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Characterizations of Continuous Distributions by Truncated Moment

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A probability distribution can be characterized through various methods. In this paper, some new characterizations of continuous distribution by truncated moment have been established. We have considered standard normal distribution, Student's *t*, exponentiated exponential, power function, Pareto, and Weibull distributions and characterized them by truncated moment.

Keywords: Characterization, exponentiated exponential distribution, power function distribution, standard normal distribution, Student's t distribution, Pareto distribution, truncated moment

Introduction

Before a particular probability distribution model is applied to fit real world data, it is essential to confirm whether the given probability distribution satisfies the underlying requirements of its characterization. Thus, characterization of a probability distribution plays an important role in statistics and mathematical sciences. A probability distribution can be characterized through various methods, see, for example, Ahsanullah, Kibria, and Shakil (2014), Huang and Su (2012), Nair and Sudheesh (2010), Nanda (2010), Gupta and Ahsanullah (2006), and Su and Huang (2000), among others. In recent years, there has been a great interest in the characterizations of probability distributions by truncated moments. For example, the development of the general theory of the characterizations of probability distributions by truncated moment began with the work of Galambos and Kotz (1978). Further development on the characterizations of probability distributions

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by truncated moments continued with the contributions of many authors and researchers, among them Kotz and Shanbhag (1980), Glänzel (1987, 1990), and Glänzel, Telcs, and Schubert (1984), are notable. However, most of these characterizations are based on a simple proportionality between two different moments truncated from the left at the same point. It appears from literature that not much attention has been paid on the characterizations of continuous distributions by using truncated moment. As pointed out by Glänzel (1987) these characterizations may also serve as a basis for parameter estimation. In this paper, some new characterizations of continuous distributions by truncated moment have been established. The organization of this paper is as follows: We will first state the assumptions and establish a lemma which will be needed for the characterizations of continuous distributions by truncated moment. The following section contains our main results for the new characterizations of continuous distributions by truncated moment. Finally, concluding remarks are presented.

A Lemma

This section will state the assumptions and establish a lemma (Lemma 1) which will be useful in proving our main results for the characterizations of continuous distributions by truncated moment.

Assumptions. Let X be a random variable having absolutely continuous (with respect to Lebesgue measure) cumulative distribution function (cdf) $\mathrm{F}(x)$ and the probability density function (pdf) $\mathrm{f}(x)$. We assume $\alpha = \inf\{x \mid \mathrm{F}(x) > 0\}$ and $\beta = \sup\{x \mid \mathrm{F}(x) < 1\}$. We define

$$\eta(x) = \frac{\mathrm{f}(x)}{\mathrm{F}(x)} ,$$

and $\mathrm{g}(x)$ is a differentiable function with respect to x for all real $x \in (\alpha, \beta)$.

Lemma 1. Suppose that X has an absolutely continuous (with respect to Lebesgue measure) cdf $\mathrm{F}(x)$, with corresponding pdf $\mathrm{f}(x)$, and $\mathrm{E}(X \mid X \le x)$ exists for all real $x \in (\alpha, \beta)$. Then $\mathrm{E}(X \mid X \le x) = \mathrm{g}(x)\eta(x)$, where $\mathrm{g}(x)$ is a differentiable function and $\eta(x) = \mathrm{f}(x) / \mathrm{F}(x)$ for all real $x \in (\alpha, \beta)$, if

$$\mathrm{f}(x) = ce^{\int_{\alpha}^{x} \frac{u - \mathrm{g}'(u)}{\mathrm{g}(u)} du}$$

where c is determined such that $\int_{\alpha}^{\beta} \mathrm{f}(x)\,dx = 1$.

Note: Since the cdf F(x) is absolutely continuous (with respect to Lebesgue measure), then by Radon-Nikodym Theorem the pdf f(x) exists and hence $\int_{-\infty}^{x} \frac{u - \mathrm{g}'(u)}{\mathrm{g}(u)} du$ exists. Also note that, in Lemma 1 above, the left truncated conditional expectation of X considers a product of reverse hazard rate and another function of the truncated point.

