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Bayesian Analysis Under Progressively Censored Rayleigh Data

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The one-parameter Rayleigh model is considered as an underlying model for evaluating the properties of Bayes estimator under Progressive Type-II right censored data. The One-Sample Bayes prediction bound length (OSBPBL) is also measured. Based on two different asymmetric loss functions a comparative study presented for Bayes estimation. A simulation study was used to evaluate their comparative properties.

Keywords: Rayleigh model, Bayes estimator, Progressive Type-II right censoring scheme, ISELF, LLF, OSBPBL.

Introduction

The Rayleigh distribution is considered as a useful life distribution. It plays an important role in statistics and operations research. Rayleigh model is applied in several areas such as health, agriculture, biology and physics. It often used in physics, related fields to model processes such as sound and light radiation, wave heights, as well as in communication theory to describe hourly median and instantaneous peak power of received radio signals. The model for frequency of different wind speeds over a year at wind turbine sites and daily average wind speed are considered under the Rayleigh model.

The probability density function and distribution function of Rayleigh distribution are

$$f\left(x;\sigma\right)=\frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right);x\geq 0,\sigma>0 \tag{1}$$

and

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$$F(x;\sigma)=1-\exp\left(-\frac{x^2}{2\sigma^2}\right); x\geq 0, \sigma>0. \tag{2}$$

Here, the parameter σ is known as location parameter. The considered model is useful in life testing experiments, in which age with time as its failure rate and is a linear function of time. The present distribution also plays an important role in communication engineering and electro-vacuum device.

The focus is on measurement of One-Sample Bayes prediction bound length based on Progressive Type-II right censored data. A comparative study of Bayes estimation under two different asymmetric loss functions is presented. For evaluation of performances of the proposed procedures, a simulation study carries out also.

A great deal of literature is available on Rayleigh model under different criterions, such as Sinha (1990), Bhattacharya & Tyagi (1990), Fernandez (2000), Hisada & Arizino (2002), Ali-Mousa & Al–Sagheer (2005), Wu, Chen, and Chen (2006), Kim & Han (2009), Prakash & Prasad (2010), Prakash & Singh (2013). Soliman, Amin, and Abd-El Aziz (2010) presented results on estimation and prediction of inverse Rayleigh distribution based on lower record values. Recently, Prakash (2013) presented Bayes estimators for inverse Rayleigh model. Bayesian analysis for Rayleigh distribution was also discussed by Ahmed, Ahmad, and Reshi (2013).

The progressive Type-II right censoring

The progressive censoring appears to be a great importance in planned duration experiments in reliability studies. In many industrial experiments involving lifetimes of machines or units, experiments have to be terminated early and the number of failures must be limited for various reasons. In addition, some life tests require removal of functioning test specimens to collect degradation related information to failure time data.

Progressive censored sampling is an important method of obtaining data in lifetime studies. Live units removed early on can be readily used in others tests, thereby saving cost to experimenter and a compromise can be achieved between time consumption and the observation of some extreme values. The Progressive Type-II right censoring scheme is describes as follows.

Suppose an experiment in which n independent and identical units $X_1, X_2, \ldots, X_n$ are placed on a life test at the beginning time and first r; $(1 \leq r \leq n)$ failure times are observed. At time of each failure occurring prior to the termination point, one or more surviving units removed from the test. The experiment is terminated at time of r^{th} failure, and all remaining surviving units are removed from the test.

Let $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(r)}$ be the lifetimes of completely observed units to fail and $R_1, R_2, \ldots, R_r$; $(r \leq n)$ are the numbers of units withdrawn at these failure times. Here, $R_1, R_2, \ldots, R_r$; $(r \leq n)$ all are predefined integers follows the relation $R_1 + R_2 + \ldots + R_r + r = n$.

At the first failure time $x_{(1)}$, withdraw R_1 units randomly from remaining $n - 1$ surviving units. Immediately after second observed failure time $x_{(2)}$, R_2 units are withdrawn from remaining $n - 2 - R_1$ surviving units at random, and so on. The experiments continue until at r^{th} failure time x_r, remaining units $R_r = n - r - \sum_{j=1}^{r-1} R_j$ are withdrawn. Here, $X_{1:r:n}^{(R_1, R_2, \ldots, R_r)}, X_{2:r:n}^{(R_1, R_2, \ldots, R_r)}, \ldots, X_{r:r:n}^{(R_1, R_2, \ldots, R_r)}$ be the r ordered failure items and $(R_1, R_2, \ldots, R_r)$ be progressive censoring scheme.

