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Bayesian estimation of $P[Y < X]$ Based on Record Values from the Lomax Distribution and MCMC Technique

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Cover Page Footnote

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Our interest is in estimating the stress-strength reliability $R = P[Y < X]$, where X and Y follow the Lomax distribution with common scale parameter. We discuss the problem in the situation where the stress measurements and the strength measurements are both in terms of records. Firstly, we obtain the MLE of R in general case (the common scale parameter is unknown). The MLE of the three unknown parameters can be obtained by solving one non-linear equation. We provide a simple fixed point type algorithm to find the MLE. We propose percentile bootstrap confidence intervals of R . A Bayes point estimator of R , and the corresponding credible interval using the MCMC sampling technique have been proposed. Secondly, assuming the common scale parameter is known, the MLE of R is obtained. Using exact distributions of the MLEs of the two unknown parameters, we construct the exact confidence interval of R . In this case, Bayes estimators have been obtained using Lindley's approximations. Analysis of a simulated data set has been presented for illustrative purposes. Finally, the different proposed methods have been compared via Monte Carlo simulation study.

Keywords: Stress-strength model, Lomax distribution, maximum likelihood estimation, bootstrap confidence intervals, credible intervals, Gibbs sampling, Markov chain Monte Carlo

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Introduction

The problem of estimating $R = P[Y < X]$ arises in the context of mechanical reliability of a system with strength (or supply) X and stress (or demand) Y , and R is chosen as a measure of system reliability. The system fails if and only if at any time the applied stress is greater than its strength. This type of reliability model is known as the stress-strength model. This problem also arises in situations where X and Y represent lifetimes of two devices and one wants to estimate the probability that one fails before the other. For example, in biometrical studies, the random variable X may represent the remaining lifetime of a patient treated with a certain drug, and Y represent the remaining lifetime when treated by another drug.

Review of literature

Parametric and nonparametric inferences on R for several specific distributions of X and Y under different conditions have been found in the literature. Nadarajah (2004a; 2004b) estimated $R = P[Y < X]$ from Logistic and Laplace distributions. Kundu and Gupta (2005) derived the maximum likelihood estimator of R and its asymptotic distribution when X and Y are independently distributed as generalized exponential distribution. Surles and Padgett (2001) considered the estimation of R where X and Y are Burr-X random variables. The theoretical and practical results on the theory and applications of the stress-strength relationships in industrial and economic systems during the last decades are collected and digested in Kotz, Lumelskii, and Pensky (2003).

The class of life-time distributions (in particular, exponential and gamma) is considered by Nadarajah (2003). Estimation of R from exponential case with common location parameter (Baklizi & El-Masri, 2004), Burr-III (Mokhlis, 2005), beta (Nadarajah, 2005a), gamma (Nadarajah, 2005b), bivariate exponential (Nadarajah & Kotz, 2006), and Weibull (Kundu & Gupta, 2006) distributions were also studied. Inferences on reliability in two-parameter exponential stress-strength model (Krishnamoorthy, Mukherjee, & Guo, 2007) and ML estimation of system reliability for Gompertz distribution (Saraçoglu & Kaya, 2007) were considered. Kakade, Shirke, and Kundu (2008) studied the exponentiated Gumbel case. Baklizi (2008a; 2008b) studied some inference problems of estimating R based on record values. Kundu and Raqab (2009) considered the estimation of R when X and Y are independent and both having three parameter Weibull distribution with common shape and location parameters, but different scale parameters.

Gupta and Peng (2009) studied the estimation of R in the context of proportional odds ratio model. Wang and Shi (2010) obtained the empirical Bayes

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inference for the Burr type XII distribution based on records. Lomax (1954) used this distribution in the analysis of business failure data. Balkema and De Haan (1974) showed that this distribution arises as a limit distribution of residual lifetime at great age. The Lomax distribution includes increasing and decreasing hazard rates as well, and was shown to be useful for modeling and analyzing the life time data in medical and biological sciences and engineering, etc.

