Journal of Modern Applied Statistical Methods

Volume 15 | Issue 1 Article 12

5-2016

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Recommended Citation

Kibria, B M Golam and Banik, Shipra (2016) "Some Ridge Regression Estimators and Their Performances," *Journal of Modern Applied Statistical Methods*: Vol. 15 : Iss. 1 , Article 12.
DOI: 10.22237/jmasm/1462075860

Journal of Modern Applied Statistical Methods
May 2016, Vol. 15, No. 1, 206-238.

ISSN 1538 − 9472

Some Ridge Regression Estimators and Their Performances

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The estimation of ridge parameter is an important problem in the ridge regression method, which is widely used to solve multicollinearity problem. A comprehensive study on 28 different available estimators and five proposed ridge estimators, KB1, KB2, KB3, KB4, and KB5, is provided. A simulation study was conducted and selected estimators were compared. Some of selected ridge estimators performed well compared to the ordinary least square (OLS) estimator and some existing popular ridge estimators. One of the proposed estimators, KB3, performed the best. Numerical examples were given.

Keywords: Linear regression, mean square error, multicollinearity, ridge regression, simulation study

Introduction

Applied researchers are often concerned about models specification under consideration, especially with regards to problems associated with errors. Models specification can be due to omission of one or several relevant variables, inclusion of unnecessary explanatory variables, wrong functional forms, autocorrelation etc. However, for modeling data, there are other problems that also might influence results. This problem occurs in situations when explanatory variables are highly inter-correlated. In practice, there may be strong or near strong linear relationship exist among explanatory variables. Thus, independence assumption of explanatory variables is no longer valid, which causes problem of multicollinearity. In the presence of multicollinearity, the OLS estimator could become unstable due to their large variance, which leads to poor prediction and wrong inference about model parameters. Empirically, problem of multicollinearity can be observed, for example, in cement production, when amount of different compounds in clinkers is

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regressed on the heat evolved of cement (See Muniz and Kibria (2009) for details). Another possible example, when a researcher is interested to predict cholesterol level of patients based on some predictors: age, body weight, blood pressure, food intake and stress causes multicollinearity. In the presence of this noise of the model, regression coefficients may be statistically insignificant or have wrong sign or have large sampling variance that may result in wide confidence interval for individual parameters. With these errors, it is very difficult to make valid statistical inferences and appropriate prediction. Therefore, resolve multicollinearity problem is a serious issue for the linear regression practitioners.

Problem of multicollinearity can be solved by various methods, namely to collect additional data, reselecting variables, principle component regression methods, re-parameterizing the model, ridge regression method, and others. In this paper, we will consider the most widely used ridge regression method. The concept of ridge regression was first proposed by Hoerl and Kennard (1970) to handle multicollinearity problem for engineering data. They found that there is a nonzero value of k (ridge parameter) for which mean squared error (MSE) for the ridge regression estimator is smaller than variance of the ordinary least squares (OLS) estimator. Many authors at different period of times worked in this area of research and developed and proposed different estimators for k. To mention a few, Hoerl and Kennard (1970), Hoerl, Kennard, and Baldwin (1975), McDonald and Galarneau (1975), Lawless and Wang (1976), Dempster, Schatzoff, and Wermuth (1977), Gibbons (1981), Kibria (2003), Khalaf and Shukur (2005), Alkhamisi and Shukur (2008), Muniz and Kibria (2009), Gruber (2010), Muniz, Kibria, Mansson, and Shukur (2012), Mansson, Shukur, and Kibria (2010), and very recently Hefnawy and Farag (2013), Aslam (2014), and Arashi and Valizadeh (2015), among others. Since aforementioned ridge regression estimators are considered by several researchers at different times and under different simulation conditions, they are not comparable as a whole. The objective of this article is to do a comprehensive study on 28 different ridge estimators those are available in literature and compare them based on minimum MSE criterion. Investigation has been carried out using a Monte Carlo simulation. A number of different models have been studied where variance of the random error, correlation among explanatory variables, sample size and unknown coefficient vector were varied. The organization of the paper is as follows. We first review the available methods for estimating k, followed by a Monte Carlo simulation study. Some applications have then been considered and, finally, some concluding remarks are presented.

Statistical Methodology

Ridge Regression Estimators

To describe the ridge regression, consider following multiple linear regression model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e} \tag{1}$$

where $\mathbf{y}$ is an $n \times 1$ vector of observations, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown regression coefficients, $\mathbf{X}$ is an $n \times p$ observed matrix of the regression, and $\mathbf{e}$ is an $n \times 1$ vector of random errors which is distributed as multivariate normal with mean $\mathbf{0}$ and covariance matrix $\sigma^2\mathbf{I}_n$, $\mathbf{I}_n$ being an identity matrix of order n. The OLS estimator of $\boldsymbol{\beta}$ is obtained as

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X'X}\right)^{-1}\mathbf{X'y}$$

and covariance matrix of $\hat{\boldsymbol{\beta}}$ is obtained as $\mathrm{Cov}\left(\hat{\boldsymbol{\beta}}\right) = \sigma^2\left(\mathbf{X'X}\right)^{-1}$. It is easy to see that both $\hat{\boldsymbol{\beta}}$ and $\mathrm{Cov}\left(\hat{\boldsymbol{\beta}}\right)$ are heavily depend on characteristics of the matrix $\mathbf{X'X}$. The standard regression model assumes that regressors are nearly independent. However, in many practical situations (e.g. engineering in particular (Hoerl & Kennard, 1970)), often find that regressors are nearly dependent. In that case, the matrix $\mathbf{X'X}$ becomes ill conditioned (i.e. $\det(\mathbf{X'X}) \approx 0$). If $\mathbf{X'X}$ is ill conditioned, then $\hat{\boldsymbol{\beta}}$ is sensitive to a number of errors and therefore meaningful statistical inference becomes very difficult for practitioners. To overcome this problem, Hoerl and Kennard (1970) suggested a small positive number to be added to diagonal elements of the matrix $\mathbf{X'X}$. Thus resulting estimators are obtained as

$$\begin{aligned}\hat{\boldsymbol{\beta}}(k) &= \left(\mathbf{X'X} + k\mathbf{I}_p\right)^{-1}\mathbf{X'y} \\ &= \mathbf{W}\hat{\boldsymbol{\beta}}\end{aligned} \tag{2}$$

where $\mathbf{W} = [\mathbf{I}_p + k\mathbf{C}^{-1}]^{-1}$, $k \geq 0$, $\mathbf{C} = \mathbf{X'X}$, and $\mathbf{I}_p$ is an identity matrix of order p. This is known as the ridge regression estimator. Since the quantity $[\mathbf{X'X} + k\mathbf{I}_p]$ in (2) is always invertible, there always exist a unique solution for $\hat{\boldsymbol{\beta}}(k)$. The ridge regression estimator is a biased estimator and, for a positive value of k, this estimator provides a smaller MSE compared to the OLS estimator. From (2), we observe that as $k \to 0$, $\hat{\boldsymbol{\beta}}(k) \to \hat{\boldsymbol{\beta}}$, and as $k \to \infty$, $\hat{\boldsymbol{\beta}}(k) \to 0$.

The bias, variance matrix, and MSE expression of $\hat{\boldsymbol{\beta}}(k)$ are respectively given as follows:

$$\text{Bias} = \text{E}\left(\hat{\boldsymbol{\beta}}(k)\right) - \boldsymbol{\beta} = -k\mathbf{C}^{-1}(k)\boldsymbol{\beta}$$

$$\text{V}\left(\hat{\boldsymbol{\beta}}(k)\right) = \sigma^2\left(\mathbf{W}\mathbf{C}^{-1}\mathbf{W}'\right)$$

$$\text{MSE}\left(\hat{\boldsymbol{\beta}}(k)\right) = \sigma^2 \,\text{tr}\left(\mathbf{W}\mathbf{C}^{-1}\mathbf{W}'\right) + k^2\boldsymbol{\beta}'C^{-2}(k)\boldsymbol{\beta}$$

where $\mathbf{C}(k) = [\mathbf{C} + k\mathbf{I}_p]$.

The parameter k is known as the “biased” or “ridge” parameter and it must be estimated using real data. Most of recent efforts in the area of multicollinearity and ridge regression estimators have concentrated on estimating the value of k. We will review statistical methodology used to analyze the estimation of k in the next section.

Estimation of Ridge Parameter *k*

Suppose there exists an orthogonal matrix $\mathbf{D}$ such that $\mathbf{D'CD} = \boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$ contains eigenvalues of the matrix $\mathbf{X'X}$. The orthogonal version of (1) is

$$\mathbf{y} = \mathbf{X}^*\boldsymbol{\alpha} + \mathbf{e} \tag{3}$$

where $\mathbf{X}^* = \mathbf{XD}$ and $\boldsymbol{\alpha} = \mathbf{D'}\boldsymbol{\beta}$. Then the generalized ridge regression estimator is given as

$$\hat{\boldsymbol{\alpha}}(k) = \left(\mathbf{X}^{*\prime}\mathbf{X}^* + \mathbf{K}\right)^{-1}\mathbf{X}^{*\prime}\mathbf{y}, \quad k \geq 0 \tag{4}$$

where $\mathbf{K} = \text{diag}(k_1, k_2, \ldots, k_p)$, $k_i > 0$ and $\hat{\boldsymbol{\alpha}}(k) = \boldsymbol{\Lambda}^{-1}\mathbf{X}^{*\prime}\mathbf{y}$ is the OLS estimators of $\boldsymbol{\alpha}$.

It follows from Hoerl and Kennard (1970) that k_i minimizes $\text{MSE}\left(\hat{\boldsymbol{\alpha}}(k)\right)$, which is defined as

$$MSE(\hat{\alpha}(\hat{k})) = \sigma^2 \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + \hat{k}_i)^2} + \sum_{i=1}^{p} \frac{\hat{k}_i^2 \hat{\alpha}_i^2}{(\lambda_i + \hat{k}_i)^2} \tag{5}$$

where the λ_i are eigenvalues of the matrix $\mathbf{X'X}$, $\hat{\alpha}_i$ is the i^{th} element of $\hat{\boldsymbol{\alpha}}$, and

$$\hat{k}_i = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} \hat{e}_i^2}{n-p}$$

$$\hat{e}_i = y_i - \mathbf{X}_j{}'\hat{\boldsymbol{\alpha}}_i$$

Now we will review available methods in literature to estimate the value of k. Hoerl and Kennard (1970) suggested k to be (denoted here by $\hat{k}_{\text{HK}}$)

$$\hat{k}_{\text{HK}} = \frac{\hat{\sigma}^2}{\hat{\alpha}_{\max}^2} \tag{6}$$

where $\hat{\alpha}_{\max}$ is the maximum element of $\hat{\boldsymbol{\alpha}}$. Hoerl and Kennard claimed that (6) gives smaller MSE than the OLS method.