Progressively Type-II right censoring scheme reduces to conventional Type-II censoring scheme when

$$R_i = 0 \forall i = 1, 2, \ldots, r-1 \Rightarrow R_r = n - r$$

and for complete sample case when

$$R_i = 0 \forall i = 1, 2, \ldots, r \Rightarrow n = r.$$

Based on progressively Type-II censoring scheme the joint probability density function of order statistics $X_{1:r:n}^{(R_1, R_2, \ldots, R_r)}, X_{2:r:n}^{(R_1, R_2, \ldots, R_r)}, \ldots, X_{r:r:n}^{(R_1, R_2, \ldots, R_r)}$ is defined as

$$f_{(X_{1:r:n}, X_{2:r:n}, \ldots, X_{r:r:n})}\left(\sigma | \underline{x}\right) = K_p \prod_{i=1}^{r} f\left(x_{(i)}; \sigma\right)\left(1 - F\left(x_{(i)}; \sigma\right)\right)^{R_i}. \tag{3}$$

Here, K_p is called as progressive normalizing constant and is defined as

$$K_p = n\left(n - R_1 - 1\right)\left(n - R_1 - R_2 - 2\right)\ldots\left(n + 1 - \sum_{j=1}^{r-1} R_j - r\right)$$

Progressive Type-II censored sample is denoted by $x \equiv (x_{(1)}, x_{(2)}, \ldots, x_{(r)})$ and $(R_1, R_2, \ldots, R_r)$ being Progressive censoring scheme for the considered model. Simplifying (3)

$$\Rightarrow f_{(X_{1:r:n}, X_{2:r:n}, \ldots, X_{r:r:n})}\left(\sigma \mid \underline{x}\right) = K_p A_r\left(\underline{x}\right)\sigma^{-2r} \exp\left(-\frac{T_r\left(\underline{x}\right)}{\sigma^2}\right); \tag{4}$$

where $A_r\left(\underline{x}\right) = \prod_{i=1}^{r} x_{(i)}$ and $T_r\left(\underline{x}\right) = \frac{1}{2}\sum_{i=1}^{r}\left(1+R_i\right)x_{(i)}^2$.

The Bayes estimation

There is no clear-cut way to determine if one prior probability estimate is better than the other. It is more frequently the case that attention is restricted to a given flexible family of priors, and one is chosen from that family that matches best with personal beliefs. However, there is adequate information about the parameter it should be used; otherwise it is preferable to use the non-informative prior. In present study, the extended Jeffrey's prior proposed by Al-Kutubi & Ibrahim (2009) is considered:

$$\pi(\sigma) \propto \left(I(\sigma)\right)^c ; c \in R^+, I(\sigma) = -nE\left[\frac{\partial^2 \log f(x,\sigma)}{\partial \sigma^2}\right] \tag{5}$$

Thus, the extended Jeffrey's prior for present model is

$$\pi(\sigma) \propto \left(\frac{n}{\sigma^2}\right)^c ; c \in R^+. \tag{6}$$

Based on Bayes theorem, the posterior density is defined as

$$\pi^*\left(\sigma \mid \underline{x}\right) = \frac{f_{(X_{1:r:n}, X_{2:r:n}, \ldots, X_{r:r:n})}\left(\sigma \mid \underline{x}\right) \cdot \pi(\sigma)}{\int_\sigma f_{(X_{1:r:n}, X_{2:r:n}, \ldots, X_{r:r:n})}\left(\sigma \mid \underline{x}\right) \cdot \pi(\sigma)\, d\sigma}. \tag{7}$$

Using (4) and (6) in (7), the posterior density is obtain as

$$\pi^*\left(\sigma|\underline{x}\right) \propto \frac{K_p A_r\left(\underline{x}\right)\sigma^{-2r}\exp\left(-\frac{T_r\left(\underline{x}\right)}{\sigma^2}\right)\cdot\left(\frac{n}{\sigma^2}\right)^c}{\int_\sigma K_p A_r\left(\underline{x}\right)\sigma^{-2r}\exp\left(-\frac{T_r\left(\underline{x}\right)}{\sigma^2}\right)\cdot\left(\frac{n}{\sigma^2}\right)^c d\sigma}$$

$$\Rightarrow \pi^*\left(\sigma|\underline{x}\right) = \frac{2\left(T_r\left(\underline{x}\right)\right)^{r+c-\frac{1}{2}}}{\Gamma\left(r+c-\frac{1}{2}\right)}\sigma^{-2(r+c)}\exp\left(-\frac{T_r\left(\underline{x}\right)}{\sigma^2}\right). \tag{8}$$

