	89.48
0.00282	0.997296	30	1.45	3.33	89.57

the estimation of c and q, 1,000 observations from a Burr distribution $(c = 3, q = 6)$ were produced. Initially uniform random numbers in the interval $[0, 1]$ were generated, and then transformed to $\{y_i\}$ using equation (2). The quartiles for this data were $(0.375, 0.655)$ which should be compared with the exact values $(0.366, 0.638)$. Employing the measured values with tail values $(0.25, 0.75)$, the iterative scheme (equation 18) results in: $c_0 = 2.000$, $c_1 = 2.497$, $c_2 = 2.843$, $c_3 = 2.982$, $c_4 = 3.000$. Successive estimates converge to the expected result for *c*, then $q = \frac{1}{\sum_{i=1}^{1000} y_i^c} + 1$ because $\varepsilon\left(y^c\right) = \frac{1}{q-1}$, which results in 5.68, a reasonable estimate for *q*.

Alternately employing the median for the sample, the value (y_M) = 0.504, which is close to the exact value 0.497. This results in $q = 5.986$, again very close to the true value. A detailed analysis would be necessary to assess which of the methods to estimate q is superior and whether selection depends on the parameters of the underlying distribution.

The final procedure requiring verification is to use a relatively small neural network to estimate the mean and standard deviation. Using the same test example, the inputs are $x_1 = 3 = c$ and $x_2 = 6 = q$. These generate the following potentials in the hidden layer $x_3 = 0.1000$ and $x_4 = -0.1615$. which result in the output $x_5 = 0.5418 = M$ and $x_6 = 0.1881 = S$.

The results exhibit reasonable agreement with the exact values (*M* = 0.5109 and *S* = 0.2022), if improved accuracy is desirable a larger network could be developed or a quadrature procedure might be employed to evaluate the required moments.

Conclusion

To help popularize the economic design of $\bar{x}$ control charts, this study employed the Burr distribution for non-normal data. Slight shortcomings in an earlier work (Chou, et al., 2000) were corrected and a series of approximations were used to reduce computational complexity. It is hoped that the reduction of the computational effort involved in the approach will encourage wider adoption.

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Appendix: Notation and Definitions

a_1: fixed cost of taking a sample

a_2: variable cost of sampling

a_3: cost of eliminating an assignable cause

a_5: cost of investigating a false alarm

a_{ij}: weights in neural networks

A: cost to rework or scrap a faulty item

B: sub-equation of the expected cost

c: first parameter of the Burr distribution

$\{c_0, c_1, c_2\}$: coefficients for B as a function of h

$\{c_0, c_1, ..., c_n\}$: successive numerical estimates for c

$\{d_0, d_1, d_2\}$: coefficients for the denominator of E as a function of h

Appendix (continued): Notation and Definitions

dE : the numerator of the derivative of E

$\{dE_0, dE_1, dE_2, dE_3, dE_4\}$: coefficients for dE as a function of h

D : time required to investigate an out-of-control signal

E : expected cost

f : density function of the Burr distribution

F : cumulative density function of the Burr distribution

g : time to measure and record the quantity of interest

h : parameter of the $\bar{x}$ control chart, interval between successive subgroups

H : Heaviside step function

k : parameter of the $\bar{x}$ control chart, control limits

$\{L_1, L_2\}$: sub-equation of the expected cost

LCL : lower control limit

M : the mean of the fitted Burr distribution

n : parameter of the $\bar{x}$ control chart, sample size

$\{n_0, n_1, n_2, n_3\}$: coefficients for the numerator of E as a function of h

P : production rate

p_3 : factor in the cubic solution

q : second parameter of the Burr distribution

q_3 : factor in the cubic solution

S : the standard deviation of the fitted Burr distribution

T : target mean

Appendix (continued): Notation and Definitions

u_3 : factor in the cubic solution

u_i : potentials for the neural networks

UCL : upper control limit

$\{v_1, v_2\}$: sub-equation of the expected cost

v_3 : factor in the cubic solution

x : random variable

x_i : inputs/outputs for the neural networks

y , $\{y_i\}$: Burr random variables

y_M : the median of the raw data

$\{y_U, y_L\}$: upper/lower cut offs of the raw data corresponding to $\{\beta_1, \beta_2\}$

z : point about which the Maclaurin expansion is performed

Z, Z' : the function (and its derivative) with a zero at c

α : probability of a Type I error

β : probability of a Type II error

$\{\beta_1, \beta_2\}$: tail areas associated with estimating c

δ : magnitude of the mean shift for an out of control process

ε : expectation (average) over the observed data

Δ : tolerance

$\{\phi_0, \phi_1\}$: decision function for the neural networks

λ : frequency of process shifts

μ : process mean

σ : process standard deviation (from past data)