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A Comparison of Shape and Scale Estimators of the Two-Parameter Weibull Distribution

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Weibull distributions are widely used in reliability and survival analysis. In this paper, different methods to estimate the shape and scale parameters of the two-parameter Weibull distribution have been reviewed and compared, based on the bias, mean square error and variance. Because a theoretical comparison is not possible, an extensive simulation study has been conducted to compare the performance of different estimators. Based on the simulation study it was observed that MLE consistently performs better than other methods.

Keywords: Two-parameter Weibull distribution, scale parameters, shape parameters

Introduction

The Weibull distribution is a commonly used model in reliability, life time and environmental data analysis. A considerable literature discussing the methods of estimation of Weibull parameters exists (Sharoon, et al., 2012; Saralees et al., 2011; Saralees et al., 2008) because of its applications in different fields. Kantar and Senoglu (2008) did a simulation comparison of different estimators for scale parameter when shape is known. Balakrishanan and Kateri (2008) showed the existence and uniqueness of maximum likelihood estimates (MLE) of Weibull distribution. Dubey (1967) derived the percentile estimators (Percentile 1) which uses 4 different percentiles to estimate the shape and scale parameters. Seki and Yokoyama (1993) proposed a simple and robust method that uses only two percentiles, 31st and 63rd percentile (Percentile 2) to estimate both parameters. Moment estimators (MOM) and median rank regression estimators (MRRS) are also commonly used in literature (Kantar and Senoglu, 2008) because of their

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easiness in computation. Existing methods (namely MLE, MOM, MRRS, Percentile 1, and Percentile 2) for estimating both shape and scale parameters of two-parameter Weibull distribution are here reviewed and compared. A simulation study has been conducted to compare the performance of these methods under same simulation conditions.

Statistical Methodology

The Weibull distribution has the probability density function, $f(x) = \alpha\beta^{-\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$ for $x \geq 0, \propto > 0, \beta > 0,$ where $\propto$ is the shape parameter and β is the scale parameter. The cumulative distribution function is given by

$$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \text{ for } x \geq 0.$$

The distribution is reversed J-shaped when $\propto < 1$, exponential when $\propto = 1$ and bell-shaped when $\propto > 1$ (Kantar and Senoglu, 2008). Because of its wide-variety of shapes it is used extensively in practice for modeling real life data in different fields.

Maximum Likelihood Estimators (MLE)

The log-likelihood function of a random sample from the two-parameter Weibull distribution is given by

$$\ln L = n\ln\alpha - n\alpha\ln\beta + (\alpha-1)\sum\ln x - \sum\left(\frac{x}{\beta}\right)^{\alpha}.$$

This will yield the following two score equations

$$\frac{\partial \ln L}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{\alpha}{\beta^{\alpha+1}}\sum x^{a} = 0 \text{ and}$$

$$\frac{\partial \ln L}{\partial \alpha} = -\frac{n}{\alpha} - n\ln\beta + \sum\ln x + \frac{\ln\beta}{\beta^{a}}\sum x^{\alpha} - \frac{1}{\beta^{a}}\sum x^{\alpha}\ln x = 0$$

The above two equations can be solved numerically to obtain MLEs.

Moment Estimators (MOM)

The moment estimators are obtained by equating the population moments to the corresponding sample moments. The first and second moments of Weibull distribution are respectively

$$\mu_1' = \beta\Gamma(1+\alpha^{-1}) \text{ and}$$
$$\mu_2' = \beta^2\Gamma(1+2\alpha^{-1})$$

The first two moments from the sample are $m_1' = \frac{1}{n}\sum x$ and $m_2' = \frac{1}{n}\sum x^2$.