Hoerl et al. (1975) proposed k to be (denoted here by $\hat{k}_{\text{HKB}}$)

$$\hat{k}_{\text{HKB}} = \frac{p\hat{\sigma}^2}{\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\alpha}}} \tag{7}$$

Lawless and Wang (1976) suggested k to be (denoted here by $\hat{k}_{\text{LW}}$)

$$\hat{k}_{\text{LW}} = \frac{p\sigma^2}{\hat{\boldsymbol{\alpha}}'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\alpha}}} \tag{8}$$

Hocking, Speed, and Lynn (1979) suggested k to be (denoted here by $\hat{k}_{\text{HSL}}$)

$$\hat{k}_{\text{HSL}} = \hat{\sigma}^2 \frac{\sum_{i=1}^{p} \left(\lambda_i \hat{\alpha}_i\right)^2}{\left(\sum_{i=1}^{p} \lambda_i \hat{\alpha}_i^2\right)^2} \tag{9}$$

Kibria (2003) proposed the following estimators for k based on arithmetic mean (AM), geometric mean (GM), and median of $\hat{\sigma}^2 / \hat{\alpha}_i^2$. These are defined as follows: The estimator based on AM (denoted by $\hat{k}_{\text{AM}}$)

$$\hat{k}_{\text{AM}} = \frac{1}{p} \sum_{i=1}^{p} \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} \tag{10}$$

The estimator based on GM (denoted by $\hat{k}_{\text{GM}}$)

$$\hat{k}_{\text{GM}} = \frac{\hat{\sigma}^2}{\left(\prod_{i=1}^{p} \hat{\alpha}_i^2\right)^{\frac{1}{p}}} \tag{11}$$

The estimator based on median (denoted by $\hat{k}_{\text{MED}}$)

$$Median \left\{ \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} \right\}, \quad i = 1, 2, \ldots, p \tag{12}$$

Based on modification of $\hat{k}_{\text{HK}}$, Khalaf and Shukur (2005) suggested k to be (denoted by $\hat{k}_{\text{KS}}$)

$$\hat{k}_{\text{KS}} = \frac{\lambda_{\max} \hat{\sigma}^2}{\left(n - p\right) \hat{\sigma}^2 + \lambda_{\max} \hat{\alpha}_{\max}^2} \tag{13}$$

where $\lambda_{\max}$ is the maximum eigenvalue of the matrix **X'X**.

Following Kibria (2003) and Khalaf and Shukur (2005), Alkhamisi, Khalaf, and Shukur (2006) proposed the following three estimators of k:

$$\hat{k}_{\text{arith}}^{\text{KS}} = \frac{1}{p}\sum_{i=1}^{p}\frac{\lambda_i\hat{\sigma}_i^2}{(n-p)\hat{\sigma}_i^2+\lambda_i\hat{\alpha}_i^2} \tag{14}$$

$$\hat{k}_{\text{max}}^{\text{KS}} = \max\left(\frac{\lambda_i\hat{\sigma}_i^2}{(n-p)\hat{\sigma}_i^2+\lambda_i\hat{\alpha}_i^2}\right) \tag{15}$$

$$\hat{k}_{\text{md}}^{\text{KS}} = \text{median}\left(\frac{\lambda_i\hat{\sigma}_i^2}{(n-p)\hat{\sigma}_i^2+\lambda_i\hat{\alpha}_i^2}\right) \tag{16}$$

Applying algorithm of GM and square root to Khalaf and Shukur (2005), Kibria (2003), and Alkhamisi et. al (2006), Muniz and Kibria (2009) proposed the following seven estimators of k:

$$\hat{k}_{\text{gm}}^{\text{KS}} = \prod_{i=1}^{p}\left(\frac{\lambda_i\hat{\sigma}_i^2}{(n-p)\hat{\sigma}_i^2+\lambda_i\hat{\alpha}_i^2}\right)^{\frac{1}{p}} \tag{17}$$

$$\hat{k}_{\text{KM2}} = \max\left(\frac{1}{\sqrt{\hat{\sigma}^2/\hat{\alpha}_i^2}}\right) \tag{18}$$

$$\hat{k}_{\text{KM3}} = \max\left(\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}}\right) \tag{19}$$

$$\hat{k}_{\text{KM4}} = \left(\prod_{i=1}^{p}\frac{1}{\sqrt{\hat{\sigma}^2/\hat{\alpha}_i^2}}\right)^{\frac{1}{p}} \tag{20}$$

$$\hat{k}_{\text{KM5}} = \left(\prod_{i=1}^{p}\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\right)^{\frac{1}{p}} \tag{21}$$

$$\hat{k}_{\text{KM6}} = \text{median}\left(\frac{1}{\sqrt{\hat{\sigma}^2/\hat{\alpha}_i^2}}\right) \tag{22}$$

$$\hat{k}_{\text{KM7}} = \text{median}\left(\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}}\right) \tag{23}$$

Following Alkhamisi and Shukur (2008) and based square root transformations, Muniz et al. (2012) proposed the following five estimators of k:

$$\hat{k}_{\text{KM8}} = \max\left(\frac{1}{q_i}\right) \tag{24}$$

$$\hat{k}_{\text{KM9}} = \max\left(q_i\right) \tag{25}$$

$$\hat{k}_{\text{KM10}} = \left(\prod_{i=1}^{p}\frac{1}{q_i}\right)^{\frac{1}{p}} \tag{26}$$

$$\hat{k}_{\text{KM11}} = \left(\prod_{i=1}^{p} q_i\right)^{\frac{1}{p}} \tag{27}$$

$$\hat{k}_{\text{KM12}} = \text{median}\left(\frac{1}{q_i}\right) \tag{28}$$

where $q_i = \dfrac{\lambda_{\max}\hat{\sigma}^2}{\left(n-p\right)\hat{\sigma}^2 + \lambda_{\max}\hat{\alpha}_i^2}$.

Khalaf (2012), based on modification of $\hat{k}_{\text{HK}}$, proposed k to be (denoted by $\hat{k}_{\text{GK}}$)

$$\hat{k}_{\text{GK}} = \hat{k}_{\text{HK}} + \frac{2}{\left(\lambda_{\max} + \lambda_{\min}\right)'} \tag{29}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of the matrix **X'X**, respectively.

Nomura (1988) suggested k to be (denoted by $\hat{k}_{\text{HMO}}$)

$$\hat{k}_{\text{HMO}} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\left[\hat{\alpha}_i^2 \Big/ 1+\left(1+\lambda_i\left(\frac{\hat{\alpha}_i^2}{\hat{\sigma}^2}\right)^{\frac{1}{2}}\right)\right]} \tag{30}$$

Dorugade and Kashid (2010), based on (7), suggested k to be (denoted by $\hat{k}_{\text{D}}$)

$$\hat{k}_{\text{D}} = \max\left(0, \hat{k}_{\text{HKB}} - \frac{1}{n\left(\text{VIF}_i\right)_{\max}}\right) \tag{31}$$

where $\text{VIF}_i = 1\big/1-R_i^2$, $i = 1, 2,\ldots, p$ is variance inflation factor of the i^{th} regressor and R_i^2 is the coefficient of determination for the regression of X_i on other covariates, $X_1, X_2,\ldots, X_i, X_{i+1},\ldots, X_p$ (a regression equation without response variable).

Crouse, Jin, and Hanumara (1995), for $k > 0$ and using unbiased ridge regression (URR) estimator $(k, \mathbf{J}) = (\mathbf{X'X} + k\mathbf{I}_p)^{-1}(\mathbf{X'y} + \mathbf{J}k)$, $k \geq 0$, where $\mathbf{J} \sim \left(\boldsymbol{\beta}, \frac{\sigma^2}{k}\right)$, proposed k to be (denoted by $\hat{k}_{\text{CJH}}$)

$$\hat{k}_{\text{CJH}} = \begin{cases} \dfrac{p\hat{\sigma}^2}{\left(\hat{\boldsymbol{\beta}}_{\text{OLS}}-\mathbf{J}\right)'\left(\hat{\boldsymbol{\beta}}_{\text{OLS}}-\mathbf{J}\right)} & \text{if } \left(\hat{\boldsymbol{\beta}}_{\text{OLS}}-\mathbf{J}\right)'\left(\hat{\boldsymbol{\beta}}_{\text{OLS}}-\mathbf{J}\right) > \hat{\sigma}^2\,\text{tr}\left(\mathbf{X'X}\right)^{-1} \\ \dfrac{p\hat{\sigma}^2}{\left(\hat{\boldsymbol{\beta}}_{\text{OLS}}-\mathbf{J}\right)'\left(\hat{\boldsymbol{\beta}}_{\text{OLS}}-\mathbf{J}\right)-\hat{\sigma}^2\,\text{tr}\left(\mathbf{X'X}\right)^{-1}} & \text{otherwise} \end{cases} \tag{32}$$

Batah and Gore (2009), using modified URR (known as MUR) estimator for $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta}_{\mathbf{J}}(k) = \left[\mathbf{I} - k\left(\mathbf{X'X}+k\mathbf{I}_p\right)^{-1}\right]\left[\left(\mathbf{X'X}+k\mathbf{I}_p\right)^{-1}\left(\mathbf{X'y}+k\mathbf{J}\right)\right],$$

suggested k to be (denoted by $\hat{k}_{\text{FG}}$)

$$\hat{k}_{\text{FG}} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\left[\hat{\alpha}_i^2 \Big/ \left(\frac{\hat{\alpha}_i^4\lambda_i^2}{4\hat{\sigma}^4} + \frac{6\hat{\alpha}_i^2\lambda_i}{\hat{\sigma}^2}\right)^{\frac{1}{2}} - \frac{6\hat{\alpha}_i^2\lambda_i}{\hat{\sigma}^2}\right]} \tag{33}$$

In the next section, we evaluated 28 different ridge estimators that are defined in equations (6) to (33) to know which estimators show better performances under our simulation study flowchart.

The Monte Carlo Simulation

The aim of this study is to compare the performance of different ridge estimators and find some good estimators for practitioners. Because a theoretical comparison is not possible, a simulation study has been conducted using MATLAB 8.0. The design of this simulation study depends on what factors are expected to affect properties of estimators under investigation and what criteria are being used to judge results. Because the degree of collinearity among explanatory variables (*X*s) is of central importance, we followed Kibria (2003) in generating *X*s using the following equation:

$$X_{ij} = \left(1-\gamma^2\right)^{\frac{1}{2}} z_{ij} + \gamma z_{ip}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, p \tag{34}$$

where z_{ij} are independent standard normal pseudo-random numbers and γ represents correlation between any two *X*s. These variables are standardized so that **X'X** and **X'y** are in correlation forms. The *n* observations for **y** are determined by the following equation:

$$y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_p X_{ip} + e_i, \text{i} = 1, 2, \ldots, \text{n} \tag{35}$$

where the e_i are i.i.d. N(0, σ^2) and, without loss of any generality, we will assume zero intercept for (35).

Correlation Coefficient, Sample Size, and Replications

A number of factors such as γ, n, σ, and number of replications can affect properties of the estimators. Since our objective is to compare performance of estimators according to the strength of multicollinearity, we used different degrees of correlation between variables and let γ=0.70, 0.80, and 0.90. Eigenvalues and eigenvectors of the correlation matrix indicate the degree of multicollinearity. One of the possible widely used estimators to measure the strength of multicollinearity called condition number (Vinod & Uallh, 1981) is defined as follows

$$\kappa = \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \tag{36}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and the smallest eigenvalues of the matrix **X'X**, respectively. If $\lambda_{\min} = 0$, then κ is infinite, which means perfect multicollinearity among *X*s. If $\lambda_{\max} = \lambda_{\min}$, then $\kappa = 1$ and the *X*s are said to be orthogonal. Large values of κ indicate serious multicollinearity. Usually, a κ between 30 and 100 indicates a moderate to strong correlation, and a κ greater than 100 suggests severe multicollinearity. An eigenvalue that approaches 0 indicates a very strong linear dependency between *X*s.

Because a purpose of the study is to see the effect of n on the performance of the estimators, $n = 20$ and $n = 50$ were considered. The number of *X*s is also of great importance since the bad impact of the collinearity on MSE might be stronger when there are more *X*s in the model. Also, $p = 5$ is used in our study. To see whether the magnitude of σ has a significant effect on the performance of the proposed estimators, we used $\sigma = 0.01, 0.5, 1.0$, and 5.0. For each set of *X*s, we selected coefficients $\beta_1, \beta_2, \ldots, \beta_p$ as normalized eigenvectors corresponding to the largest eigenvalue of the matrix **X'X** subject to constraint $\boldsymbol{\beta}'\boldsymbol{\beta} = 1$. Thus, for n, p, $\boldsymbol{\beta}$, λ, γ, and σ, sets of *X*s are generated. Then the experiment was repeated 5000 times by generating new error terms. Values of k of different selected estimators and average MSEs are estimated and presented them in Tables 5 to 10. In these tables, average k was calculated for ridge estimators and the proportion of replications for which OLS estimators produce a smaller MSE than selected ridge regression estimators and are presented in parenthesis.