Many statistical methods have been developed for this distribution. For a review of the Lomax distribution, see Habibullah and Ahsanullah (2000), Upadhyay and Peshwani (2003), and Abd Ellah (2003; 2006) and references therein. A great deal of research was done on estimating the parameters of a Lomax distribution using both classical and Bayesian techniques. The form of the probability density function (pdf) and cumulative distribution function (cdf) with the scale parameter λ and the shape parameter α of the Lomax distribution, denoted by $\text{Lomax}(\lambda, \alpha)$, are given, respectively, by

$$f(x) = \alpha \lambda^\alpha (x + \lambda)^{-(\alpha+1)}, \quad x > 0, \quad \alpha, \lambda > 0 \quad (1)$$

$$F(x) = 1 - \lambda^\alpha (x + \lambda)^{-\alpha}, \quad x > 0, \quad \alpha, \lambda > 0 \quad (2)$$

Record data arise in a wide variety of practical situations. Examples include industrial stress testing, meteorological analysis, hydrology, seismology, sporting, athletic events, and oil and mining surveys. Specifically, Let $\{X_j, j \geq 1\}$ be a sequence of independent identically continuous random variables. An observation X_j will be called an upper record value if its value exceeds that of all previous observations. That is, X_j is an upper record if $X_j > X_i$ for every $i < j$. An analogous definition can be given for lower record values. Record values can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of the observations.

Maximum Likelihood Estimator of R

Suppose that X is the strength of a component which is subject to stress Y . The system fails if and only if, at any time, the applied stress is greater than strength. Let X be a random variable whose pdf is a Lomax distribution with parameters λ and α , denoted by $\text{Lomax}(\lambda, \alpha)$, and Y is another Lomax distribution random variable with parameters λ and β , denoted by $\text{Lomax}(\lambda, \beta)$, where X and Y are independent. Thus,

$$\begin{aligned}
 R &= P[Y < X] = \int_0^{\infty} P(Y < X | X = x)P(X = x)dx \\
 &= \frac{\alpha}{\beta + \alpha}
 \end{aligned}
 \tag{3}$$

The interest is in estimating R when the data available on both X and Y are in the form of upper record values. To compute the MLE of R , compute the MLE of α and β . Suppose $\mathbf{x} = x_{U(1)}, x_{U(2)}, \dots, x_{U(n)}$ is the first upper record values of size n from $Lomax(\lambda, \alpha)$, and $\mathbf{y} = y_{U(1)}, y_{U(2)}, \dots, y_{U(m)}$ is an independent set of the first upper record values of size m from $Lomax(\lambda, \beta)$. The likelihood functions for both observed records \mathbf{x} and \mathbf{y} are given, respectively, by

$$L_1(\lambda, \alpha | \mathbf{x}) = f(x_{U(n)}) \prod_{i=1}^{n-1} \frac{f(x_{U(i)})}{1 - F(x_{U(i)})}
 \tag{4}$$

and

$$L_2(\lambda, \beta | \mathbf{y}) = g(y_{U(m)}) \prod_{j=1}^{m-1} \frac{g(x_{U(j)})}{1 - G(x_{U(j)})},
 \tag{5}$$

where f and F are the pdf and cdf of $X \sim Lomax(\lambda, \alpha)$, respectively, and g and G are the pdf and cdf of $Y \sim Lomax(\lambda, \beta)$, respectively (Arnold, Balakrishnan, & Nagaraja, 1998). Substituting f, F, g and G in the likelihood functions,

$$L_1(\lambda, \alpha | \mathbf{x}) = \alpha^n \lambda^\alpha \eta_1(\mathbf{x}, \lambda) \exp[-\alpha \ln(x_{U(n)} + \lambda)]
 \tag{6}$$

$$L_2(\lambda, \beta | \mathbf{y}) = \beta^m \lambda^\beta \eta_2(\mathbf{y}, \lambda) \exp[-\beta \ln(y_{U(m)} + \lambda)],
 \tag{7}$$

where

$$\eta_1(\mathbf{x}, \lambda) = \prod_{i=1}^n (x_{U(i)} + \lambda)^{-1}, \eta_2(\mathbf{y}, \lambda) = \prod_{j=1}^m (y_{U(j)} + \lambda)^{-1}.
 \tag{8}$$

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Therefore, the joint Log-likelihood function of the observed records \mathbf{x} and \mathbf{y} under Lomax distribution is

$$\begin{aligned} \ell(\lambda, \alpha, \beta | \text{data}) = & n \ln \alpha + m \ln \beta + (\alpha + \beta) \ln \lambda - \alpha \ln(x_{U(n)} + \lambda) \\ & - \beta \ln(y_{U(m)} + \lambda) - \sum_{i=1}^n \ln(x_{U(i)} + \lambda) - \sum_{j=1}^m \ln(y_{U(j)} + \lambda) \end{aligned} \quad (9)$$