The moment estimates are obtained by solving the following two equations

$$m_1' = \beta\Gamma(1+\alpha^{-1})$$
$$m_2' = \beta^2\Gamma(1+2\alpha^{-1})$$

Median Rank Regression Estimators (MRRS)

MRR is a procedure for estimating the Weibull parameters by fitting a least squares regression line through the points on a probability plot. Thus,

$$\log\left(1-F(x)\right) = -\left(\frac{x}{\beta}\right)^{\alpha} \text{ and hence}$$
$$\log\left(\log\left(1-F(x)\right)\right) = \alpha\log x = \alpha\log\beta.$$

This is now a linear model and method of least squares can be used to estimate α and β. The sample data are first sorted in ascending order and then following Abernethy (2006), the distribution function, $F(x_i)$ is approximated for each point (x_i) in the sorted sample as $F(x_i) = \frac{i-0.3}{n+0.4}$, where I is the ascending rank of the data point x_i.

Percentile Estimators (Percentile 1)

Percentile estimators for both shape and scale parameters were derived by Dubey (1967). He proposed an estimator based on 17th and 97th percentiles for shape parameter and one based on 40th and 82nd percentile for scale parameter. The formulae for the shape and scale percentile estimators are presented here; for details refer to Dubey (1967). Let $p_1 = 0.1673$ and $p_2 = 0.9737$. Define $k_1 = \log(-\log(1-p_1)) - \log(-\log(1-p_2))$. Let y_1 and y_2 represent the $100p_1$th percentile from the data. Then

$$\hat{\alpha} = \frac{-k_1}{\log(y_1) - \log(y_2)}$$

Similarly to estimate β, define $p_3 = 0.3978$ and $p_4 = 0.8211$. Let $k_2 = \log(-\log(1-p_3)) - \log(-\log(1-p_4))$; $k_3 = -\log(1-p_3)$ and $w = 1 - \frac{\log(k_3)}{k_2}$. Let y_3 and y_4 represent the $100p_3$th and $100p_4$th percentile from the data. Then

$$\beta = \exp(w\log(y_3) + (1-w)\log(y_4)\,.$$

Improved Percentile Estimators (Percentile 2)

Seki and Yokoyama (1993) proposed this simple and robust method that uses only two percentiles, 31st and 63rd percentile to estimate α and β. The Weibull cumulative distribution function is given by

$$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \quad \text{for } x \geq 0\,.$$

Hence the $100p$th percentile of the Weibull distribution can be written as $x_p = \beta(-\log(1-p))^{1/\beta}$. Then the $100(1 - e^{-1}) = 63.2$th percentile is $x_{0.632} = \beta$ for any Weibull distribution. This can be used to compute $\hat{\beta}$. Therefore, the estimate of the shape parameter can be obtained as $\hat{\alpha} = \frac{\log\left(-\log(1-p)\right)}{\log\left(\frac{x_p}{x_{0.632}}\right)}$. Seki and Yokoyama (1993) approximated the numerator of this estimator as –1 and then obtained $p = 0.31$, approximately, to obtain $\hat{\alpha}$.

Simulation Study

A simulation study has been conducted to explore the performances of the different methods discussed in this article.

Simulation Technique

The main objective of this study is to compare the performance of five different methods to estimate the shape and scale parameters of two-parameter Weibull distribution. Weibull distribution with parameters scale = 10 and shape = 0.5, 1, 1.5,2,3 and 4 were used to generate 5,000 samples of sizes n = 5, 10, 20, 30, 50 and 100. The estimates are compared using the values of average bias, mean squared error (MSE) and variance. The simulation was done using statistical software R version 2.15.2.

Results and Discussion

The results of the simulation are shown in Tables 1 to 3. The bias and MSEs from Weibull (10, 0.5) and Weibull (10, 3) are also presented in Figures 1 to 4. From Tables 1 to 3, it can be observed that as sample size increases, bias, MSE and variance decrease. For small sample size, the performance of methods differs significantly. For all methods, absolute bias, MSE and variance decrease as sample size increases. It can be observed from Tables 1 to 3 and Figures 1 to 4, in almost all cases MLE performed better than the other 4 methods and percentile method-1 performed the worst. In some situations, MRRS also performs well, especially for shape estimates. It can also be observed that both percentile estimators perform poorly in estimation of shape. There is no consistency in the performance of estimates by the method of moments. Because MLE is performing consistently better than the other 4 methods practitioners are encouraged to use MLE whenever possible.