Results

Performance as a Function of *σ*

In Tables 5 to 10, the MSEs of selected estimators are provided as a function of σ. To understand very clearly for $\gamma = 0.70$ and $n = 20$, performance of estimators as a function of σ is provided in Figure 1. From results, we observed as σ increases, MSEs also increases. Also for smaller σ (e.g. $\sigma = 0.1$), performances of selected estimators do not differ greatly. It is noticeable that all ridge estimators have smaller MSE than the OLS estimator except $\sigma = 0.1$. The performance of the GM, KM2, KM3, KM4, KM5, KM6, KM7, KM8, KM9, KM10, KM11, HMO, and FG estimators are better compared to the rest of estimators.

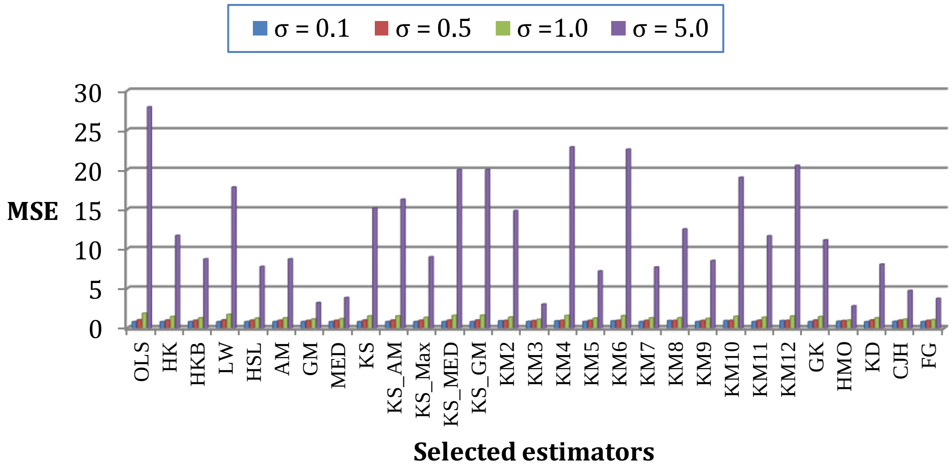

Figure 1. Performance of estimators as a function of σ

However, when σ is large (e.g. $\sigma = 5.0$), the GM, MED, KM3, HMO, CJH, and FG estimators outperform all other estimators in the sense of smaller MSE (see Figure 1). A significant increase in MSEs were observed when a shifting from $\sigma = 1.0$ to $\sigma = 5.0$.

Performance as a Function of *γ*

MSEs of selected estimators were also analyzed as a function of γ for selected values of n, p, and σ. These results are available on request from the authors. For a clear understanding, for ($\sigma = 1$, $n = 20$) and ($\sigma = 5$, $n = 50$), performances of estimators are provided in Figure 2 and Figure 3, respectively. It is clear that, as γ increases, the MSEs also increase (see Figures 2 and 3). When γ increases (see Figure 3), higher correlation between Xs resulted in an increase of MSEs of ridge estimators. In general, HSL, GM, MED, KS_Max, KM2, KM3, KM5, KM8, KM9, HMO, and FG performed better than other estimators.

Performance as a Function of *n*

MSEs of selected estimators were evaluated as a function of *n*, for which tabulated results are available from the authors on request. For given $\gamma = 0.8$, $p = 5$, performances of estimators as a function of *n* for $\sigma = 1$ and $\sigma = 5$ are provided in Figure 4 and Figure 5, respectively. We observed that, as *n* increases, MSEs decrease and the performance of estimators do not vary significantly. An important change has been observed in MSEs when σ shifts from 1 to 5. We observed that, in general, when *n* increases, MSEs decrease, which is true for large values of γ and σ. Performance of estimators does not vary greatly for small values of σ and γ.

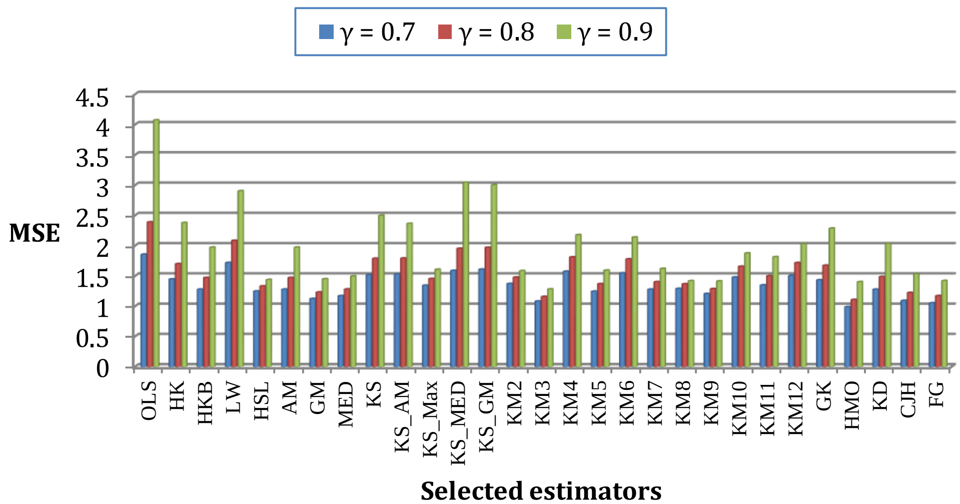

Figure 2. Performance of estimators as a function of γ for $\sigma = 1$ and $n = 20$

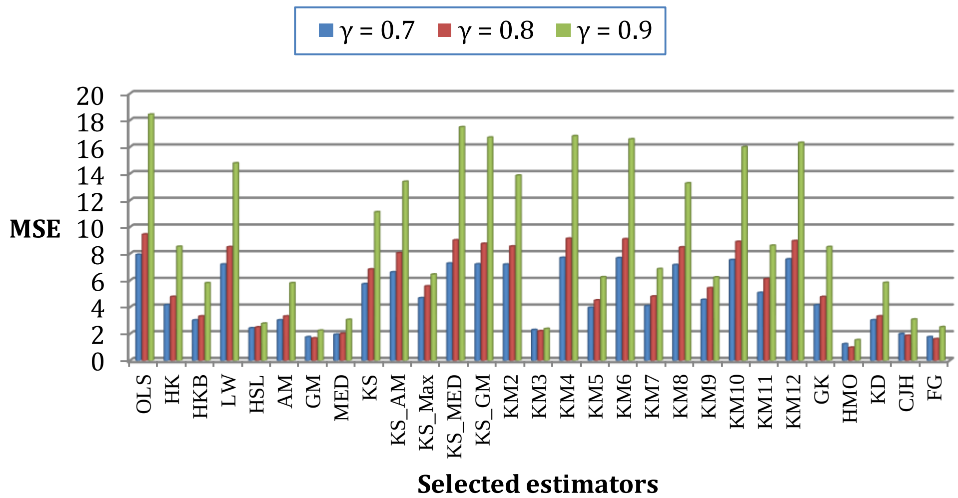

Figure 3. Performance of estimators as a function of γ for $\sigma = 5$ and $n = 50$

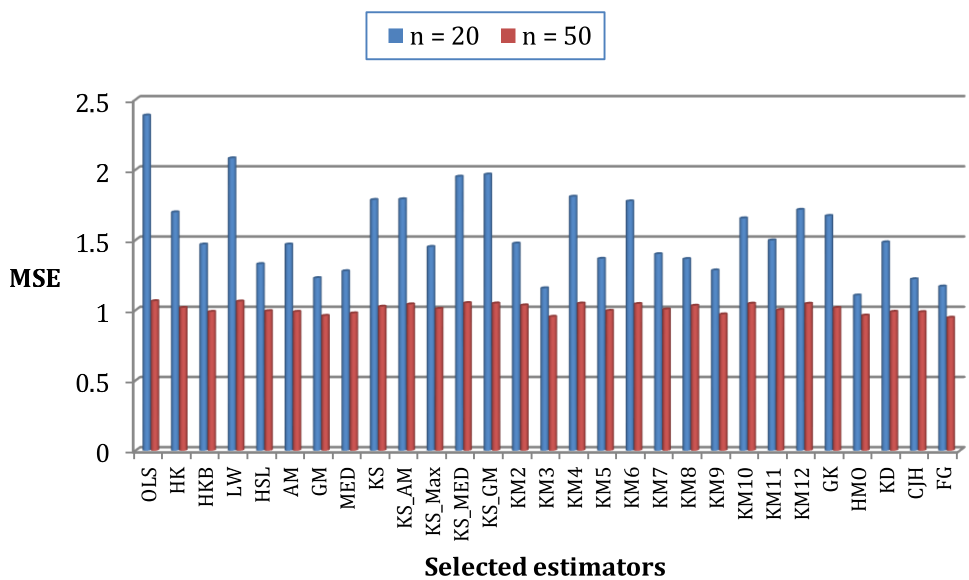

Figure 4. Performance of estimators as a function of n for $\gamma = 0.8$ and $\sigma = 1.0$

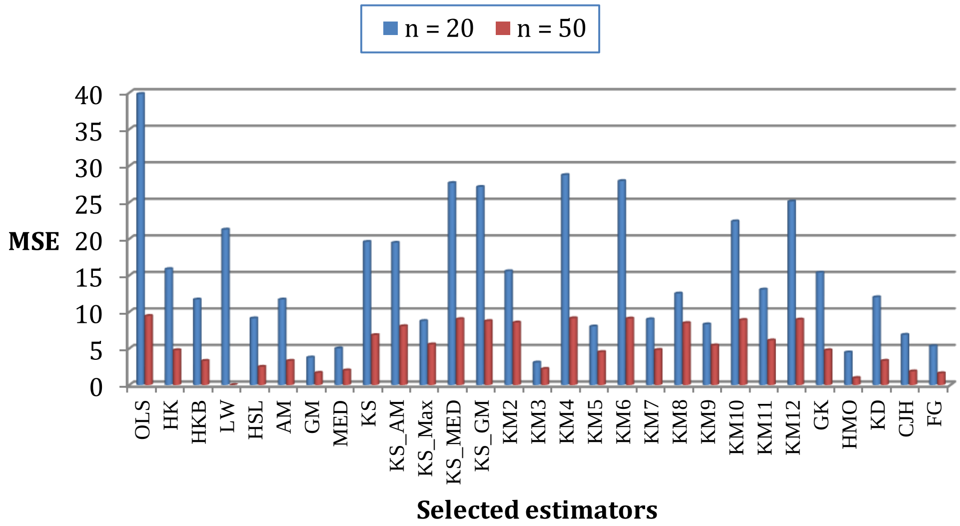

Figure 5. Performance of estimators as a function of *n* for *γ* = 0.8 and *σ* = 5.0

Some Proposed Ridge Estimators

Based on the above, the following five new estimators of *k* are proposed:

1. KB1 = Arithmetic mean of (GM, MED, KM3, HMO, CJH, FG)
2. KB2 = Median(GM, MED, KM3, HMO, CJH, FG)
3. KB3 = Max(GM, MED, KM3, HMO, CJH, FG)
4. KB4 = Geometric mean of (GM, MED, KM3, HMO, CJH, FG)
5. KB5 = Harmonic mean of (GM, MED, KM3, HMO, CJH, FG)

MSEs values for n = 10, 20, and 30, γ = 0.9, and p = 5 are reported for σ = 3 and σ = 10 in Table A7 and Table A8, respectively, for 28 selected existing estimators and our proposed 5 ridge estimators. For better understanding, MSEs are plotted in Figures 6 and 7. It appears from these results that all proposed estimators are performing well under some conditions. However, proposed KB3 performed the best followed by KB1 (See Figures 6 and 7).

