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K. A. Adepoju
University of Ibadan, ka.adepoju@ui.edu.ng

O. I. Chukwu
University of ibadan, unnachuks2002@yahoo.co.uk

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Maximum Likelihood Estimation of the Kumaraswamy Exponential Distribution with Applications

K. A. Adepoju
University of Ibadan
Ibadan, Nigeria

O. I. Chukwu
University of Ibadan
Ibadan, Nigeria

The Kumaraswamy exponential distribution, a generalization of the exponential, is developed as a model for problems in environmental studies, survival analysis and reliability. The estimation of parameters is approached by maximum likelihood and the observed information matrix is derived. The proposed models are applied to three real data sets.

Keywords: Information matrix, Maximum likelihood, Moment generating function.

Introduction

A random variable X has the exponential distribution if its cumulative distribution function for $x > 0$ is given by

$$F(x) = 1 - \ell^{-\lambda x} \tag{1}$$

where $\lambda > 0$ is a scale parameter, the probability density function is

$$f(x) = \lambda \ell^{-\lambda x} \tag{2}$$

Using the Kumaraswamy link function by Cordeiro and de Castro (2011) given as

$$g(x) = a,b\, f(x)\left[F(x)\right]^{b-1}\left[1 - F(x)^{b}\right]^{a-1} \tag{3}$$

K.A. Adepoju is in the Department of Statistics. Email at ka.adepoju@ui.edu.ng. O.I. Chukwu is in the Department of Statistics. Email at unnachuks2002@yahoo.co.uk.

By inserting (1) and (2) in (3) we have

$$g(x) = a,b\,\lambda \ell^{-\lambda\ell}\left(1-\ell^{-\lambda x}\right)^{b-1}\left[1-\left(1-\ell^{-\lambda x}\right)^{b}\right]^{a-1} \qquad (4)$$
$$a,\, b,\, \lambda > 0$$

Another term of Kumaraswamy distribution can be obtained using the binomial series expansion. The Kumaraswamy exponential distribution in equation (4) can be expanded as follows:

$$\left(1-m\right)^{K} = \sum_{j=1}^{K}\left(-1\right)^{j}\binom{K}{j}M^{j} \qquad (5)$$

as

$$g(x) = a\,b\lambda\ell^{-\lambda x}\left(1-\ell^{-\lambda x}\right)^{b-1}\sum_{j=1}^{K}\left(-1\right)^{j}\binom{K}{j}\left(1-\ell^{-\lambda x}\right)^{bj}$$
$$= a\,b\lambda\ell^{-\lambda x}\sum_{j=1}^{K}\left(-1\right)^{j}\binom{K}{j}\left(1-\ell^{-\lambda x}\right)^{bj+b-1}$$

Statistical inference

Given a random variable X following equation (4), the likelihood function is obtained as

$$L = a^{n}b^{n}\lambda^{n}\,\ell\prod_{i=1}^{n}\ell^{-\lambda x}\left(1-\ell^{-\lambda x}\right)^{b-1}\left[1-\left(1-\ell^{-\lambda x}\right)^{b}\right]^{a-1}$$

Taking log-likelihood of the above

$$\log L = n\log a + n\log b + n\log\lambda - \lambda\sum_{i=1}^{n}x + \left(b-1\right)\sum_{i=1}^{n}\log\left(1-\ell^{-\lambda x}\right)$$
$$+\left(a-1\right)\sum_{i=1}^{n}\log\left[1-\left(1-\ell^{-\lambda x}\right)^{b}\right]$$

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \log\left[1-\left(1-\ell^{-\lambda x}\right)^{b}\right]$$

$$\frac{\partial \log L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log\left(1-\ell^{-\lambda x}\right) - (a-1)\sum_{i=1}^{n} \frac{\left(1-\ell^{-\lambda x}\right)^{b} \log\left(1-\ell^{-\lambda x}\right)}{1-\left(1-\ell^{-\lambda x}\right)^{b}}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x + (b-1)\sum_{i=1}^{n} \frac{x\ell^{-\lambda x}}{1-\ell^{-\lambda x}} - (a-1)b\sum_{i=1}^{n} \frac{x\ell^{-\lambda x}\left(1-\ell^{-\lambda x}\right)^{b-1}}{1-\left(1-\ell^{-\lambda x}\right)^{b}}$$