Proof of Lemma 1. It is known that

$$\frac{\int_{\alpha}^{x} u\,\mathrm{f}(u)\,du}{\mathrm{F}(x)} = \frac{\mathrm{g}(x)\mathrm{f}(x)}{\mathrm{F}(x)} .$$

Thus

$$\int_{\alpha}^{x} u\,\mathrm{f}(u)\,du = \mathrm{g}(x)\mathrm{f}(x) .$$

Differentiating both sides of the equation produces the following

$$x\mathrm{f}(x) = \mathrm{g}'(x)\mathrm{f}(x) + \mathrm{g}(x)\mathrm{f}'(x) .$$

On simplification, one gets

$$\frac{\mathrm{f}'(x)}{\mathrm{f}(x)} = \frac{x - \mathrm{g}'(x)}{\mathrm{g}(x)} .$$

Integrating the above equation gives

$$\mathrm{f}(x) = ce^{\int_{\alpha}^{x} \frac{u - \mathrm{g}'(u)}{\mathrm{g}(u)} du}$$

where c is determined such that $\int_{\alpha}^{\beta} \mathrm{f}(x)\,dx = 1$. This completes the proof of Lemma 1.

Characterizations of some Continuous Distributions by Truncated Moments

Standard Normal Distribution

The characterization of standard normal distribution is provided in Theorem 1 below.

Theorem 1. Suppose that an absolutely continuous (with respect to Lebesgue measure) random variable X has cdf $\mathrm{F}(x)$ and pdf $\mathrm{f}(x)$ for $-\infty < x < \infty$.We assume that $\mathrm{f}'(t)$ and $\mathrm{E}(X \mid X \le t)$ exist for all t, $-\infty < t < \infty$. Then

$$\mathrm{E}\left(X \mid X \le x\right) = \mathrm{g}(x)\eta(x) ,$$

where

$$\eta(x) = \frac{\mathrm{f}(x)}{\mathrm{F}(x)}$$

and $\mathrm{g}(x) = -1$, if and only if

$$\mathrm{f}(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2} \quad \forall x \in (-\infty, \infty) ,$$

which is the probability density function of the standard normal distribution.

Proof: Suppose

$$\mathrm{f}(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2}$$

Then it is easily seen that g(x) = -1. Consequently, the proof of the "if" part of the Theorem 1 follows from Lemma 1. We will now prove the "only if" condition of the Theorem 1. Suppose that g(x) = -1.

Then it easily follows that

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)} = -x \ .$$

On integrating the above equation,

$$f(x) = c e^{-\frac{1}{2}x^2} \ ,$$

where $c = 1/\sqrt{2\pi}$. This completes the proof of Theorem 1.

Student's *t* Distribution

The characterization of the Student's *t* is provided in Theorem 2 below.

Theorem 2. Suppose that an absolutely continuous (with respect to Lebesgue measure) random variable X has the cdf F(x) and pdf f(x) for $-\infty < x < \infty$. We assume that f′(x) and $E(X \mid X \le x)$ exist for all x, $-\infty < x < \infty$. Then X has the Student's *t* distribution if and only if

$$E(X \mid X \le x) = g(x)\tau(x) \ ,$$

where

$$g(x) = -\frac{n}{n-1}\left(1 + \frac{x^2}{n}\right), n > 1$$

and

$$\tau(x) = \frac{f(x)}{F(x)} \ .$$

Proof: Suppose the random variable X has the t distribution with n degrees of freedom. The pdf f(x) of X is

$$\mathrm{f}\left(x\right)=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\left(\frac{n}{2}\right)}\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}},-\infty<x<\infty\ .$$

Then it is easily seen that

$$\mathrm{g}\left(x\right)=\frac{\int_{-\infty}^{x}u\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\left(\frac{n}{2}\right)}\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{u^2}{n}\right)^{-\frac{n+1}{2}}du}{\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\left(\frac{n}{2}\right)}\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}}=-\frac{n}{n-1}\left(1+\frac{x^2}{n}\right)\ .$$

Consequently, the proof of "if" part of Theorem 2 follows from Lemma 1. Now prove the "only if" condition of the Theorem 2. Suppose that

$$\mathrm{g}\left(x\right)=-\frac{n}{n-1}\left(1+\frac{x^2}{n}\right)\ .$$

Then, one easily has

$$\mathrm{g}'\left(x\right)=-\frac{2x}{n-1}$$

Thus, after simplification, one obtains the following:

$$\frac{x-\mathrm{g}'(x)}{\mathrm{g}(x)}=\frac{\frac{n+1}{n-1}x}{\frac{n}{n-1}\left(1+\frac{x^2}{n}\right)}=\frac{\frac{n+1}{2}\frac{2x}{n}}{\left(1+\frac{x^2}{n}\right)} .$$