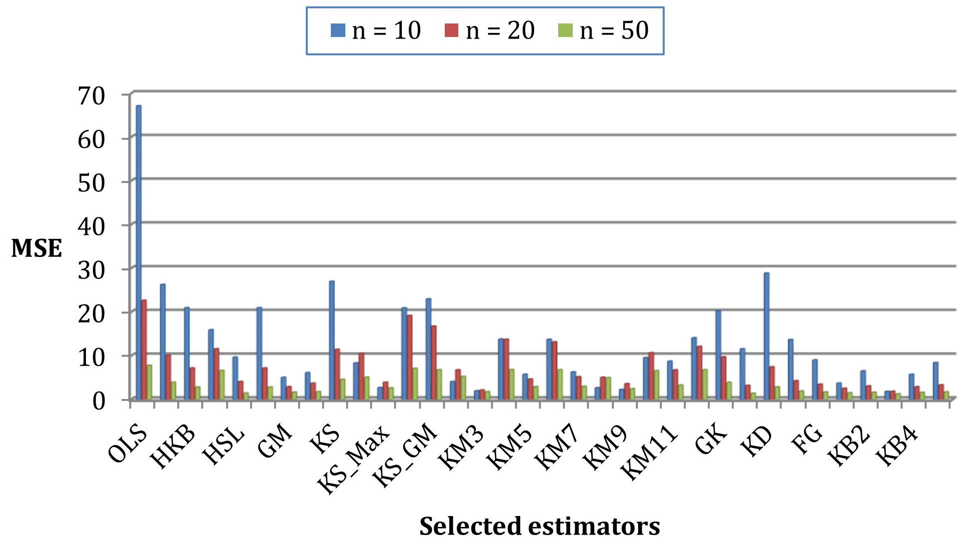

Figure 6. Performance of estimators as a function of *n* for $\gamma = 0.9$ and $\sigma = 3.0$

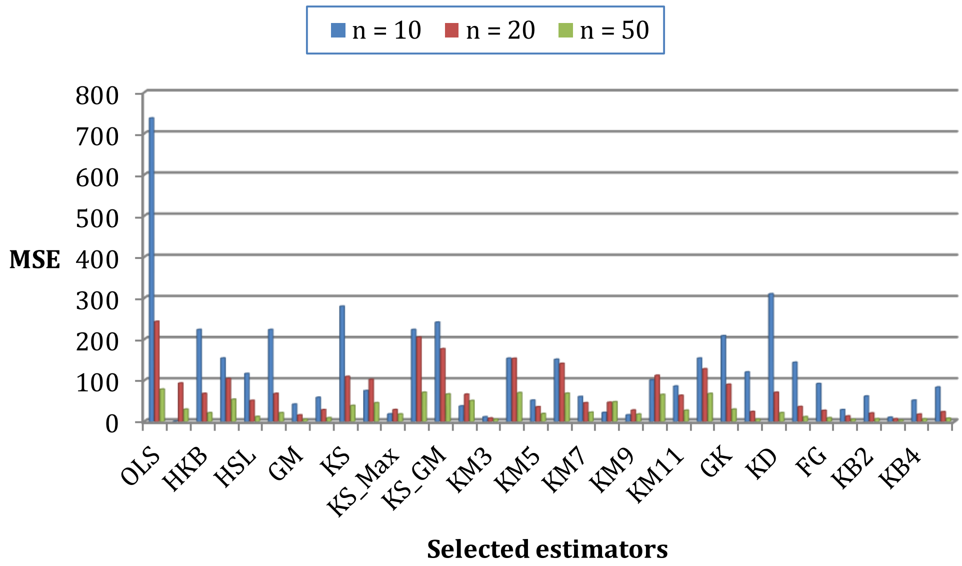

Figure 7. Performance of estimators as a function of *n* for $\gamma = 0.9$ and $\sigma = 10.0$

Application

Example 1

Consider an example which has been taken from Pasha and Shah (2004) to compare the performances of the selected estimators. The following regression model is considered:

$$y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \beta_4 X_{i4} + \beta_5 X_{i5} + e_i,\ i = 1, 2, \ldots n \tag{37}$$

where y_i = number of persons employed (million), X_{i1} = land cultivated (million hectares), X_{i2} = inflation rate (%), X_{i3} = number of establishments, X_{i4} = population (million), X_{i5} = literacy rate (%), and n=28. For details about the data set, see Pasha and Shah (2004).

Table 1. Correclations among exclamatory variables

	X_{i1}	X_{i2}	X_{i3}	X_{i4}	X_{i5}	y_i
X_{i1}	1.0000	0.6573	0.9427	0.9761	0.9564	0.9731
X_{i2}	0.6573	1.0000	0.6232	0.7062	0.6905	0.6926
X_{i3}	0.9427	0.6232	1.0000	0.9633	0.8672	0.9437
X_{i4}	0.9761	0.7062	0.9633	1.0000	0.9506	0.9930
X_{i5}	0.9564	0.6905	0.8672	0.9506	1.0000	0.9572
y_i	0.9731	0.6926	0.9437	0.9930	0.9572	1.0000

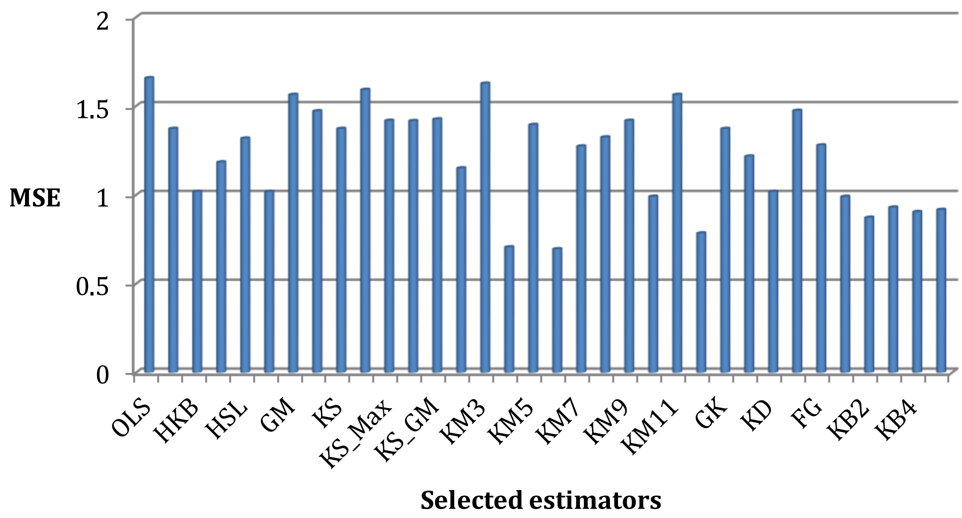

Figure 8. MSE of selected ridge estimators

The correlation matrix of Xs in (37) is presented in Table 1. It is observed that the Xs are highly correlated. Moreover, κ = 38115.32, which implies the existence of multicollinearity in the data set. So it is adequate to compare proposed ridge estimators with the real data set. Estimated MSEs along with ridge regression coefficients are presented in Table 2 and, for a better presentation, MSEs are plotted in Figure 8.

The MSE of estimators is estimated by

$$\mathrm{MSE}\left(\hat{\boldsymbol{\beta}}\right) = \sigma^2 \sum_{i=1}^{p} \frac{\lambda_i}{\left(\lambda_i + \hat{k}\right)^2} + \sum_{i=1}^{p} \frac{\hat{k}_i \beta_i^2}{\left(\lambda_i + \hat{k}_i\right)^2} \tag{38}$$

where $\hat{k}$ is one of $\hat{k}_{\mathrm{HK}}, \hat{k}_{\mathrm{HKB}}, \ldots, \hat{k}_{\mathrm{KB5}}$, and other terms are explained in (5). It is evident from Table 2 and Figure 8 that all ridge estimators perform better than the OLS estimator. However, HKB, AM, KM4, KM6, KM10, KM12, KD, and our five proposed estimators are performing better as compared to other ridge estimators.

Table 2. MSE and estimated ridge regression coefficients of the estimators

Estimators	MSE	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$
OLS	1.6578	-1.2600	0.3123	-0.0623	-0.2276	0.0068
HK	1.3726	-0.0969	0.2812	-0.0604	-0.2272	0.0069
HKB	1.0182	-0.2889	0.3029	-0.0618	-0.2275	0.0069
LW	1.1841	-1.2598	0.3123	-0.0623	-0.2276	0.0069
HSL	1.3180	-0.0001	0.0013	-0.0009	-0.0547	0.0068
AM	1.0182	-0.2889	0.3029	-0.0618	-0.2275	0.0068
GM	1.5638	-0.0181	0.1913	-0.0526	-0.2256	0.0067
MED	1.4714	-0.0534	0.2584	-0.0588	-0.2269	0.0066
KS	1.3726	-0.0969	0.2812	-0.0604	-0.2272	0.0063
KS_AM	1.5918	-0.0003	0.0066	-0.0043	-0.1394	0.0064
KS_MAX	1.4179	-0.0001	0.0010	-0.0009	-0.0547	0.0068
KS_MED	1.4164	-0.0772	0.2737	-0.0599	-0.2271	0.0066
KS_GM	1.4258	-0.0730	0.2716	-0.0597	-0.2271	0.0068
KM2	1.1505	-0.2088	0.2984	-0.0615	-0.2274	0.0060
KM3	1.6268	-0.0049	0.0924	-0.0368	-0.2204	0.0068
KM4	0.7061	-0.8279	0.3108	-0.0622	-0.2275	0.0059
KM5	1.3945	-0.0870	0.2777	-0.0602	-0.2272	0.0068
KM6	0.6951	-0.6600	0.3097	-0.0622	-0.2275	0.0066
KM7	1.2737	-0.1441	0.2915	-0.0611	-0.2273	0.0063
KM8	1.3244	-0.1194	0.2870	-0.0608	-0.2273	0.0068
KM9	1.4179	-0.0001	0.0013	-0.0009	-0.0547	0.0067
KM10	0.9905	-1.1420	0.3120	-0.0623	-0.2275	0.0066
KM11	1.5638	-0.0181	0.1913	-0.0526	-0.2256	0.0069
KM12	0.7842	-0.9608	0.3114	-0.0623	-0.2275	0.0068
GK	1.3726	-0.0969	0.2812	-0.0604	-0.2272	0.0068
HMO	1.2170	-0.1729	0.2952	-0.0613	-0.2274	0.0070
KD	1.0182	-0.2889	0.3029	-0.0618	-0.2275	0.0062
CJH	1.4740	-0.0523	0.2574	-0.0587	-0.2269	0.0064
FG	1.2795	-0.1412	0.2910	-0.0610	-0.2273	0.0069
KB1	0.9902	-0.0193	0.1959	-0.0531	-0.2257	0.0065
KB2	0.8727	-0.0528	0.2579	-0.0587	-0.2269	0.0068
KB3	0.9296	-0.0049	0.0926	-0.0368	-0.2204	0.0066
KB4	0.9045	-0.0438	0.2486	-0.0580	-0.2268	0.0062
KB5	0.9165	-0.0771	0.2736	-0.0599	-0.2271	0.0068

Example 2

Consider the data set on total national research and development expenditures as a percent of gross national product originally due to Gruber (1998) and later by Akdeniz and Erol (2003), among others. The regression model is defined as

$$y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \beta_4 X_{i4} + e_i,\ i = 1, 2, \ldots n \tag{39}$$

where y = percent spent by United States, X_1 = percent spent by France, X_2 = percent spent by West Germany, X_3 = percent spent by Japan, and X_4 = percent spent by the Soviet Union. The correlation matrix of Xs in (39) is tabulated in Table 3. We found that the Xs are highly correlated. Moreover, κ = 93.6823 implies the existence of multicollinearity in the data set so it is reasonable to evaluate proposed ridge estimators with the real data set. Estimated MSEs along with regression coefficients are tabulated in Table 4 and, for a better presentation, MSEs are presented in Figure 9. It is evident from Table 4 and Figure 9 that all ridge estimators outperformed the OLS estimator. However, all ridge estimators except KM2, KM3, KM4, KM5, KM6, KM7, KM8, KM10, and KM12 have smaller MSE than the OLS estimator.