Table 1. Bias, Variance and MSE of both Scale and Shape estimates α=10 and β=0.5; 1

α, β	n		Scale MLE	Scale MOM	Scale MRRS	Scale Perctle1	Scale Perctle2	Shape MLE	Shape MOM	Shape MRRS	Shape Perctle1	Shape Perctle2
10, 0.5	5	Bias	2.929	6.216	5.361	5.340	4.394	0.219	0.351	0.025	0.661	0.497
		Vars	156.930	224.302	219.866	282.068	197.209	0.147	0.092	0.085	5.231	0.313
		MSEs	165.511	262.939	248.610	310.581	216.516	0.195	0.216	0.085	5.668	0.560
	10	Bias	1.525	4.606	2.998	2.334	1.773	0.085	0.218	-0.015	0.283	0.193
		Vars	65.147	96.815	84.821	110.433	73.834	0.031	0.033	0.026	0.349	0.050
		MSEs	67.473	118.026	93.811	115.879	76.977	0.038	0.081	0.026	0.429	0.087
	20	Bias	0.705	3.217	1.643	0.979	0.656	0.036	0.136	-0.021	0.112	0.079
		Vars	25.634	40.337	32.389	43.623	30.013	0.010	0.016	0.012	0.069	0.015
		MSEs	26.131	50.689	35.088	44.582	30.444	0.012	0.035	0.013	0.082	0.021
	30	Bias	0.556	2.621	1.258	0.707	0.442	0.026	0.107	-0.017	0.072	0.054
		Vars	17.111	25.216	20.965	26.613	19.722	0.006	0.013	0.008	0.036	0.009
		MSEs	17.420	32.084	22.548	27.113	19.917	0.007	0.024	0.008	0.041	0.012
	50	Bias	0.283	1.912	0.783	0.405	0.207	0.014	0.076	-0.016	0.039	0.034
		Vars	9.585	15.077	11.172	15.220	11.347	0.003	0.008	0.005	0.017	0.005
		MSEs	9.665	18.731	11.785	15.384	11.390	0.004	0.014	0.005	0.019	0.006
	100	Bias	0.174	1.257	0.487	0.236	0.158	0.008	0.048	-0.011	0.021	0.018
		Vars	4.664	8.153	5.320	7.373	5.727	0.002	0.005	0.003	0.007	0.002
		MSEs	4.694	9.734	5.557	7.429	5.752	0.002	0.008	0.003	0.008	0.003
10, 1	5	Bias	0.284	0.312	1.232	0.485	0.304	0.443	0.353	0.049	1.297	0.926
		Vars	23.380	23.097	27.835	29.087	24.408	0.621	0.411	0.347	10.932	1.147
		MSEs	23.461	23.194	29.353	29.322	24.501	0.817	0.535	0.349	12.614	2.005
	10	Bias	0.190	0.222	0.840	0.223	0.060	0.173	0.173	-0.032	0.507	0.371
		Vars	11.082	11.192	12.944	15.822	12.013	0.127	0.108	0.106	0.991	0.199
		MSEs	11.118	11.241	13.650	15.872	12.016	0.157	0.137	0.107	1.247	0.336
	20	Bias	0.072	0.100	0.486	0.067	-0.047	0.074	0.087	-0.042	0.225	0.158
		Vars	5.576	5.707	6.335	8.386	6.348	0.043	0.046	0.048	0.281	0.063
		MSEs	5.581	5.717	6.571	8.390	6.350	0.049	0.053	0.050	0.331	0.088
	30	Bias	0.075	0.098	0.412	0.049	-0.006	0.049	0.062	-0.040	0.141	0.108
		Vars	3.671	3.796	4.182	5.412	4.340	0.025	0.029	0.032	0.134	0.037
		MSEs	3.676	3.805	4.351	5.414	4.340	0.027	0.033	0.033	0.154	0.048
	50	Bias	0.050	0.062	0.281	0.028	-0.010	0.029	0.039	-0.031	0.087	0.069
		Vars	2.173	2.269	2.426	3.332	2.635	0.014	0.018	0.020	0.072	0.020
		MSEs	2.175	2.273	2.504	3.332	2.635	0.015	0.019	0.021	0.080	0.025
	100	Bias	0.032	0.034	0.179	0.012	0.002	0.016	0.021	-0.021	0.045	0.037
		Vars	1.117	1.178	1.232	1.698	1.349	0.007	0.010	0.010	0.031	0.010
		MSEs	1.118	1.179	1.264	1.698	1.349	0.007	0.010	0.011	0.033	0.011