Therefore, by Lemma 1, one has

$$\mathrm{f}(x)=c\mathrm{e}^{\int_{-\infty}^{x}\frac{\frac{n+1}{2}\frac{2u}{n}}{-\left(1+\frac{u^2}{n}\right)}du}=c\mathrm{e}^{-\left(\frac{n+1}{2}\right)\ln\left(1+\frac{x^2}{n}\right)}=c\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}} .$$

Now, using the condition $\int_{-\infty}^{\infty}\mathrm{f}(x)\,dx=1$, one obtains

$$\mathrm{f}(x)=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\left(\frac{n}{2}\right)}\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}},-\infty<x<\infty .$$

Note the condition $n \geq 1$ is needed for $\mathrm{E}(X \mid X \leq x)$ to exist. This completes the proof of Theorem 2.

Exponentiated Exponential Distribution

The Characterization of exponentiated exponential distribution is presented in the Theorem 3 below.

Theorem 3. Suppose an absolutely continuous (with respect to Lebesgue measure) random variable X has the cdf $\mathrm{F}(x)$ and pdf $\mathrm{f}(x)$ for $0 < x < \infty$ such that $\mathrm{f}'(x)$ and $\mathrm{E}(X \mid X \leq x)$ exist for all x, $0 < x < \infty$. Then X has the exponentiated exponential distribution

$$\mathrm{f}(x)=\alpha\lambda\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\right)^{\alpha-1}, \alpha>1, x>0$$

if and only if

$$\mathrm{E}\left(X \mid X \le x\right) = \mathrm{g}\left(x\right)\eta\left(x\right) ,$$

where

$$\eta\left(x\right) = \frac{\mathrm{f}\left(x\right)}{\mathrm{F}\left(x\right)}$$

and

$$\mathrm{g}\left(x\right) = \frac{x\left(1-\mathrm{e}^{-\lambda x}\right)^{\alpha}}{\lambda\alpha\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\right)^{\alpha-1}} - \frac{\int_0^x \left(1-\mathrm{e}^{-\lambda u}\right)^{\alpha} du}{\lambda\alpha\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\right)^{\alpha-1}}$$

Proof: Suppose

$$\mathrm{f}\left(x\right) = \alpha\lambda\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\right)^{\alpha-1}, \alpha > 1, x > 0 \ .$$

Then it is easily seen that

$$\mathrm{g}\left(x\right) = \frac{x\left(1-\mathrm{e}^{-\lambda x}\right)}{\lambda\alpha\mathrm{e}^{-\lambda x}} - \frac{\int_0^x \left(1-\mathrm{e}^{-\lambda u}\right)^{\alpha} du}{\lambda\alpha\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\right)^{\alpha-1}} \ .$$

Consequently, the proof of the “if” part of the Theorem 3 follows from Lemma 1. Now prove the “only if” condition of the Theorem 3. Suppose that

$$\mathrm{g}\left(x\right) = \frac{x\left(1-\mathrm{e}^{-\lambda x}\right)}{\lambda\alpha\mathrm{e}^{-\lambda x}} - \frac{\int_0^x \left(1-\mathrm{e}^{-\lambda u}\right)^{\alpha} du}{\lambda\alpha\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\right)^{\alpha-1}} \ .$$

Simple differentiation and simplification gives $\mathrm{g}'(x) = x + \mathrm{g}(x)\mathrm{A}(x)$, where

$$\mathrm{A}\left(x\right) = -\lambda + \frac{\left(\alpha-1\right)\lambda\mathrm{e}^{-\lambda x}}{1-\mathrm{e}^{-\lambda x}}, \alpha > 1 \ .$$

Thus

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)} = -A(x) = -\lambda + \frac{(\alpha - 1)\lambda e^{-\lambda x}}{1 - e^{-\lambda x}} .$$

On integrating the above equation, if follows that

$$f(x) = c e^{\int_0^x A(u) du} .$$

But

$$\int_0^x A(u)\, du = \int_0^x \left(-\lambda \frac{(\alpha - 1)\lambda e^{-\lambda u}}{1 - e^{-\lambda u}} \right) du$$
$$= -\lambda x + (\alpha - 1)\ln\left(1 - e^{-\lambda x}\right)$$

Thus $f(x) = ce^{-\lambda x}(1 - e^{-\lambda x})^{\alpha - 1}$, where

$$\frac{1}{c} = \int_0^\infty e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{\alpha - 1} dx = \frac{1}{\alpha\lambda} .$$

This completes the proof of Theorem 3.