Table 3. Correlations among the variables.

	X_{i1}	X_{i2}	X_{i3}	X_{i4}	y_i
X_{i1}	1.0000	0.8877	0.9248	0.3090	0.9776
X_{i2}	0.8877	1.0000	0.9621	0.1573	0.9080
X_{i3}	0.9248	0.9621	1.0000	0.3276	0.9565
X_{i4}	0.3090	0.1573	0.3276	1.0000	0.3482
y_i	0.9776	0.9080	0.9565	0.3482	1.0000

Table 4. MSEs and the estimated ridge regression coefficients of the estimators

Estimators	MSE	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$
OLS	1.2595	-0.1623	0.4616	0.1733	0.4462
HK	0.0628	-0.1343	0.3975	0.1716	0.4462
HKB	0.0588	-0.1177	0.3569	0.1702	0.4462
LW	0.0804	-0.1618	0.4605	0.1733	0.4462
HSL	0.0622	-0.1327	0.3937	0.1715	0.4462
AM	0.0588	-0.1177	0.3569	0.1702	0.4462
GM	0.0602	-0.1026	0.3181	0.1686	0.4462
MED	0.0588	-0.1170	0.3534	0.1702	0.4462
KS	0.0628	-0.1344	0.3975	0.1716	0.4462
KS_AM	0.0588	-0.1176	0.3566	0.1702	0.4462
KS_MAX	0.0724	-0.0798	0.2557	0.1652	0.4462
KS_MED	0.0638	-0.1367	0.4031	0.1718	0.4462
KS_GM	0.0613	-0.1302	0.3875	0.1713	0.4462
KM2	0.2646	-0.0005	0.0017	0.0101	0.4295
KM3	0.1765	-0.0203	0.0720	0.1302	0.4459
KM4	0.2613	-0.0008	0.0029	0.0162	0.4361
KM5	0.1463	-0.0315	0.1104	0.1451	0.4460
KM6	0.2621	-0.0007	0.0026	0.0146	0.4349
KM7	0.1523	-0.0293	0.1022	0.1426	0.4460
KM8	0.2887	-0.0006	0.0001	0.0009	0.3057
KM9	0.0931	-0.0605	0.2005	0.1605	0.4462
KM10	0.2721	-0.0001	0.0004	0.0025	0.3833
KM11	0.0602	-0.1026	0.3181	0.1686	0.4462
KM12	0.2764	-0.0001	0.0003	0.0017	0.3578
GK	0.0588	-0.1160	0.3525	0.1701	0.4462
HMO	0.0726	-0.0794	0.2552	0.1651	0.4462
KD	0.0643	-0.1377	0.4056	0.1718	0.4462
CJH	0.0713	-0.0810	0.2597	0.1654	0.4462
FG	0.0896	-0.0632	0.2084	0.1613	0.4462
KB1	0.0705	-0.0554	0.1851	0.1588	0.4461
KB2	0.0819	-0.0802	0.2574	0.1653	0.4462
KB3	0.0765	-0.0203	0.0720	0.1302	0.4459
KB4	0.0843	-0.0747	0.2420	0.1642	0.4460
KB5	0.0867	-0.0876	0.2780	0.1666	0.4462

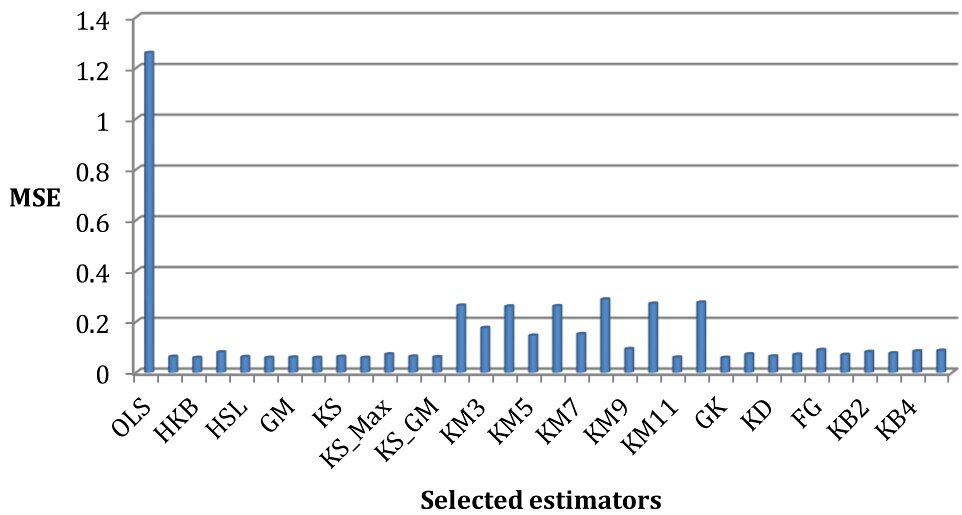

Figure 9. MSE of selected ridge estimators.

Conclusions

Based on our simulation results, the following conclusions can be drawn: As σ increases, MSE have a negative effect, meaning that MSE increases. As γ increases, MSE also increases. When n increases, MSE decreases even when γ and σ are large. In all situations, all ridge estimators have smaller MSE than the OLS estimator. When $\sigma = 5.0$, GM, KM3, MED, KMO, CJH, and FG outperformed all other estimators in the sense of producing smaller MSE. Two real life examples have been studied. Based on the results of simulations and numerical examples, estimators HSL, AM, GM, MED, KS_MAX, KM2, KM3, KM5, KM8, KM9, KMO, CJH, FG, and proposed KB1, KB2, KB3, KB4, and KB5 performed better than the rest in the sense of small MSE and may be recommended to practitioners.

Acknowledgements

This paper is dedicated to all who sacrificed themselves during the liberation war that started on March 26, 1971 and ended on December 16, 1971 to bring out the freedom of our beautiful Bangladesh.

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Appendix

Table A1. Simulated MSE, average *k*s and proportion of time (%) LSE perform better than ridge estimators for $n = 20$, $p = 5$, and $\gamma = 0.7$. Condition number $\kappa = 26.53$

Estimator	$\sigma = 0.1$	$\sigma = 0.5$	$\sigma = 1.0$	$\sigma = 5.0$
OLS	0.8001	1.0294	1.8532	27.9287
HK	0.8017 (0.0275, 96.3)	0.9957 (0.5307, 37.6)	1.4446 (1.5525, 18.80)	11.3570 (82.1698, 0.08)
HKB	0.8031 (0.0504, 96.3)	0.9720 (1.1334, 39.6)	1.2765 (3.4790, 20.60)	8.7257 (9.7970, 0.08)
LW	0.8023 (0.0025, 96.3)	1.0218 (0.0620, 35.3)	1.7174 (0.2358, 16.60)	17.8015 (1.4258, 0.16)
HSL	0.8018 (0.0282, 96.3)	0.9759 (0.7261, 39.5)	1.2452 (2.9035, 21.60)	7.7684 (12.0529, 0.08)
AM	0.8031 (0.0504, 96.3)	0.9720 (1.1334, 39.6)	1.2765 (3.4790, 20.60)	8.7257 (9.7970, 0.08)
GM	0.8036 (0.0605, 96.3)	0.9488 (2.3970, 41.4)	1.1223 (12.4590, 20.60)	3.2072 (46.3951, 0.08)
MED	0.8034 (0.0582, 96.3)	0.9600 (1.7633, 40.5)	1.1681 (10.4330, 22.30)	3.8485 (43.8037, 0.08)
KS	0.8017 (0.0272, 96.3)	0.9985 (0.4358, 37.3)	1.5160 (0.9018, 21.80)	15.1509 (1.5806, 0.20)
KS_AM	0.8035 (0.0588, 96.3)	0.9904 (0.4064, 37.8)	1.5265 (0.6275, 18.40)	16.2535 (0.9005, 0.20)
KS_MAX	0.8076 (0.1326, 96.3)	0.9708 (0.7990, 39.8)	1.3412 (1.4414, 17.70)	8.9894 (2.6993, 0.20)
KS_MED	0.8026 (0.0435, 96.3)	0.9922 (0.3735, 37.6)	1.5860 (0.4658, 19.80)	19.9875 (0.4988, 0.16)
KS_GM	0.8030 (0.0490, 96.3)	0.9975 (0.3156, 37.4)	1.6066 (0.4259, 17.40)	20.005 (0.5052, 0.16)
KM2	0.9044 (6.3585, 96.3)	0.9481 (1.5384, 43.7)	1.3699 (1.0418, 17.20)	14.8189 (0.8429, 0.12)
KM3	0.8200 (0.3931, 92.6)	0.9308 (9.4316, 45.0)	1.0795 (217.0360, 18.50)	3.0143 (76.7550, 0.08)
KM4	0.8922 (4.3038, 95.9)	0.9680 (0.7983, 40.6)	1.5727 (0.4285, 22.40)	22.8690 (0.2390, 0.12)
KM5	0.8131 (0.2414, 92.9)	0.9550 (1.4246, 41.2)	1.2430 (2.9300, 17.60)	7.2046 (5.5596, 0.16)
KM6	0.8934 (4.4212, 96.1)	0.9629 (0.9310, 41.3)	1.5454 (0.5069, 20.70)	22.5850 (0.2727, 0.12)
KM7	0.8128 (0.2366, 92.8)	0.9608 (0.2220, 40.8)	1.2757 (2.4513, 17.48)	7.6965 (5.0834, 0.16)
KM8	0.9338 (42.3390, 96.1)	0.9416 (12.9320, 46.8)	1.2885 (1.6238, 20.60)	12.5045 (1.2550, 0.12)
KM9	0.8083 (0.1521, 92.2)	0.9422 (2.1465, 43.6)	1.2042 (2.6788, 18.90)	8.5190 (2.9051, 0.20)
KM10	0.9277 (19.6900, 96.3)	0.9556 (1.2124, 42.5)	1.4762 (0.6735, 21.00)	19.0340 (0.5044, 0.12)
KM11	0.8035 (0.0587, 92.5)	0.9704 (0.9319, 39.5)	1.3472 (1.6037, 17.78)	11.6359 (2.0797, 0.20)
KM12	0.9287 (20.7930, 96.3)	0.9554 (1.3005, 42.5)	1.5019 (0.6313, 19.60)	20.5370 (0.4299, 0.16)
GK	0.8042 (0.0705, 96.3)	0.9923 (0.5737, 37.7)	1.4313 (1.5955, 17.60)	11.1205 (82.2120, 0.08)
HMO	0.8456 (1.2774, 95.2)	0.9255 (8.9203, 43.7)	0.9871 (18.1095, 22.80)	2.7934 (28.6862, 0.20)
KD	0.7967 (0.0077, 45.2)	1.0030 (1.0936, 38.8)	1.2757 (3.4679, 20.72)	8.0586 (9.8342, 0.16)
CJH	0.8338 (0.7991, 96.4)	0.9738 (24.1710, 45.8)	1.0909 (23.3341, 23.80)	4.7328 (28.8315, 0.20)
FG	0.8116 (0.2832, 96.5)	0.9483 (4.2218, 42.5)	1.0504 (9.0792, 22.64)	3.7383 (15.1262, 0.16)

Table A2. Simulated MSE, average *k*s and proportion of time (%) LSE perform better than ridge estimators for n = 20, p = 5, and γ = 0.8. Condition number κ = 44.43