The MLEs of λ , α , and β , say $\hat{\lambda}$, $\hat{\alpha}$, and $\hat{\beta}$, respectively, can be obtained as the solution of

$$\frac{\hat{\alpha} + \hat{\beta}}{\hat{\lambda}} - \frac{\hat{\alpha}}{x_{U(n)} + \hat{\lambda}} - \frac{\hat{\beta}}{y_{U(m)} + \hat{\lambda}} - \sum_{i=1}^n \frac{1}{x_{U(i)} + \hat{\lambda}} - \sum_{j=1}^m \frac{1}{y_{U(j)} + \hat{\lambda}} = 0 \quad (10)$$

$$\frac{n}{\hat{\alpha}} - \ln(x_{U(n)} + \hat{\lambda}) + \ln \hat{\lambda} = 0 \quad (11)$$

$$\frac{m}{\hat{\beta}} - \ln(y_{U(m)} + \hat{\lambda}) + \ln \hat{\lambda} = 0 . \quad (12)$$

From (11) and (12),

$$\hat{\alpha} = n \left[\ln \left(\frac{x_{U(n)}}{\hat{\lambda}} \right) + 1 \right]^{-1} \quad (13)$$

$$\hat{\beta} = m \left[\ln \left(\frac{y_{U(m)}}{\hat{\lambda}} \right) + 1 \right]^{-1} , \quad (14)$$

and $\hat{\lambda}$ can be obtained as the solution of the following non-linear equation:

$$\begin{aligned}
 f(\lambda) &= \frac{n}{\lambda} \left[\ln \left(\frac{x_{U(n)}}{\lambda} + 1 \right) \right]^{-1} + \frac{m}{\lambda} \left[\ln \left(\frac{y_{U(m)}}{\lambda} + 1 \right) \right]^{-1} \\
 &\quad - \frac{n}{x_{U(n)} + \lambda} \left[\ln \left(\frac{x_{U(n)}}{\lambda} + 1 \right) \right]^{-1} - \frac{m}{y_{U(m)} + \lambda} \left[\ln \left(\frac{y_{U(m)}}{\lambda} + 1 \right) \right]^{-1} \\
 &\quad - \sum_{i=1}^n \frac{1}{x_{U(i)} + \lambda} - \sum_{j=1}^m \frac{1}{y_{U(j)} + \lambda}
 \end{aligned} \tag{15}$$

Therefore, $\hat{\lambda}$ can be obtained as the solution of the non-linear equation of the form

$$h(\lambda) = \lambda \tag{16}$$

where

$$\begin{aligned}
 h(\lambda) &= \left(n \left[\ln \left(\frac{x_{U(n)}}{\lambda} + 1 \right) \right]^{-1} + m \left[\ln \left(\frac{y_{U(m)}}{\lambda} + 1 \right) \right]^{-1} \right) \left[\frac{n}{x_{U(n)} + \lambda} \left[\ln \left(\frac{x_{U(n)}}{\lambda} + 1 \right) \right]^{-1} \right. \\
 &\quad \left. + \frac{m}{y_{U(m)} + \lambda} \left[\ln \left(\frac{y_{U(m)}}{\lambda} + 1 \right) \right]^{-1} + \sum_{i=1}^n \frac{1}{x_{U(i)} + \lambda} + \sum_{j=1}^m \frac{1}{y_{U(j)} + \lambda} \right]^{-1}
 \end{aligned}$$

Since $\hat{\lambda}$ is a fixed point solution of non-linear equation (15) and, therefore, it can be obtained by using a simple iterative scheme as follows:

$$h(\lambda_j) = \lambda_{j+1}, \tag{17}$$

where λ_j is the j^{th} iterate of $\hat{\lambda}$. The iteration procedure should be stopped when $|\lambda_{j+1} - \lambda_j|$ is sufficiently small. Therefore, the MLE of R becomes

$$\hat{R} = \frac{\hat{\alpha}}{\hat{\beta} + \hat{\alpha}} \tag{18}$$

Bootstrap Confidence Intervals

Confidence intervals are proposed based on the parametric bootstrap methods (i) percentile bootstrap method (Boot-p) based on the idea of Efron (1982) and (ii) bootstrap-t method (Boot-t) based on the idea of Hall (1988). The algorithms for estimating the confidence intervals of R using both methods are illustrated as follows:

Percentile Bootstrap Method

1. From the original two samples of upper records $\{x_{U(1)}, x_{U(2)}, \dots, x_{U(m)}\}$ and $\{y_{U(1)}, y_{U(2)}, \dots, y_{U(m)}\}$, compute ML estimates $\hat{\lambda}$, $\hat{\alpha}$, $\hat{\beta}$, and \hat{R}
2. Using $\hat{\lambda}$ and $\hat{\alpha}$, generate a bootstrap upper record sample $\{x_{U(1)}^*, x_{U(2)}^*, \dots, x_{U(m)}^*\}$ and, similarly, using $\hat{\lambda}$ and $\hat{\beta}$, generate a bootstrap upper record sample $\{y_{U(1)}^*, y_{U(2)}^*, \dots, y_{U(m)}^*\}$. Based on these data, we compute the bootstrap estimates, say $\hat{\lambda}^*$, $\hat{\alpha}^*$, $\hat{\beta}^*$, and \hat{R}^*
3. Repeat step 2 N_{boot} times
4. Let $V(x) = P(\hat{R}^* \leq x)$ be the cdf of \hat{R}^* . Define $\hat{R}_{boot} = V^{-1}(x)$ for a given x . The approximate $100(1 - \gamma)\%$ confidence interval of R is given by

$$\left(\hat{R}_{Boot-p} \left(\frac{\gamma}{2} \right), \hat{R}_{Boot-p} \left(1 - \frac{\gamma}{2} \right) \right). \quad (19)$$

Bootstrap-t Method

1. From the original two samples of upper records $\{x_{U(1)}, x_{U(2)}, \dots, x_{U(m)}\}$ and $\{y_{U(1)}, y_{U(2)}, \dots, y_{U(m)}\}$, compute ML estimates $\hat{\lambda}$, $\hat{\alpha}$, $\hat{\beta}$, and \hat{R}
2. Using $\hat{\lambda}$ and $\hat{\alpha}$, generate a bootstrap upper record sample $\{x_{U(1)}^*, x_{U(2)}^*, \dots, x_{U(m)}^*\}$ and, similarly, using $\hat{\lambda}$ and $\hat{\beta}$, generate a bootstrap upper record sample $\{y_{U(1)}^*, y_{U(2)}^*, \dots, y_{U(m)}^*\}$. Based on these data, we compute the bootstrap estimates, say $\hat{\lambda}^*$, $\hat{\alpha}^*$, $\hat{\beta}^*$, and compute the bootstrap estimate of R using (18), \hat{R}^* , and following statistic:

$$T^* = \frac{\sqrt{n}(\hat{R}^* - \hat{R})}{\sqrt{\text{Var}(\hat{R}^*)}},$$

where $\text{Var}(\hat{R}^*)$ is obtained using the Delta method (Greene, 2000)

3. Repeat step 2 N boot times
4. For the T^* values obtained in step 2, determine the upper and lower bounds of the $100(1 - \gamma)\%$ confidence interval of R as follows: let $H(x) = P(T^* \leq x)$ be the cdf of T^* . For a given x , define

$$\hat{R}_{\text{Boot-t}}(x) = \hat{R} + n^{-1/2} \sqrt{\text{Var}(\hat{R})} H^{-1}(x).$$

Here also, $\text{Var}(\hat{R})$ can be computed as same as computing the $\text{Var}(\hat{R}^*)$.

The approximate $100(1 - \gamma)\%$ confidence interval of R is given by

$$\left(\hat{R}_{\text{Boot-t}}\left(\frac{\gamma}{2}\right), \hat{R}_{\text{Boot-t}}\left(1 - \frac{\gamma}{2}\right) \right) \tag{20}$$

Bayes Estimation of R Using MCMC

The advantage of MCMC is that it not only gives a point estimate of the parameter, but also gives an interval estimation based on the final simulated empirical distribution. MCMC is essentially an iterative sampling algorithm, drawing values from the posterior distributions of the parameter in the concerned model. Consider the MCMC method to generate samples from the posterior distributions and then compute the Bayes estimates of R under upper record values from Lomax distribution. A wide variety of MCMC schemes are available. An important subclass of MCMC methods are Gibbs sampling and more general Metropolis-Hastings (M-H) algorithm (Metropolis, Rosenbluth, Rosenbluth, Teller, & Teller, 1953; Hastings, 1970). For more details about MCMC and the related methodologies, one can refer to Gentle (1998), Chen, Shao, and Ibrahim (2000), and Robert and Casella (2004).