Power Function Distribution

The characterization of the power function distribution is provided in Theorem 4 below.

Theorem 4. Suppose an absolutely continuous (with respect to Lebesgue measure) random variable X has the cdf $F(x)$ and pdf $f(x)$ for $0 \leq x < 1$. Assume that $f'(x)$ and $E(X \mid X \leq x)$ exist for all x, $0 < x < 1$. Then

$$E(X \mid X \leq x) = g(x)\eta(x) ,$$

where

$$\eta(x) = \frac{\mathrm{f}(x)}{\mathrm{F}(x)}$$

and

$$\mathrm{g}(x) = \frac{x^2}{\alpha+1} \ ,$$

if and only if

$$\mathrm{f}(x) = \alpha x^{\alpha-1}, \alpha > 1, 0 < x < 1 \ ,$$

which is the pdf of the power function distribution.

Proof: Suppose f(x) = $\alpha x^{\alpha - 1}$, $\alpha > 1$, $0 < x < 1$. Then it is easily seen that

$$\mathrm{g}(x) = \frac{x^2}{\alpha+1} \ .$$

If

$$\mathrm{g}(x) = \frac{x^2}{\alpha+1} \ ,$$

then

$$x - \mathrm{g}'(x) = \frac{(\alpha-1)x}{\alpha+1},$$
$$\frac{x - \mathrm{g}'(x)}{\mathrm{g}(x)} = \frac{\alpha-1}{x}$$

Thus, by Lemma 1,

$$\mathrm{f}(x) = c\mathrm{e}^{\int_1^x \frac{\alpha-1}{u} du} = cx^{\alpha-1} \ ,$$

where c is a constant. Using the condition $\int_{-\infty}^{\infty} f(x)\,dx = 1$, we obtain $f(x) = \alpha x^{\alpha-1}$, $\alpha > 1$, $0 < x < 1$. This completes the proof of Theorem 4.

Remark 1. If $\alpha = 1$, then $g(x) = x^2/2$ and one gets a characterization of the uniform distribution in [0, 1].

Pareto Distribution

The characterization of Pareto distribution is provided in Theorem 5 below.

Theorem 5. Suppose the random variable X has an absolutely continuous (with respect to Lebesgue measure) cdf $F(x)$ and pdf $f(x)$. We assume that $F(1) = 0$, $F(x) > 0$ for all $x > 1$, and $E(X)$ exists. Then X has a Pareto distribution if and only if

$$E(X \mid X \le x) = g(x)\tau(x) ,$$

where

$$g(x) = \frac{x^{\alpha+1} - x^2}{\alpha - 1}, \; x > 1, \alpha > 1$$

and

$$\tau(x) = \frac{f(x)}{F(x)}$$

Proof: Suppose the random variable X has the Pareto distribution. The pdf $f(x)$ of X is given by

$$f(x) = \frac{\alpha}{x^{\alpha+1}}, \; x \ge 1, \alpha > 1 \; .$$

Then it is easily seen that

$$\mathrm{g}(x)=\frac{\int_1^x \frac{\alpha u}{u^{\alpha+1}}du}{\frac{\alpha}{x^{\alpha+1}}}=\frac{x^{\alpha+1}-x^2}{\alpha-1} \ .$$

Consequently, the proof of the “if” part of Theorem 5 follows from Lemma 1. Now prove the “only if” condition of the Theorem 5. Suppose that

$$\mathrm{g}(x)=\frac{x^{\alpha+1}-x^2}{\alpha-1},\ x>1, \alpha>1 \ .$$

Then it is easy to show that

$$\mathrm{g}'(x)=\frac{(\alpha+1)x^{\alpha}-2x}{\alpha-1}$$

and

$$x-\mathrm{g}'(x)=\frac{(\alpha+1)\left(x-x^{\alpha}\right)}{\alpha-1} \ .$$

Consequently,

$$\frac{x-\mathrm{g}'(x)}{\mathrm{g}(x)}=-\frac{\alpha+1}{x} \ .$$

Therefore, by Lemma 1, one obtains

$$\mathrm{f}(x)=c\mathrm{e}^{-\int_1^x \frac{\alpha+1}{u}du}=\frac{c}{x^{\alpha+1}} \ .$$

Now, using the condition $\int_{-\infty}^{\infty}\mathrm{f}(x)\,dx=1$,

$$\mathrm{f}(x)=\frac{\alpha}{x^{\alpha+1}},\ x\geq 1, \alpha>1 \ .$$

This completes the proof of Theorem 5.