Estimator	σ = 0.1	σ = 0.5	σ = 1.0	σ = 5.0
OLS	0.8456	1.2497	2.3874	39.8284
HK	0.8489 (0.0293, 97.8)	1.1623 (0.5070, 34.4)	1.6979 (1.4373, 14.6)	15.8668 (539.9400, 0.08)
HKB	0.8513 (0.0500, 97.8)	1.1153 (1.0862, 36.6)	1.4686 (3.0389, 15.9)	11.6959 (7.2387, 0.12)
LW	0.8459 (0.0026, 97.8)	1.2279 (0.0647, 32.2)	2.0825 (0.2464, 13.2)	21.2712 (1.4494, 0.08)
HSL	0.8490 (0.0299, 97.7)	1.1160 (0.7767, 37.0)	1.3292 (3.1169, 17.0)	9.1223 (11.3958, 0.16)
AM	0.8513 (0.0500, 97.8)	1.1153 (1.0862, 36.6)	1.4686 (3.0389, 15.9)	11.6959 (7.2387, 0.12)
GM	0.8520 (0.0571, 97.7)	1.0750 (2.5046, 38.2)	1.2290 (9.8068, 17.0)	3.7616 (36.1258, 0.16)
MED	0.8518 (0.0547, 97.8)	1.0927 (1.8202, 37.2)	1.2786 (7.0498, 16.6)	5.0127 (33.3226, 0.16)
KS	0.8489 (0.0290, 97.8)	1.1694 (0.4212, 34.2)	1.7861 (0.8497, 14.4)	19.5872 (1.5105, 0.08)
KS_AM	0.8512 (0.0494, 97.8)	1.1639 (0.3346, 34.6)	1.7905 (0.5518, 13.8)	19.4642 (0.8356, 0.08)
KS_MAX	0.8565 (0.0987, 97.6)	1.1211 (0.6671, 36.9)	1.4515 (1.5492, 15.9)	8.7602 (2.9129, 0.08)
KS_MED	0.8500 (0.0380, 97.8)	1.1720 (0.2890, 34.1)	1.9516 (0.3353, 13.6)	27.6417 (0.3535, 0.04)
KS_GM	0.8506 (0.0437, 97.8)	1.1816 (0.2461, 33.9)	1.9671 (0.3243, 13.5)	27.1016 (0.3818, 0.04)
KM2	0.9848 (6.1657, 93.6)	1.0671 (1.6257, 40.6)	1.4754 (1.1924, 15.0)	15.5751 (0.9925, 0.04)
KM3	0.8786 (0.3496, 96.9)	1.0371 (10.4610, 40.2)	1.1570 (29.6414, 17.7)	3.0807 (114.9800, 0.12)
KM4	0.9756 (4.4381, 93.8)	1.1038 (0.8180, 37.4)	1.8091 (0.4621, 13.8)	28.7349 (0.2695, 0.04)
KM5	0.8697 (0.2345, 97.2)	1.0834 (1.4215, 38.2)	1.3671 (2.6928, 16.6)	8.0125 (4.9376, 0.08)
KM6	0.9766 (4.5711, 93.7)	1.0946 (0.9472, 38.2)	1.7758 (0.5336, 14.2)	27.9144 (0.3165, 0.04)
KM7	0.8693 (0.2289, 97.2)	1.0927 (1.2106, 37.6)	1.3998 (2.2861, 16.1)	8.9963 (4.4526, 0.08)
KM8	0.9855 (39.8268, 89.9)	1.0499 (3.2641, 43.2)	1.3653 (2.0516, 15.7)	12.5204 (1.6050, 0.04)
KM9	0.8582 (0.1235, 97.2)	1.0554 (2.2557, 39.6)	1.2844 (2.8385, 16.6)	8.3068 (3.1120, 0.08)
KM10	0.9948 (20.8949, 92.1)	1.0804 (1.2369, 39.4)	1.6549 (0.6999, 14.3)	22.3960 (0.5245, 0.04)
KM11	0.8519 (0.0558, 97.7)	1.1111 (0.9332, 36.9)	1.4979 (1.5784, 15.6)	13.0713 (2.0425, 0.08)
KM12	0.9946 (22.2324, 92.0)	1.0793 (1.3071, 39.7)	1.7157 (0.6401, 14.3)	25.1129 (0.4413, 0.04)
GK	0.8534 (0.0697, 97.6)	1.1548 (0.5474, 34.8)	1.6721 (1.4778, 14.6)	15.3714 (539.9800, 0.08)
HMO	0.9096 (0.8998, 95.6)	1.0082 (6.6626, 39.4)	1.1057 (12.5907, 17.1)	4.4618 (18.1221, 0.16)
KD	0.8465 (0.0077, 45.3)	1.1209 (1.0368, 36.6)	1.4844 (2.9895, 15.9)	12.0123 (7.1893, 0.12)
CJH	0.9308 (2.3565, 96.6)	1.0571 (17.7803, 41.5)	1.2212 (15.4247, 17.4)	6.8747 (19.0881, 0.16)
FG	0.8732 (0.2763, 97.2)	1.0367 (3.6773, 39.4)	1.1695 (7.1088, 17.1)	5.3217 (10.4628, 0.12)

Table A3. Simulated MSE, average *k*s and proportion of time (%) LSE perform better than ridge estimators for n = 20, p = 5, and γ = 0.9. Condition number κ = 99.57

Estimator	σ = 0.1	σ = 0.5	σ = 1.0	σ = 5.0
OLS	0.9252	1.7372	4.072	78.5675
HK	0.9337 (0.0312, 93.84)	1.4437 (0.4401, 25.60)	2.3761 (0.4630, 8.16)	29.8253 (1.0378, 0.08)
HKB	0.9383 (0.0498, 93.76)	1.3376 (0.9440, 27.76)	1.9708 (1.6720, 8.96)	20.9941 (0.0039, 0.04)
LW	0.9260 (0.0030, 93.64)	1.6249 (0.0734, 23.40)	2.9017 (0.1812, 8.08)	26.8522 (0.0015, 0.08)
HSL	0.9339 (0.0332, 93.84)	1.2837 (0.8665, 29.40)	1.4359 (0.1949, 9.84)	12.2574 (0.0102, 0.04)
AM	0.9383 (0.0498, 93.76)	1.3376 (0.9440, 27.76)	1.9708 (1.5685, 8.96)	20.9941 (0.0039, 0.04)
GM	0.9392 (0.0565, 93.40)	1.2386 (2.7928, 30.20)	1.4474 (31.5700, 9.32)	05.0707 (0.0212, 0.04)
MED	0.9392 (0.0532, 93.60)	1.2723 (1.8752, 29.76)	1.4933 (31.5700, 9.52)	07.0164 (0.0203, 0.04)
KS	0.9336 (0.0309, 93.84)	1.4599 (0.3692, 25.68)	2.4969 (0.5450, 8.20)	33.5740 (0.0012, 0.08)
KS_AM	0.9358 (0.0309, 93.84)	1.4577 (0.2530, 26.08)	2.3629 (2.3302, 8.24)	25.4480 (0.7430, 0.04)
KS_MAX	0.9435 (0.0398, 93.92)	1.3101 (0.6745, 28.72)	1.6055 (0.6007, 9.16)	8.1732 (3.0535, 0.08)
KS_MED	0.9345 (0.0715, 93.84)	1.5204 (0.1668, 24.80)	3.0406 (2.0300, 7.84)	51.505 (0.1871, 0.04)
KS_GM	0.9348 (0.0345, 93.84)	1.5345 (0.1538, 24.72)	3.0023 (3.4393, 8.00)	48.3841 (0.0002, 0.04)
KM2	1.0903 (0.0359, 88.28)	1.1898 (1.9181, 32.64)	1.5825 (2.8401, 9.20)	14.5482 (0.0014, 0.04)
KM3	0.9840 (6.0050, 92.96)	1.1547 (58.5970, 32.12)	1.2792 (0.8296, 9.56)	3.0370 (0.0543, 0.04)
KM4	1.0865 (0.3674, 88.56)	1.2609 (0.8602, 29.72)	2.1760 (1.3919, 8.28)	38.6266 (0.0003, 0.04)
KM5	0.9740 (4.5079, 93.80)	1.2399 (1.4093, 30.68)	1.5905 (0.7200, 9.16)	09.6299 (0.0038, 0.08)
KM6	1.0868 (0.2322, 88.52)	1.2409 (1.0000, 30.24)	2.1364 (1.2528, 8.52)	37.4332 (0.0004, 0.04)
KM7	0.9737 (4.6621, 93.92)	1.2581 (1.1738, 30.36)	1.6190 (6.2880, 9.24)	10.9557 (0.0036, 0.08)
KM8	1.0492 (0.2254, 80.16)	1.1564 (4.6383, 34.44)	1.4168 (2.1108, 9.88)	10.2354 (0.0030, 0.04)
KM9	0.9473 (37.9604, 93.40)	1.1851 (2.3532, 32.00)	1.4116 (13.8060, 9.44)	7.7777 (0.0032, 0.08)
KM10	1.0730 (0.1355, 84.16)	1.2115 (2.3532, 31.20)	1.8736 (4.2749, 8.56)	25.5669 (0.0006, 0.04)
KM11	0.9391 (0.0549, 93.44)	1.3040 (1.3710, 28.84)	1.8124 (3.4567, 9.20)	16.4846 (0.0018, 0.08)
KM12	1.0709 (23.184, 83.84)	1.2097 (0.8781, 30.80)	2.0275 (1.2345, 8.48)	32.1960 (0.0005, 0.04)
GK	0.9433 (0.0705, 93.92)	1.4161 (1.4160, 25.72)	2.2847 (2.4356, 8.20)	27.8409 (1.0378, 0.08)
HMO	1.0005 (0.5086, 92.45)	1.1512 (0.4794, 26.54)	1.3997 (0.5679, 8.76)	9.2514 (8.0253, 0.04)
KD	0.9275 (0.0079, 91.98)	1.3581 (0.8946, 25.78)	2.0319 (0.9879, 8.92)	22.2854 (3.8341, 0.04)
CJH	1.0653 (11.7230, 92.34)	1.2061 (12.8190, 27.89)	1.5307 (13.5460, 8.76)	12.0446 (30.1750, 0.04)
FG	0.9777 (0.2594, 91.23)	1.1753 (2.6850, 27.92)	1.4181 (2.8790, 9.12)	8.9451 (5.4004, 0.08)

Table A4. Simulated MSE, average *k*s and proportion of time (%) LSE perform better than ridge estimators for n = 50, p = 5, and γ = 0.7. Condition number κ = 28.37