Table 2. Bias, Variance and MSE of both Scale and Shape estimates α=10 and β=1.5; 2

			Scale					Shape				
α, β	***n***		**MLE**	**MOM**	**MRRS**	**Perctle1**	**Perctle2**	**MLE**	**MOM**	**MRRS**	**Perctle1**	**Perctle2**
10, 1.5	**5**	**Bias**	-0.081	-0.149	0.513	-0.149	-0.226	0.650	0.398	0.059	2.045	1.298
		Vars	9.333	9.331	10.477	11.069	9.546	1.374	0.947	0.713	47.203	2.365
		MSEs	9.340	9.353	10.740	11.091	9.597	1.797	1.106	0.717	51.386	4.050
	10	**Bias**	0.038	-0.002	0.451	-0.016	-0.092	0.257	0.171	-0.048	0.755	0.537
		Vars	4.948	4.986	5.542	6.870	5.493	0.275	0.231	0.232	2.331	0.427
		MSEs	4.950	4.986	5.746	6.870	5.501	0.342	0.260	0.235	2.901	0.715
	20	**Bias**	0.039	0.016	0.317	0.016	-0.046	0.114	0.081	-0.065	0.339	0.248
		Vars	2.508	2.517	2.763	3.802	2.851	0.097	0.090	0.109	0.606	0.144
		MSEs	2.509	2.517	2.863	3.802	2.853	0.110	0.096	0.113	0.721	0.205
	30	**Bias**	0.014	-0.003	0.227	-0.006	-0.064	0.075	0.054	-0.054	0.209	0.167
		Vars	1.669	1.677	1.816	2.439	1.947	0.055	0.054	0.074	0.324	0.083
		MSEs	1.669	1.677	1.867	2.439	1.952	0.061	0.057	0.077	0.368	0.111
	50	**Bias**	-0.008	-0.019	0.145	-0.021	-0.050	0.041	0.030	-0.050	0.121	0.099
		Vars	0.973	0.979	1.048	1.486	1.166	0.031	0.032	0.045	0.155	0.048
		MSEs	0.973	0.979	1.069	1.486	1.168	0.033	0.033	0.048	0.170	0.058
	100	**Bias**	0.003	-0.004	0.102	-0.020	-0.027	0.022	0.016	-0.032	0.067	0.054
		Vars	0.493	0.495	0.533	0.748	0.591	0.014	0.016	0.023	0.070	0.021
		MSEs	0.493	0.495	0.543	0.749	0.592	0.015	0.016	0.024	0.074	0.024
10, 2	**5**	**Bias**	-0.102	-0.123	0.339	-0.200	-0.263	0.868	0.510	0.089	2.499	1.673
		Vars	5.328	5.376	5.763	6.306	5.441	2.301	1.833	1.321	43.582	3.716
		MSEs	5.339	5.391	5.878	6.345	5.511	3.055	2.093	1.329	49.828	6.516
	10	**Bias**	-0.049	-0.065	0.254	-0.132	-0.189	0.338	0.191	-0.060	1.024	0.699
		Vars	2.797	2.830	3.090	3.811	3.062	0.459	0.390	0.394	4.718	0.699
		MSEs	2.799	2.834	3.154	3.829	3.098	0.573	0.427	0.398	5.767	1.187
	20	**Bias**	-0.041	-0.051	0.157	-0.089	-0.136	0.151	0.087	-0.080	0.473	0.331
		Vars	1.359	1.368	1.461	2.068	1.560	0.171	0.159	0.197	1.296	0.262
		MSEs	1.361	1.371	1.486	2.076	1.578	0.194	0.166	0.203	1.520	0.371
	30	**Bias**	-0.004	-0.009	0.157	-0.023	-0.059	0.096	0.054	-0.081	0.275	0.222
		Vars	0.911	0.915	0.991	1.347	1.059	0.096	0.090	0.125	0.540	0.142
		MSEs	0.911	0.915	1.016	1.348	1.062	0.105	0.093	0.131	0.615	0.191
	50	**Bias**	-0.007	-0.010	0.110	-0.028	-0.049	0.059	0.035	-0.063	0.166	0.141
		Vars	0.543	0.546	0.596	0.829	0.654	0.055	0.054	0.080	0.269	0.084
		MSEs	0.543	0.546	0.608	0.830	0.656	0.059	0.055	0.084	0.297	0.104
	100	**Bias**	0.001	-0.002	0.071	-0.020	-0.022	0.031	0.019	-0.040	0.088	0.073
		Vars	0.276	0.276	0.296	0.427	0.332	0.025	0.025	0.041	0.120	0.037
		MSEs	0.276	0.276	0.301	0.427	0.333	0.026	0.026	0.042	0.128	0.043