Weibull Distribution

The characterization of Weibull distribution is provided in Theorem 6 below.

Theorem 6. Suppose an absolutely continuous (with respect to Lebesgue measure) random variable X has the cdf $F(x)$ and pdf $f(x)$ for $0 < x < \infty$, and that $f'(x)$ and $E(X \mid X \le x)$ exist for all x, $0 < x < \infty$. Then

$$E\left(X \mid X \le x\right) = g\left(x\right)\eta\left(x\right)$$

where

$$\eta\left(x\right) = \frac{f\left(x\right)}{F\left(x\right)}$$

and

$$g\left(x\right) = \frac{x^{2-\lambda}}{\lambda} + \frac{x^{1-\lambda}}{\lambda} e^{-x^{\lambda}}\, h\left(x\right) ,$$

with

$$h\left(x\right) = \gamma\left(x^{\lambda}, \frac{2\lambda - 1}{\lambda}\right) ,$$

where $\gamma\left(x, n\right) = \int_0^x u^{n-1} e^{-u} du$, if and only if

$$f\left(x\right) = \lambda x^{\lambda-1} e^{-x^{\lambda}} , 0 < x < \infty, \lambda > 0$$

which is the pdf of the Weibull distribution.

Proof: Note that

$$\mathrm{g}(x)=\frac{\int_0^x \lambda u^{\lambda}\mathrm{e}^{-u^{\lambda}}du}{\lambda x^{\lambda-1}\mathrm{e}^{-x^{\lambda}}}=\frac{x^{2-\lambda}}{\lambda}+\frac{\int_0^x \mathrm{e}^{-u^{\lambda}}du}{\lambda x^{\lambda-1}\mathrm{e}^{-x^{\lambda}}}$$
$$=-\frac{x^{2-\lambda}}{\lambda}+\frac{x^{1-\lambda}e^{-x^{\lambda}}}{\lambda}\gamma\left(x^{\lambda},\frac{2\lambda-1}{\lambda}\right)$$

Suppose that

$$\mathrm{g}(x)=-\frac{x^{2-\lambda}}{\lambda}+\frac{x^{1-\lambda}e^{-x^{\lambda}}}{\lambda}\gamma\left(x^{\lambda},\frac{2\lambda-1}{\lambda}\right).$$

Then

$$\mathrm{g}'(x)=-\frac{2-\lambda}{\lambda}x^{1-\lambda}+\frac{x^{1-\lambda}}{\lambda}+\left(\frac{1-\lambda}{\lambda}x^{-\lambda}-\lambda x^{1-\lambda}\right)\mathrm{e}^{-x^{\lambda}}\gamma\left(x^{\lambda},\frac{2\lambda-1}{\lambda}\right)$$
$$=-\frac{1-\lambda}{\lambda}x^{1-\lambda}+\left(\frac{1-\lambda}{\lambda}x^{-\lambda}-\lambda x^{1-\lambda}\right)\mathrm{e}^{-x^{\lambda}}\gamma\left(x^{\lambda},\frac{2\lambda-1}{\lambda}\right)$$

Also,

$$x-\mathrm{g}(x)=x-\frac{\lambda-1}{\lambda}x^{1-\lambda}-\left(\frac{1-\lambda}{\lambda}x^{-\lambda}-\lambda x^{\lambda-1}\right)\mathrm{e}^{-x^{\lambda}}\gamma\left(x^{\lambda},\frac{2\lambda-1}{\lambda}\right)$$
$$=\left(\frac{\lambda-1}{x}-\lambda x^{\lambda-1}\right)\mathrm{g}(x)$$

Thus

$$\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}=\frac{x-\mathrm{g}'(x)}{\mathrm{g}(x)}=\frac{\lambda-1}{x}-\lambda x^{\lambda-1}.$$

On integrating with respect to x from 0 to x, one obtains $\mathrm{f}(x)=\lambda x^{\lambda-1}\mathrm{e}^{-x^{\lambda}}$, where c is constant. On using the boundary conditions F(0) = 0 and F(∞) = 1, we have $c = \lambda$ and $\mathrm{F}(x)=\mathrm{e}^{-x^{\lambda}}$. This completes the proof of Theorem 6.

Conclusions

Some continuous probability distributions, namely, standard normal, Student's t, exponentiated exponential, power functions, Pareto, and Weibull distributions, are considered. Their corresponding characterizations are provided by truncated moments, which may be useful in applied and physical sciences.

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