Estimator	σ = 0.1	σ = 0.5	σ = 1.0	σ = 5.0
OLS	0.7067	0.7181	0.9257	7.9302
HK	0.7069 (0.0240, 91.44)	0.7022 (0.5570, 8.12)	0.8447 (2.1865, 1.60)	4.1563 (44.5510, 1.24)
HKB	0.7071 (0.0501, 91.80)	0.6898 (1.2241, 9.64)	0.7865 (4.6033, 2.36)	3.0189 (28.4550, 1.32)
LW	0.7067 (0.0005, 90.80)	0.7178 (0.0108, 7.04)	0.9235 (0.0472, 1.48)	7.2028 (0.9094, 0.48)
HSL	0.7069 (0.0242, 91.44)	0.7016 (0.5657, 8.12)	0.8247 (2.5638, 1.48)	2.4236 (39.4080, 1.60)
AM	0.7071 (0.0501, 91.80)	0.6898 (1.2241, 9.64)	0.7865 (2.5638, 1.64)	3.0189 (28.4550, 1.32)
GM	0.7071 (0.0571, 91.88)	0.6806 (2.0736, 11.84)	0.7195 (4.6033, 2.36)	1.7582 (118.3500, 1.52)
MED	0.7072 (0.0641, 92.04)	0.6868 (1.6400, 10.96)	0.7675 (10.2230, 3.28)	1.9448 (106.1700, 1.64)
KS	0.7069 (0.0238, 91.44)	0.7036 (0.5006, 8.08)	0.8660 (7.0966, 3.08)	5.7339 (2.9518, 0.56)
KS_AM	0.7071 (0.0530, 91.88)	0.7066 (0.3863, 7.84)	0.8934 (1.4835, 1.56)	6.6141 (1.1676, 0.40)
KS_MAX	0.7073 (0.0796, 92.56)	0.7024 (0.5342, 8.08)	0.8549 (0.7058, 1.52)	4.6770 (4.1927, 0.56)
KS_MED	0.7071 (0.0492, 91.76)	0.7056 (0.4169, 7.84)	0.9030 (1.6563, 1.56)	7.2753 (0.5269, 0.36)
KS_GM	0.7071 (0.0487, 91.76)	0.7073 (0.3631, 7.80)	0.8991 (0.4985, 1.48)	7.2165 (0.5824, 0.36)
KM2	0.7959 (6.5729, 100.00)	0.6831 (1.3668, 9.56)	0.8911 (0.7070, 1.52)	7.1954 (0.4214, 0.36)
KM3	0.7091 (0.3012, 94.68)	0.6676 (7.2693, 15.12)	0.7073 (23.8640, 1.48)	2.2938 (137.6900, 1.40)
KM4	0.7636 (4.2624, 100.00)	0.6952 (0.7612, 8.16)	0.9061 (0.3795, 3.48)	7.7002 (0.1448, 0.36)
KM5	0.7086 (0.2375, 94.12)	0.6858 (0.3872, 9.76)	0.8209 (2.9563, 1.48)	3.9628 (8.9179, 0.92)
KM6	0.7603 (4.0412, 100.00)	0.6927 (0.8443, 8.28)	0.9021 (0.4517, 2.00)	7.6841 (0.1637, 0.36)
KM7	0.7087 (0.2513, 94.24)	0.6889 (1.2445, 9.76)	0.8376 (2.4653, 1.48)	4.0883 (8.0111, 0.92)
KM8	0.9818 (43.9325, 100.00)	0.6709 (2.0870, 11.20)	0.8897 (0.7168, 1.96)	7.1626 (0.4352, 0.36)
KM9	0.7074 (0.0901, 92.76)	0.6632 (3.1074, 13.52)	0.7776 (4.2204, 1.48)	4.5522 (4.4955, 0.56)
KM10	0.9030 (18.6343, 100.00)	0.6913 (0.8901, 8.24)	0.9045 (0.4149, 2.12)	7.5370 (0.2732, 0.36)
KM11	0.7071 (0.0563, 91.88)	0.6901 (1.1899, 9.56)	0.8317 (2.4992, 1.48)	5.0763 (3.7189, 0.56)
KM12	0.8911 (16.8071, 100.00)	0.6894 (0.9386, 8.28)	0.9036 (0.4220, 1.88)	7.5913 (0.2505, 0.36)
GK	0.7069 (0.0333, 91.60)	0.7020 (0.5652, 8.12)	0.8444 (2.1949, 1.48)	4.1521 (44.5610, 1.24)
HMO	0.7656 (4.3850,100.00)	0.7364 (32.7499, 58.40)	0.5435 (81.3278, 1.60)	1.2348 (138.5700, 1.60)
KD	0.7069 (0.0301, 91.48)	0.6903 (1.2042, 9.64)	0.7870 (4.5830, 11.16)	3.0229 (28.4350, 1.32)
CJH	0.7144 (0.8160, 97.96)	0.6944 (70.5420, 45.00)	0.5912 (147.4000, 10.28)	1.9891 (114.2100, 1.48)
FG	0.7091 (0.2956, 94.60)	0.6517 (05.9959, 20.80)	0.6153 (18.3570, 4.60)	1.7616 (49.5940, 1.48)

Table A5. Simulated MSE, average *k*s and proportion of time (%) LSE perform better than ridge estimators for n = 50, p = 5, and γ = 0.8. Condition number κ = 50.12

Estimator	σ = 0.1	σ = 0.5	σ = 1.0	σ = 5.0
OLS	0.7754	0.8161	1.0648	9.4533
HK	0.7754 (0.0266, 60.32)	0.8087 (0.6000, 38.56)	1.01750 (2.1083, 23.56)	4.77240 (352.2000, 0.36)
HKB	0.7755 (0.0501, 61.40)	0.8076 (1.2116, 42.52)	0.98930 (4.4460, 26.12)	3.31000 (22.5420, 0.56)
LW	0.7754 (0.0005, 59.48)	0.8159 (0.0111, 34.12)	1.06300 (0.0484, 19.44)	8.50309 (0.9530, 0.20)
HSL	0.7754 (0.0267, 60.32)	0.8070 (0.6291, 38.68)	0.99420 (2.7644, 24.20)	2.49500 (32.8540, 0.48)
AM	0.7755 (0.0501, 61.40)	0.8076 (1.2116, 42.52)	0.98930 (4.4467, 26.12)	3.31000 (22.5490, 0.56)
GM	0.7755 (0.0548, 61.56)	0.8120 (1.9840, 46.04)	0.96060 (10.7210, 29.36)	1.66830 (91.4920, 0.76)
MED	0.7756 (0.0614, 61.72)	0.8095 (1.5500, 44.16)	0.97850 (7.0245, 27.20)	2.01050 (72.2960, 0.68)
KS	0.7754 (0.0265, 60.32)	0.8091 (0.5407, 38.2)	1.02619 (1.4740, 22.08)	6.83390 (2.5615, 0.20)
KS_AM	0.7755 (0.0470, 61.28)	0.8108 (0.3207, 36.40)	1.04240 (0.6254, 20.60)	8.05320 (0.9630, 0.16)
KS_MAX	0.7755 (0.0683, 62.12)	0.8077 (0.5641, 38.30)	1.01080 (1.8020, 22.48)	5.56520 (3.6691, 0.20)
KS_MED	0.7755 (0.0445, 61.12)	0.8110 (0.3127, 36.30)	1.05170 (0.3672, 19.96)	9.01690 (0.2741, 0.16)
KS_GM	0.7755 (0.0443, 61.16)	0.8114 (0.2841, 36.00)	1.04860 (0.4421, 20.16)	8.75150 (0.4528, 0.16)
KM2	0.8822 (6.2317, 100.00)	0.8017 (1.3228, 43.20)	1.03520 (0.7341, 20.56)	8.55220 (0.4577, 0.16)
KM3	0.7767 (0.2857, 70.48)	0.8252 (10.1870, 50.80)	0.95400 (46.5690, 29.72)	2.20640 (328.1800, 0.48)
KM4	0.8494 (4.3497,100.00)	0.8033 (0.7904, 40.24)	1.04830 (0.3872, 20.00)	9.13830 (0.1613, 0.16)
KM5	0.7764 (0.2328, 68.64)	0.8064 (1.3477, 43.28)	0.99660 (2.9730, 23.92)	4.51400 (7.9870, 0.28)
KM6	0.8452 (4.1265, 100.00)	0.8029 (0.8755, 40.80)	1.04510 (0.4659, 20.24)	9.09630 (0.1874, 0.16)
KM7	0.7765 (0.2460, 69.08)	0.8061 (1.2070, 42.36)	1.00630 (2.4243, 23.36)	4.80110 (6.9678, 0.28)
KM8	1.0671 (39.4900, 100.00)	0.8008 (1.9541, 45.84)	1.03280 (0.7473, 20.56)	8.48080 (0.4980, 0.16)
KM9	0.7756 (0.0814, 62.88)	0.8153 (3.0747, 51.40)	0.97120 (4.5613, 26.60)	5.42800 (3.8988, 0.20)
KM10	1.0068 (19.378, 100.00)	0.8016 (0.9057, 40.96)	1.04730 (0.4091, 19.96)	8.89900 (0.3189, 0.16)
KM11	0.7755 (0.0542, 61.52)	0.8074 (1.1896, 42.04)	1.00220 (2.5720, 23.88)	6.11030 (3.1890, 0.20)
KM12	0.9953 (17.4860, 100.00)	0.8012 (0.9787, 41.20)	1.04680 (0.4198, 20.00)	8.95850 (0.2940, 0.16)
GK	0.7754 (0.0352, 60.68)	0.8086 (0.6075, 38.60)	1.01740 (2.1162, 23.56)	4.76620 (352.2900, 0.36)
HMO	0.8216 (2.9092, 99.20)	0.9433 (22.3590, 83.60)	0.96280 (56.9328, 45.6)	0.96700 (103.5200, 0.88)
KD	0.7755 (0.0301, 60.48)	0.8077 (1.1917, 42.42)	0.98960 (4.4268, 26.08)	3.31580 (22.5230, 0.56)
CJH	0.7894 (1.2570, 92.56)	0.9368 (59.4930, 84.31)	0.98740 (82.1500, 45.16)	1.86680 (762.3000, 0.68)
FG	0.7769 (0.2931, 70.40)	0.8302 (5.5836, 61.35)	0.94740 (16.3068, 36.6)	1.61010 (40.0510, 0.56)

Table A6. Simulated MSE, average *k*s and proportion of time (%) LSE perform better than ridge estimators for n = 50, p = 5, and γ = 0.9. Condition number κ = 119.60