The selection of loss function may be crucial in Bayesian analysis. If most commonly used loss function, squared error loss function (SELF) is taken as a measure of inaccuracy, and then the resulting risk is often too sensitive to assumptions about behavior of tail of probability distribution. In Bayesian point of view, SELF is inappropriate in many situations. To overcome this difficulty, a useful asymmetric loss function based on SELF has selected. This asymmetric loss function is known as invariant squared error loss function (ISELF) and is defined for any estimate $\hat{\sigma}$ corresponding to the parameter σ as

$$L\left(\hat{\sigma},\sigma\right) = \left(\sigma^{-1}\partial\right)^2; \partial = \hat{\sigma}-\sigma. \tag{9}$$

The Bayes estimator $\hat{\sigma}_I$ for location parameter σ under ISELF is obtained as

$$\begin{aligned}\hat{\sigma}_1 &= \left[E\left(\sigma^{-1}\right)\right]\left[E\left(\sigma^{-2}\right)\right]^{-1} \\ &= \left[\int_\sigma \sigma^{-1}\pi^*\left(\sigma|\underline{x}\right)d\sigma\right]\left[\int_\sigma \sigma^{-2}\pi^*\left(\sigma|\underline{x}\right)d\sigma\right]^{-1}\end{aligned}$$

$$\Rightarrow \hat{\sigma}_1 = \frac{\Gamma\left(r+c\right)}{\Gamma\left(r+c+\frac{1}{2}\right)}\sqrt{T_r\left(\underline{x}\right)}. \tag{10}$$

Some estimation problems overestimation is more serious than the underestimation, or vice-versa. In addition, there are some cases when the positive and negative errors have different consequences. In such cases, a useful and flexible class of asymmetric loss function (LINEX loss function (LLF)) is defined as

$$L\left(\partial^*\right) = e^{a\partial^*} - a\partial^* - 1; a \neq 0, \partial^* = \left(\sigma^{-1}\partial\right). \tag{11}$$

The shape parameter of LLF is denoted by 'a'. Negative (positive) value of 'a' gives more weight to overestimation (underestimation) and its magnitude reflect the degree of asymmetry. It is seen that, for $a = 1$ the function is quite asymmetric with overestimation being more costly than underestimation. For small values of $|a|$, LLF is almost symmetric and is not far from SELF.

The Bayes estimator $\hat{\sigma}_L$ of location parameter under LLF is obtain by simplifying following equality

$$E\left\{\frac{1}{\sigma}e^{-a\frac{\hat{\sigma}_L}{\sigma}}\right\} = e^a E\left\{\frac{1}{\sigma}\right\}$$

$$\Rightarrow \int_\sigma \sigma^{-1} e^{-a\frac{\hat{\sigma}_L}{\sigma}} \pi^*\left(\sigma|\underline{x}\right) d\sigma = e^a \int_\sigma \sigma^{-1}\pi^*\left(\sigma|\underline{x}\right) d\sigma$$

$$\Rightarrow \int_\sigma \sigma^{-(2r+2c+1)} \exp\left(-\left(\frac{T_r\left(\underline{x}\right)}{\sigma^2} + a\frac{\hat{\sigma}_L}{\sigma}\right)\right) d\sigma = \frac{e^a}{2}\Gamma\left(r+c\right) T_r\left(\underline{x}\right)^{-(r+c)}. \tag{12}$$

A closed form of Bayes estimator $\hat{\sigma}_L$ does not exist. A numerical technique is applied here for obtaining the risk for the Bayes estimator corresponding to their loss.

One-sample Bayes prediction bound length

Consider the nature of future behavior of the observation when sufficient information about the past and the present behavior of an event or an observation is known or given. The Bayesian statistical analysis to predict the future statistic from the considered model is based on the Progressive Type-II right ordered data.