Table 3. Bias, Variance and MSE of both Scale and Shape estimates α=10 and β=3; 4

			Scale					Shape				
α, β	***n***		**MLE**	**MOM**	**MRRS**	**Perctle1**	**Perctle2**	**MLE**	**MOM**	**MRRS**	**Perctle1**	**Perctle2**
10, 3	5	**Bias**	-0.166	-0.138	0.122	-0.275	-0.320	1.295	0.767	0.131	3.899	2.417
		Vars	2.392	2.439	2.515	2.865	2.489	5.768	5.041	3.285	243.739	8.976
		MSEs	2.419	2.458	2.530	2.941	2.591	7.446	5.629	3.302	258.938	14.818
	10	**Bias**	-0.053	-0.037	0.147	-0.129	-0.152	0.505	0.276	-0.102	1.461	1.048
		Vars	1.221	1.234	1.290	1.659	1.352	1.100	0.998	0.911	9.896	1.722
		MSEs	1.224	1.235	1.312	1.676	1.374	1.354	1.074	0.922	12.032	2.820
	20	**Bias**	-0.017	-0.009	0.120	-0.060	-0.070	0.218	0.113	-0.134	0.689	0.487
		Vars	0.602	0.605	0.649	0.896	0.690	0.374	0.359	0.428	2.594	0.572
		MSEs	0.602	0.605	0.663	0.900	0.695	0.422	0.372	0.445	3.070	0.810
	30	**Bias**	-0.022	-0.017	0.082	-0.049	-0.057	0.153	0.084	-0.110	0.452	0.341
		Vars	0.415	0.417	0.442	0.618	0.487	0.230	0.227	0.294	1.387	0.343
		MSEs	0.416	0.417	0.448	0.620	0.491	0.253	0.234	0.306	1.592	0.459
	50	**Bias**	-0.020	-0.017	0.058	-0.039	-0.044	0.083	0.044	-0.101	0.254	0.209
		Vars	0.243	0.243	0.259	0.358	0.288	0.121	0.123	0.185	0.643	0.183
		MSEs	0.243	0.243	0.262	0.360	0.290	0.128	0.125	0.196	0.707	0.226
	100	**Bias**	0.000	0.001	0.047	-0.003	-0.014	0.035	0.014	-0.075	0.098	0.096
		Vars	0.118	0.118	0.128	0.186	0.144	0.058	0.059	0.092	0.265	0.087
		MSEs	0.118	0.118	0.130	0.186	0.144	0.059	0.059	0.098	0.274	0.096
10, 4	5	**Bias**	-0.125	-0.081	0.091	-0.230	-0.258	1.716	1.051	0.164	4.996	3.156
		Vars	1.388	1.411	1.442	1.683	1.471	9.001	7.677	5.003	141.435	13.999
		MSEs	1.404	1.418	1.451	1.736	1.537	11.945	8.782	5.029	166.393	23.962
	10	**Bias**	-0.078	-0.054	0.076	-0.140	-0.156	0.686	0.401	-0.131	2.017	1.402
		Vars	0.670	0.675	0.709	0.913	0.747	2.063	1.999	1.723	16.360	3.181
		MSEs	0.676	0.678	0.714	0.932	0.772	2.535	2.159	1.740	20.429	5.146
	20	**Bias**	-0.033	-0.021	0.070	-0.056	-0.076	0.305	0.166	-0.174	0.875	0.659
		Vars	0.362	0.364	0.391	0.545	0.420	0.691	0.695	0.774	4.648	1.023
		MSEs	0.363	0.365	0.396	0.548	0.426	0.783	0.723	0.804	5.414	1.457
	30	**Bias**	-0.024	-0.016	0.055	-0.038	-0.055	0.194	0.107	-0.156	0.549	0.444
		Vars	0.229	0.230	0.245	0.339	0.271	0.388	0.411	0.523	2.300	0.567
		MSEs	0.229	0.230	0.248	0.341	0.274	0.425	0.423	0.547	2.601	0.764
	50	**Bias**	-0.016	-0.011	0.042	-0.025	-0.035	0.112	0.060	-0.132	0.322	0.271
		Vars	0.138	0.139	0.149	0.209	0.163	0.224	0.237	0.323	1.087	0.342
		MSEs	0.138	0.139	0.151	0.209	0.165	0.237	0.241	0.341	1.190	0.415
	100	**Bias**	-0.013	-0.011	0.023	-0.021	-0.025	0.060	0.036	-0.087	0.156	0.147
		Vars	0.066	0.067	0.071	0.104	0.082	0.108	0.116	0.174	0.478	0.157
		MSEs	0.066	0.067	0.072	0.105	0.083	0.111	0.118	0.181	0.503	0.178