Fisher information

$$\frac{\partial^2 \log L}{\partial a^2} = -\frac{n}{a^2}$$

$$\frac{\partial^2 \log L}{\partial b^2} = -\frac{n}{b^2} - (a-1)\sum_{i=1}^{n} \frac{\left(1-\ell^{-\lambda x}\right)^{b}\left[\log\left(1-\ell^{-\lambda x}\right)\right]^{2}}{\left[1-\left(1-\ell^{-\lambda x}\right)^{b}\right]^{2}}$$

$$\frac{\partial^2 \log L}{\partial \lambda^2} = -\frac{n}{\lambda^2} - (b-1)\sum_{i=1}^{n} \frac{x}{\left(1-\ell^{-\lambda x}\right)^{2}} -$$

$$b(a-1)\sum_{i=1}^{n} \frac{x^2\ell^{-\lambda x}\left(1-\ell^{-\lambda x}\right)^{b-1}}{1-\left(1-\ell^{-\lambda x}\right)^{b}}\left[\frac{(b-1)\ell^{-\lambda x}\left(1-\ell^{-\lambda x}\right)^{b-2}}{\left(1-\ell^{-\lambda x}\right)^{b-1}} - \frac{b\ell^{-\lambda x}\left(1-\ell^{-\lambda x}\right)^{b-1}}{\left(1-\ell^{-\lambda x}\right)^{b}}\right]$$

$$\frac{\partial^2 \log L}{\partial a \partial b} = -\left(1-\ell^{-\lambda x}\right)^b \log\left(1-\ell^{-\lambda x}\right)$$

$$\frac{\partial^2 \log L}{\partial b \partial \lambda} = \sum_{i=1}^{n} \frac{x_i \ell^{-\lambda x_i}}{1-\ell^{-\lambda x_i}}$$

$$\frac{\partial^2 \log L}{\partial a \partial \lambda} = -b \sum_{i=1}^{n} \frac{x_i \ell^{-\lambda x_i}\left(1-\ell^{-\lambda x_i}\right)^{b-1}}{1-\left(1-\ell^{-\lambda x_i}\right)^b}$$

Application

For the sake of numerical illustrations, the two data sets used by Raja and Mir (2011) are considered. The first data set is on the failure time of the conditioning system of an airplane and the second is the runs scored by a Cricketer in 27 innings at national level.

Table 1. Descriptive Statistics on Failure Time on Conditional System

Min	Q_1	Q_2	Mean	Q_3	Max	Var
1.0	12.5	22.0	59.6	83.0	261.0	5167.421

Skewness	Kurtosis
1.693605	4.966655

Table 2. Descriptive Statistics in runs scored by a Cricketer

Min	Q_1	Q_2	Mean	Q_3	Max	Var
2.00	8.00	25.00	36.41	50.00	127.00	1149.02

Skewness	Kurtosis
1.124548	3.492725

Table 3. Failure Time on Conditional System

Model	Estimates	Statistics	
		Log-likelihood	AIC
Weibull	$\hat{\alpha}$ =0.8536, $\hat{\lambda}$ =0.0183	-151.970	305.94
Lognormal	$\hat{\mu}$ =3.358, $\hat{\lambda}$ =1.3190	151.621	305.242
Exponentiated Weibull	$\hat{\alpha}$ =3.824, $\hat{\theta}$ =0.1732, $\hat{\delta}$ =82.235	-149.567	308.134
Exponentiated Gumbel	$\hat{\alpha}$ =1.9881, $\hat{\lambda}$ =49.0638	-148537	299.074
Exponentiated Lognormal	$\hat{\alpha}$ =0.1542, $\hat{\mu}$ =3.1353, $\hat{\delta}$ =0.3648	-148.659	303.318
Lehman Type II Exponential	$\hat{\alpha}$ =0.3439, $\hat{\lambda}$ =0.0057	-152.6097	309.2593
Exponential	$\hat{\lambda}$ =0.0168	-152.6297	307.2593
Kumaraswamy Exponential Distribution	$\hat{\alpha}$ =10.142, $\hat{b}$ =0.9129, $\hat{\lambda}$ =0.0005	-107/9653	221.9306

Table 4. Runs Scored by a Cricketer

Model	Estimates	Statistics	
		Log-likelihood	AIC
Gamma	$\hat{\alpha}$ =0.7235, λ=0.0127	-125.654	253.308
Weibull	$\hat{\alpha}$ =1.040, λ=36.985	-124.021	250.042
Lognormal	$\hat{\mu}$ =3.0534, λ =1.174	-125.059	252.118
Exponentiated exponential	$\hat{\alpha}$ =0.8126, λ=0.0153	-125.945	253.93
Exponentiated Lognormal	$\hat{\alpha}$ =0.578, $\hat{\mu}$ =3.1836, $\hat{\delta}$ =0.7834	-125.965	257.93
Exponentiated Gumbel	$\hat{\alpha}$ =1.873, λ=45.264	-124.843	251.686
Exponential	$\hat{\lambda}$ =0.0275	-124.0589	250.1177
Kumaraswamy Exponential	$\hat{\alpha}$ =0.13006, $\hat{b}$ =0.9557, $\hat{c}$ =0.00014	-108.7224	223.4449

Conclusion

The probability density function of Kumaraswamy-exponential distribution was discussed and applied for two data sets. In first data set regarding failure times of the conditioning system of an aeroplane. Kumaraswamy exponential provided the best fit followed by exponentiated Gumbel. In second data set regarding runs scored by a cricketer Kumaraswamy exponential, Weibull and exponential distributions provided better fit.

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