Let $x_{(1)}, x_{(2)}, \ldots, x_{(r)}$ be the first r observed failure units from a sample of size n under the Progressive Type-II right censoring scheme from underlying model (1). If $y \equiv (y_{(1)}, y_{(2)}, \ldots, y_{(s)})$ be the second independent random sample of future observations from same model. Then Bayes predicative density of future observation Y is denoted by $h\left(Y|\underline{x}\right)$ and obtained by simplifying the following relation

$$h\left(Y|\underline{x}\right)=\int_{\sigma} f\left(y;\sigma\right)\cdot\pi^{*}\left(\sigma|\underline{x}\right)d\sigma$$

$$=\frac{2y\left(T_r\left(\underline{x}\right)\right)^{r+c-\frac{1}{2}}}{\Gamma\left(r+c-\frac{1}{2}\right)}\int_{\sigma}\exp\left(-\frac{1}{\sigma^2}\left(\frac{y^2}{2}+T_r\left(\underline{x}\right)\right)\right)\cdot\sigma^{-2(r+c+1)}d\sigma$$

$$\Rightarrow h\left(Y|\underline{x}\right)=\left(r+c-\frac{1}{2}\right)y\left(T_r\left(\underline{x}\right)\right)^{r+c-\frac{1}{2}}\left(T_r\left(\underline{x}\right)+\frac{y^2}{2}\right)^{-r-c-\frac{1}{2}}. \tag{13}$$

Let l_1 and l_2 are the lower and upper Bayes prediction limits for the random variable Y and $1 - \vartheta$ is called the confidence prediction coefficient. Then (l_1, l_2) be the $100(1 - \vartheta)$ % prediction limits for future random variable Y, if

$$\Pr\left(l_1\le Y\le l_2\right)=1-\vartheta. \tag{14}$$

Now, the Central Coverage Bayes Prediction lower and upper limits are obtain by solving following equality

$$\Pr\left(Y\le l_1\right)=\frac{1-\vartheta}{2}=\Pr\left(Y\ge l_2\right). \tag{15}$$

Solving (15), the lower and upper Bayes prediction limits for the future random observation Y are obtain as

$$l_1=\sqrt{\left(2T_r\left(\underline{x}\right)\left(\vartheta^{*}-1\right)\right)} \text{ and } l_2=\sqrt{\left(2T_r\left(\underline{x}\right)\left(\vartheta^{**}-1\right)\right)};$$

where

$$\vartheta^{*}=\left(\frac{1+\vartheta}{2}\right)^{-\lambda},\vartheta^{**}=\left(\frac{1-\vartheta}{2}\right)^{-\lambda} \text{ and } \lambda=\left(r+c-\frac{1}{2}\right)^{-1}.$$

The One-Sample Bayes Prediction bound length under the Central Coverage is obtained as

$$I = l_2 - l_1. \tag{16}$$

Numerical illustration

The procedure is illustrated by presenting a complete analysis under a simulated data set in present section. A comparative study of Bayes estimators based on simulation in terms of risk ratios under Progressively Type-II right censored data is presented as follows:

1) Random values of parameter σ are generated from prior density (6) for selected parametric values of c (= 0, 0.50, 1.50, 2.00, 5.00) and $n = 20$.
2) The value of $c = 0$ is used for Uniform distribution. For the values of $c = 0.50$ and $c = 1.50$ the analysis corresponding to the Jeffrey's prior and Hartigan's prior (Hartigan (1964)) respectively.
3) Using generated values of σ obtained in step (1), generate a Progressively Type-II censored sample of size m form given values of censoring scheme R_i ; $i = 1, 2, \ldots, m$, for considered model, according to an algorithm proposed by Balakrishnan and Aggarwala (2000).
4) The censoring scheme for different values of m is presented in Table 1.
5) The risk ratio of the Bayes estimators are calculated form 1,00,000 generated future ordered samples each of size $n = 20$ of Rayleigh model.
6) For selected values of shape parameter a (= 0.25, 0.50, 1.00, 1.50) of LLF, a risk ratio between the Bayes estimator $\hat{\sigma}_L$ and $\hat{\sigma}_I$ are obtained for considered parametric values and presented in Tables 2-3 under ISELF and LLF respectively.
7) From both tables, note the risk ratios are smaller than unity. This shows that the magnitude of risk with respect to LLF is smaller than the ISELF, when other parameters values considered to be fixed.

8) A decreasing trend has been seen for risk ratio when c increases in both cases. Similar behavior also seen when censoring scheme m changed.
9) Further, it is noted also that the risk ratios tend to be wider as shape parameter 'a' increases when other parametric values are consider to be fixed.
10) The magnitude of risk ratio will be wider for ISELF as compared to LLF when other parametric values considered to be fixed.
11) Further, the magnitude of the risk ratio for both case are robust.