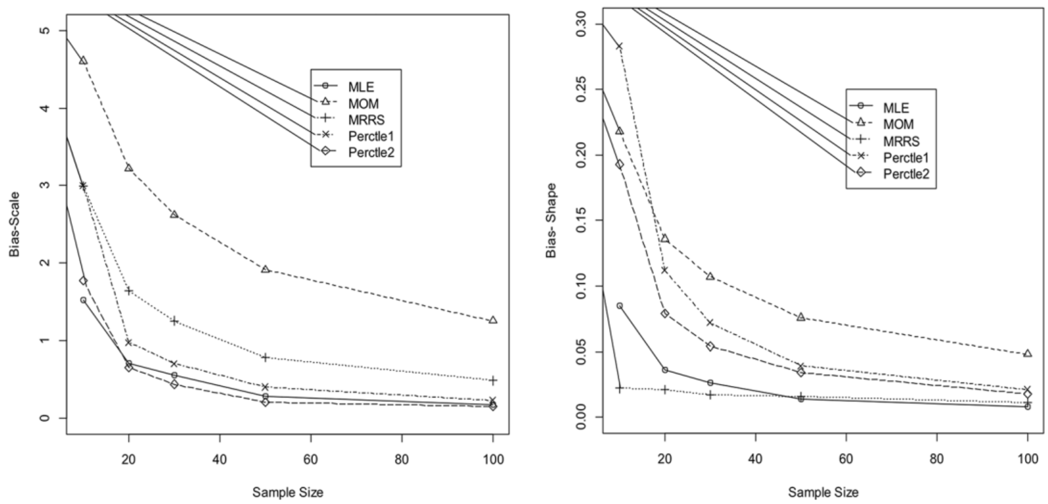

Figure 1. Absolute Bias of Scale parameter estimate (left), Shape parameter estimate (right) vs. Sample size from Weibull (10, 0.5)