Estimator	σ = 0.1	σ = 0.5	σ = 1.0	σ = 5.0
OLS	0.8678	1.0066	1.4255	18.4356
HK	0.8670 (0.0309, 27.44)	0.9735 (0.6019, 25.16)	1.2806 (1.8241, 15.44)	8.5312 (55.9952, 0.24)
HKB	0.8667 (0.0500, 28.60)	0.9590 (1.1721, 28.80)	1.1994 (3.9608, 18.08)	5.8085 (13.0289, 0.24)
LW	0.8678 (0.0006, 25.96)	1.0053 (0.0123, 20.36)	1.4168 (0.0537, 11.64)	14.7874 (1.0500, 0.12)
HSL	0.8670 (0.0309, 27.48)	0.9562 (0.7374, 25.52)	1.1796 (3.1439, 17.08)	2.7700 (35.1922, 0.24)
AM	0.8667 (0.0500, 28.60)	0.9590 (1.1721, 28.80)	1.1994 (3.9608, 18.08)	5.8085 (13.0289, 0.24)
GM	0.8667 (0.0531, 28.64)	0.9595 (2.2919, 33.08)	1.1402 (11.9880, 21.44)	2.2477 (57.2516, 0.28)
MED	0.8666 (0.0571, 28.92)	0.9583 (1.5774, 29.88)	1.1692 (7.5610, 19.28)	3.0584 (49.7520, 0.28)
KS	0.8670 (0.0307, 27.44)	0.9749 (0.5443, 24.72)	1.3011 (1.3187, 14.64)	11.1257 (2.2876, 0.16)
KS_AM	0.8669 (0.0388, 28.16)	0.9835 (0.2472, 22.08)	1.3471 (0.5417, 12.80)	13.4029 (0.9329, 0.08)
KS_MAX	0.8665 (0.0551, 28.72)	0.9593 (0.6567, 25.16)	1.2253 (2.0058, 15.52)	6.4475 (4.0782, 0.16)
KS_MED	0.8669 (0.0363, 27.88)	0.9899 (0.1721, 21.60)	1.3934 (0.2030, 11.88)	17.4920 (0.1406, 0.04)
KS_GM	0.8669 (0.0370, 27.92)	0.9894 (0.1787, 21.68)	1.3839 (0.2675, 12.16)	16.7151 (0.2668, 0.04)
KM2	1.0119 (5.7916, 99.96)	0.9348 (1.3595, 29.56)	1.2988 (0.8391, 13.28)	13.8647 (0.6424, 0.04)
KM3	0.8651 (0.2785, 40.32)	0.9691 (9.6830, 39.04)	1.1274 (164.0300, 22.64)	2.3765 (90.5992, 0.24)
KM4	0.9819 (4.4273, 99.80)	0.9475 (0.7989, 26.08)	1.3550 (0.4078, 12.48)	16.8254 (0.2065, 0.04)
KM5	0.8650 (0.2289, 38.00)	0.9503 (1.3896, 30.36)	1.2126 (2.9486, 16.92)	6.2501 (6.2615, 0.20)
KM6	0.9784 (4.2866, 99.64)	0.9445 (0.9231, 27.04)	1.3455 (0.4886, 12.72)	16.5862 (0.2421, 0.04)
KM7	0.8650 (0.2370, 38.52)	0.9516 (1.1690, 28.60)	1.2323 (2.3762, 15.92)	6.8584 (5.6013, 0.16)
KM8	1.1763 (34.1253, 100.00)	0.9312 (2.0914, 32.76)	1.2820 (0.9396, 13.28)	13.2874 (0.7194, 0.04)
KM9	0.8664 (0.0780, 29.76)	0.9579 (3.4085, 39.32)	1.1441 (4.8109, 19.88)	6.2412 (4.2808, 0.16)
KM10	1.1438 (20.0632, 100.00)	0.9409 (0.9377, 26.92)	1.3513 (0.4309, 12.52)	15.9926 (0.3402, 0.04)
KM11	0.8667 (0.0525, 28.64)	0.9557 (1.1940, 29.08)	1.2215 (2.5096, 16.44)	8.6147 (3.0609, 0.16)
KM12	1.1380 (18.8572, 100.00)	0.9392 (1.0620, 27.84)	1.3524 (0.4330, 12.60)	16.3197 (0.3018, 0.04)
GK	0.8669 (0.0389, 27.96)	0.9731 (0.6089, 25.20)	1.2800 (1.8316, 15.44)	8.5085 (56.0051, 0.24)
HMO	0.8936 (1.4404, 82.56)	1.0294 (12.1690, 58.72)	1.0577 (29.8000, 27.80)	1.5332 (42.5577, 0.28)
KD	0.8672 (0.0300, 27.56)	0.9597 (1.1522, 28.72)	1.2004 (3.9409, 18.00)	5.8310 (13.0090, 0.24)
CJH	0.9395 (3.2631, 94.16)	1.0615 (30.2460, 65.08)	1.0986 (64.1490, 28.36)	3.0802 (840.5210, 0.24)
FG	0.8653 (0.2857, 41.08)	0.9642 (4.7476, 45.12)	1.0884 (12.3090, 24.48)	2.5102 (21.1200, 0.24)

Table A7. Simulated MSE, average ks and proportion of time (%) LSE perform better than proposed new ridge estimators for different values of n, p = 5, σ = 3.0, and γ = 0.9

Estimator	***n* = 10**	***n* = 20**	***n* = 50**
OLS	67.1158	22.6167	7.7726
HK	26.2573 (78.0160, 0.12)	10.0504 (83.0563, 0.32)	3.8897 (20.1510, 1.08)
HKB	20.9523 (1.7574, 0.12)	7.1599 (3.9477, 0.36)	2.8018 (10.1280, 1.52)
LW	15.9023 (1.3047, 0.12)	11.5401 (1.0943, 0.28)	6.5964 (0.4158, 0.36)
HSL	9.6370 (6.0834, 0.16)	4.0571 (10.3777, 0.32)	1.4353 (25.3280, 1.64)
AM	20.9523 (1.7574, 0.12)	7.1599 (3.9477, 0.36)	2.8018 (10.1280, 1.52)
GM	5.0207 (9.8421, 0.16)	2.8726 (17.2089, 0.28)	1.6304 (36.1640, 1.92)
MED	6.0914 (12.5365, 0.16)	3.6802 (16.7250, 0.36)	1.7539 (30.9620, 1.52)
KS	26.9777 (1.1632, 0.12)	11.4083 (1.2309, 0.20)	4.5503 (2.3287, 0.56)
KS_AM	8.3030 (1.0881, 0.12)	10.4941 (0.7219, 0.08)	5.0463 (1.0467, 0.32)
KS_MAX	2.6825 (4.3804, 0.12)	3.8964 (2.9958, 0.20)	2.6053 (4.5236,0.56)
KS_MED	20.9093 (0.3330, 0.12)	19.1547 (0.1196, 0.04)	7.0904 (0.1791, 0.28)
KS_GM	22.9548 (0.3001, 0.12)	16.7338 (0.2319, 0.04)	6.7395 (0.2876, 0.32)
KM2	4.0575 (2.5361, 0.12)	6.7336 (1.2331, 0.04)	5.2090 (0.7023, 0.28)
KM3	1.9326 (49.8389, 0.12)	02.1282 (61.7303, 0.36)	1.7384 (59.9134, 1.52)
KM4	13.7695 (0.5735, 0.12)	13.7225 (0.3724, 0.04)	6.7678 (0.2385, 0.28)
KM5	5.7004 (2.5166, 0.12)	4.6168 (3.4765, 0.32)	2.8852 (5.1752, 0.84)
KM6	13.7148 (0.6410, 0.12)	13.1157 (0.4334, 0.04)	6.7395 (0.2611, 0.28)
KM7	6.2518 (2.4318, 0.12)	5.1636 (3.2115, 0.32)	2.9637 (4.6849, 0.80)
KM8	2.6439 (10.0721, 0.12)	5.0384 (2.1737, 0.04)	4.9405 (0.7863, 0.32)
KM9	2.2439 (5.1076, 0.12)	3.5706 (3.3286, 0.24)	2.4176 (5.2458, 0.68)
KM10	9.5348 (0.9021, 0.12)	10.6497 (0.6062, 0.04)	6.5327 (0.3014, 0.28)
KM11	8.7200 (1.6719, 0.12)	6.7272 (1.8406, 0.20)	3.2644 (3.4917, 0.60)
KM12	14.0551 (0.7741, 0.12)	12.0833 (0.5238, 0.04)	6.7666 (0.2596, 0.28)
GK	20.2413 (78.082, 0.12)	9.6857 (83.0930, 0.32)	3.8801 (20.1590, 1.08)
HMO	11.5302 (3.0492, 0.16)	3.1834 (8.8390, 0.32)	1.3795 (37.4880, 2.28)
KD	28.8528 (1.6649, 0.12)	7.4149 (3.8977, 0.36)	2.8119 (10.1080, 1.52)
CJH	13.6394 (3.8924, 0.16)	4.2477 (13.8341, 0.28)	1.8765 (38.3210, 1.96)
FG	9.0114 (2.2648, 0.12)	3.4396 (6.1783, 0.36)	1.6613 (18.5430, 1.64)
KB1	3.6922 (13.5707, 0.16)	2.5210 (20.7526, 0.36)	1.4909 (36.8980, 1.96)
KB2	6.4905 (4.7393, 0.16)	3.0329 (9.6154, 0.36)	1.6045 (23.6380, 1.84)
KB3	1.8130 (57.2684, 0.16)	1.8326 (76.6178, 0.28)	1.2435 (107.5580, 2.32)
KB4	5.6967 (4.5998, 0.12)	2.8649 (9.7842, 0.36)	1.5804 (24.0460, 1.84)
KB5	8.3678 (3.2453, 0.12)	3.2945(7.7060, 0.36)	1.6849 (19.6580, 1.72)

Table A8. Simulated MSE, average *k*s and proportion of time (%) LSE perform better than proposed new ridge estimators for different values of n, p = 5, σ = 10.0, and γ = 0.9

Estimator	*n* = 10	*n* = 20	*n* = 50
OLS	737.0407	242.9571	78.1037
HK	272.0000 (1.3987, 0)	93.0000 (8.6474, 0)	29.5000 (4.1928, 0.00)
HKB	223.1200 (1.8062, 0)	67.8198 (4.5231, 0)	20.9440 (13.9677, 0.04)
LW	153.6900 (1.6560, 0)	103.8625 (1.6510, 0)	53.5800 (1.3756, 0.00)
HSL	116.1700 (6.1421, 0)	50.8556 (11.0230, 0)	11.7460 (48.0250, 0.04)
AM	223.1200 (1.8062, 0)	67.8198 (4.5230, 0)	20.9440 (13.9670, 0.04)
GM	41.7100 (11.4337, 0)	15.4604 (26.2730, 0)	5.2872 (76.8460, 0.00)
MED	58.2900 (12.5456, 0)	28.2886 (24.1990, 0)	8.7540 (69.4340, 0.04)
KS	279.7900 (1.3952, 0)	109.0829 (1.4290, 0)	38.4023 (2.8360, 0.00)
KS_AM	74.7100 (1.1902, 0)	101.9246 (0.7850, 0)	45.2139 (1.2170, 0.00)
KS_MAX	17.9800 (4.8874, 0)	28.5829 (3.3113, 0)	17.8669 (5.3720, 0.00)
KS_MED	223.0900 (0.3343, 0)	204.5300 (0.1190, 0)	70.6056 (0.1790, 0.00)
KS_GM	240.9700 (0.3085, 0)	176.4906 (0.2370, 0)	66.4137 (0.2980, 0.00)
KM2	37.2600 (2.5292, 0)	65.9034 (1.2148, 0)	50.4013 (0.6740, 0.00)
KM3	10.9600 (56.5641, 0)	7.9644 (70.3667, 0)	5.2666 (192.4500, 0.04)
KM4	153.2700 (0.5298, 0)	153.0248 (0.3210, 0)	69.6911 (0.1750, 0.00)
KM5	51.4400 (2.7141, 0)	35.3579 (4.1621, 0)	18.9584 (7.3710, 0.00)
KM6	150.7500 (0.6215, 0)	140.6503 (0.4110, 0)	68.2656 (0.2180, 0.00)
KM7	60.2200 (2.5963, 0)	45.2184 (3.6642, 0)	21.9880 (6.5460, 0.00)
KM8	21.5800 (10.0380, 0)	46.3192 (2.1200, 0)	47.6514 (0.7460, 0.00)
KM9	15.7200 (5.3348, 0)	27.3427 (3.4367, 0)	17.5897 (5.4840, 0.00)
KM10	100.3600 (0.8550, 0)	111.8558 (0.5840, 0)	65.2522 (0.2820, 0.00)
KM11	85.6400 (1.7224, 0)	63.1383 (1.8982, 0)	26.9247 (3.7130, 0.00)
KM12	153.5700 (0.7523, 0)	127.6506 (0.5070, 0)	67.7397 (0.2410, 0.00)
GK	208.0000 (1.3987, 0)	90.0000 (8.6474, 0)	29.5000 (4.1920, 0.00)
HMO	119.9000 (2.9913, 0)	23.8446 (9.4118, 0)	5.5032 (42.1730, 0.00)
KD	309.9300 (1.7137, 0)	70.5684 (4.4732, 0)	21.0470 (13.9400, 0.04)
CJH	143.4400 (3.7576, 0)	35.9131 (13.0910, 0)	11.3326 (37.6000, 0.00)
FG	91.7100 (2.2766, 0)	26.5555 (6.4474, 0)	8.7287 (20.7300, 0.04)
KB1	28.3800 (14.9280, 0)	12.9418 (24.9650, 0)	5.0918 (73.2000, 0.04)
KB2	61.270 (4.8625, 0)	20.2705 (11.6430, 0)	6.3075 (36.1800, 0.04)
KB3	9.9300 (63.2258, 0)	6.2206 (88.2720, 0)	2.9234 (257.3900, 0.00)
KB4	51.3900 (5.0065, 0)	17.4754 (11.9870, 0)	6.1536 (35.6300, 0.04)
KB5	83.2800 (3.3955, 0)	23.2968 (8.7820, 0)	7.5708 (26.6600, 0.04)