The random samples are generated for One-Sample Bayes Prediction Central Coverage bound length. The procedure and results are as follows.

1) A set of 1,00,000 random samples of size $n = 20$ was drawn from the model for similar set of parametric values as consider earlier in step (1) to (5).
2) For the selected values of level of significance $\vartheta = 99\%, 95\%, 90\%$; the central coverage Bayes prediction lengths of bounds were obtained and presented them in Table 4.
3) It is observed from Table 4 that the Central Coverage Bayes prediction bounds lengths under One–Sample plan tend to be wider as c increases when other parametric values are fixed (except for $c = 5.00$).
4) The bound length expended also, when progressive censoring plan m changed.
5) Note the length of bounds tends to be closer when level of significance ϑ decreases when other parametric values are fixed.
6) The magnitudes of lengths are smaller or nominal. This shows that the central Coverage Bayes prediction criterion is robust.

Table 1. Censoring scheme for different values of m

Case	m	R_i ; $i = 1, 2, \ldots, r$
1	10	1 2 1 0 0 1 2 0 0 0
2	10	1 0 0 3 0 0 1 0 0 1
3	20	1 0 2 0 0 1 0 2 0 0 0 1 0 0 0 1 0 0 1 0

Table 2. Risk ratio between $\hat{\sigma}_L$ and $\hat{\sigma}_I$ under ISELF

$m \downarrow$	$c \downarrow a \rightarrow$	**0.25**	**0.5**	**1**	**1.5**
10	0	0.7765	0.7842	0.7915	0.7988
	0.5	0.7583	0.7659	0.773	0.7802
	1.5	0.7148	0.722	0.7287	0.7354
	2	0.6124	0.6186	0.6243	0.63
	5	0.385	0.3889	0.3925	0.3961
10	0	0.7522	0.7597	0.7668	0.7738
	0.5	0.7346	0.742	0.7488	0.7556
	1.5	0.6924	0.6993	0.7059	0.7123
	2	0.5933	0.5992	0.6049	0.6104
	5	0.373	0.3767	0.3802	0.3837
20	0	0.7288	0.7359	0.7429	0.7496
	0.5	0.7117	0.7187	0.7255	0.7322
	1.5	0.6707	0.6774	0.6838	0.6901
	2	0.5747	0.5803	0.5857	0.5912
	5	0.3613	0.3649	0.3682	0.3717

Table 3. Risk ratio between $\hat{\sigma}_L$ and $\hat{\sigma}_I$ under LLF

$m \downarrow$	$c \downarrow a \rightarrow$	**0.25**	**0.5**	**1**	**1.5**
10	0	0.7741	0.7819	0.7891	0.7964
	0.5	0.7561	0.7636	0.7707	0.7776
	1.5	0.7125	0.7198	0.7265	0.7332
	2	0.6105	0.6166	0.6225	0.6281
	5	0.3838	0.3878	0.3913	0.3948
10	0	0.6748	0.6815	0.6879	0.6941
	0.5	0.659	0.6655	0.6717	0.6778
	1.5	0.6211	0.6273	0.6332	0.6389
	2	0.5321	0.5375	0.5426	0.5475
	5	0.3346	0.3378	0.3411	0.3441
20	0	0.5898	0.5957	0.6013	0.6068
	0.5	0.5759	0.5817	0.5871	0.5926
	1.5	0.5429	0.5483	0.5534	0.5585
	2	0.4651	0.4698	0.4742	0.4785
	5	0.2924	0.2952	0.2981	0.3008

Table 4. One-Sample Central Coverage Bayes Prediction Bound Length

$m \downarrow$	$c \downarrow \vartheta \rightarrow$	**99%**	**95%**	**90%**
10	0	0.4195	0.3246	0.2711
	0.5	0.6243	0.4796	0.4021
	1.5	0.7737	0.5961	0.4988
	2	1.0101	0.7785	0.6516
	5	0.385	0.3839	0.3825
10	0	0.441	0.3409	0.2853
	0.5	0.637	0.4905	0.4115
	1.5	0.7859	0.6062	0.507
	2	1.0193	0.7864	0.6578
	5	0.373	0.3707	0.3682
20	0	0.45	0.3465	0.2901
	0.5	0.6436	0.4958	0.4149
	1.5	0.7899	0.609	0.51
	2	1.0231	0.7885	0.6602
	5	0.3713	0.3699	0.3678

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