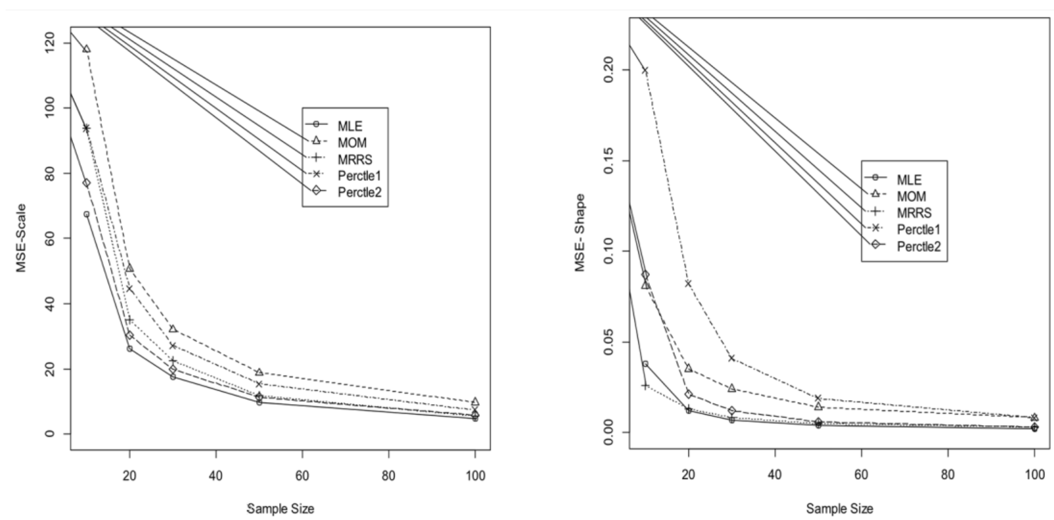

Figure 2. MSE of Scale parameter estimate (left), Shape parameter estimate (right) vs. Sample size from Weibull (10, 0.5)

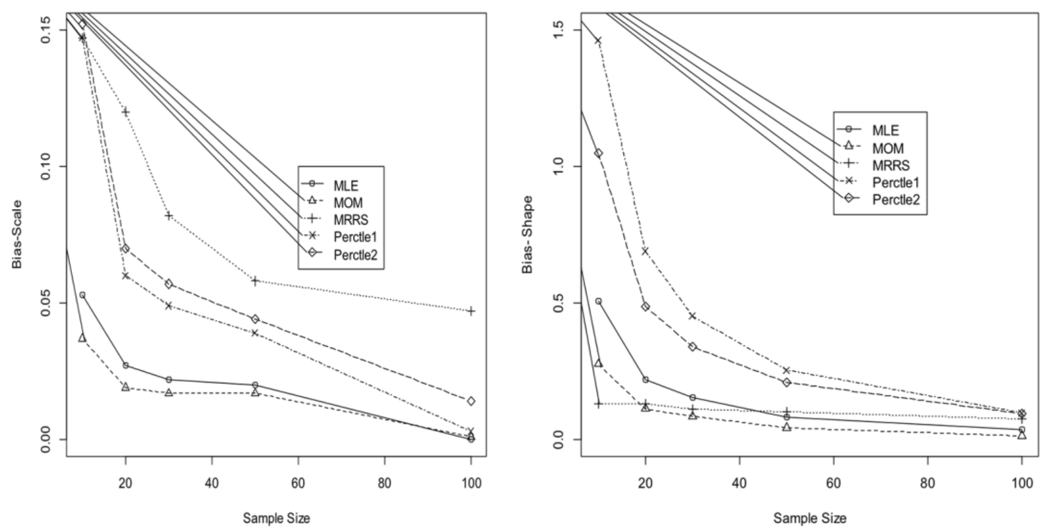

Figure 3. Absolute Bias of Scale parameter estimate (left), Shape parameter estimate (right) vs. Sample size from Weibull (10, 3)

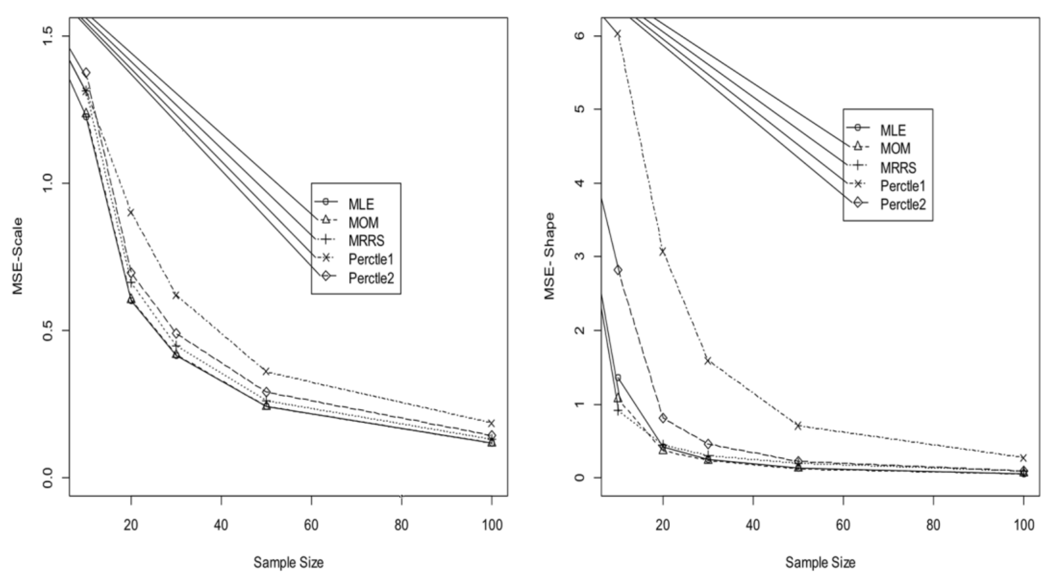

Figure 4. MSE of Scale parameter estimate (left), Shape parameter estimate (right) vs. Sample size from Weibull (10, 3)

Example

Many researchers modeled wind data using the Weibull distribution (Dorvlo, 2002; Weisser, 2003; Celik, 2003). The five methods with an example in Battacharya and Bhattacharjee (2010) will be discussed next. This example provides the average monthly wind speed (m/s) of Kolkata from 1st March 2009 to 31st March 2009. Table 4 presents the data set.

The estimates of the two-parameter Weibull distribution obtained by fitting to the data using the methods discussed in the article are given in the Table 5. It look like the Percentile 1 and Percentile 2 estimates are at the extreme ends and the MLE estimates lie somewhat between the values of other estimates.

Table 4. Average daily wind speed in Kolkata during March 2009.

Date	Speed (m/s)	Date	Speed (m/s)
1	0.56	17	0.28
2	0.28	18	0.83
3	0.56	19	1.39
4	0.56	20	1.11
5	1.11	21	1.11
6	0.83	22	0.83
7	1.11	23	0.56
8	1.94	24	0.83
9	1.11	25	1.67
10	0.83	26	1.94
11	1.11	27	1.39
12	1.39	28	0.83
13	0.28	29	2.22
14	0.56	30	1.67
15	0.28	31	2.22
16	0.28		

Table 5. Estimates of two-parameter Weibull by different methods.

Method	Scale	Shape
MLE	1.1550	1.9081
MOM	1.1501	1.8456
MRRS	1.1636	1.8031
Percentile 1	1.1100	1.8055
Percentile 2	1.1816	2.1704

Conclusion

Five different methods for the joint estimation of both scale and shape parameters of two-parameter Weibull distribution were reviewed in this article. A simulation study was conducted to compare the five methods based on bias, mean square error and variance of estimates. From simulation results, it was observed that MLE performs consistently better than MOM, MRRS, percentile method and improved percentile method and therefore MLE estimates are recommended to the practitioners.

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