Now, obtain the Bayes estimation of R under assumption that the parameters (λ, α, β) are independent random variables. The Bayes estimate of R under the squared error loss and the corresponding credible interval by the Gibbs sampling

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technique are considering. It is assumed that (λ, α, β) have independent gamma priors with the pdf's

$$\pi_1(\lambda | a_1, b_1) = \begin{cases} \frac{b_1^{a_1}}{\Gamma(a_1)} \lambda^{a_1-1} \exp(-b_1 \lambda), & \lambda > 0 \\ 0, & \lambda \leq 0 \end{cases} \quad (21)$$

$$\pi_2(\alpha | a_2, b_2) = \begin{cases} \frac{b_2^{a_2}}{\Gamma(a_2)} \alpha^{a_2-1} \exp(-b_2 \alpha), & \alpha > 0 \\ 0, & \alpha \leq 0 \end{cases} \quad (22)$$

$$\pi_3(\beta | a_3, b_3) = \begin{cases} \frac{b_3^{a_3}}{\Gamma(a_3)} \beta^{a_3-1} \exp(-b_3 \beta), & \beta > 0 \\ 0, & \beta \leq 0 \end{cases} \quad (23)$$

where, $a_1, b_1, a_2, b_2, a_3,$ and b_3 are chosen to reflect prior knowledge about $\lambda, \alpha,$ and β . The expression for the posterior can be obtained up to proportionality by multiplying the likelihood with the prior and this can be written as

$$\begin{aligned} \pi^*(\lambda, \alpha, \beta | \text{data}) &\propto \lambda^{(\alpha+\beta+a_1-1)} \alpha^{(n+a_2-1)} \beta^{(m+a_3-1)} \eta_1(\mathbf{x}, \lambda) \eta_2(\mathbf{y}, \lambda) \exp(-b_1 \lambda) \\ &\quad \times \exp\left[-\alpha\left(b_2 + \ln\left(x_{U(n)} + \lambda\right)\right)\right] \\ &\quad \times \exp\left[-\beta\left(b_3 + \ln\left(y_{U(m)} + \lambda\right)\right)\right] \end{aligned} \quad (24)$$

where $\eta_1(\mathbf{x}, \lambda), \eta_2(\mathbf{y}, \lambda)$ are given in (8) and

$$\begin{aligned} \pi_1^*(\lambda | \alpha, \beta, \text{data}) &\propto \lambda^{(\alpha+\beta+a_1-1)} \left(x_{U(n)} + \lambda\right)^{-\alpha} \left(y_{U(m)} + \lambda\right)^{-\beta} \\ &\quad \times \eta_1(\mathbf{x}, \lambda) \eta_2(\mathbf{y}, \lambda) \exp(-b_1 \lambda) \end{aligned} \quad (25)$$

Similarly, the full posterior conditional distribution for α and β are given by

$$\pi_2^*(\alpha | \lambda, \text{data}) \sim \text{Gamma} \left(n + a_2, b_2 + \ln \left(\frac{x_{U(n)}}{\lambda} + 1 \right) \right) \quad (26)$$

$$\pi_3^*(\beta | \lambda, \text{data}) \sim \text{Gamma} \left(m + a_3, b_3 + \ln \left(\frac{x_{U(m)}}{\lambda} + 1 \right) \right). \quad (27)$$

It can be seen that (26) is a gamma density with shape parameters $(n + a_2)$ and $\left(b_2 + \ln \left(\frac{x_{U(n)}}{\lambda} + 1 \right) \right)$ and (27) is a gamma density with shape parameters $(m + a_3)$ and $\left(b_3 + \ln \left(\frac{y_{U(m)}}{\lambda} + 1 \right) \right)$. Therefore, samples of α and β can be easily generated using any gamma generating routine. However, in our case, the conditional posterior distribution of λ equation (25) cannot be reduced analytically to well-known distributions and, therefore, it is not possible to sample directly by standard methods. However, the plot of it shows that it is similar to a normal distribution. To generate random numbers from this distribution, use the Metropolis-Hastings method with normal proposal distribution.

Therefore, the algorithm of Gibbs sampling is as follows:

- Start with an $(\lambda^{(0)} = \hat{\lambda})$ and set = 1
- Generate $\alpha^{(t)}$ from $\text{Gamma} \left(n + a_2, b_2 + \ln \left(\frac{x_{U(n)}}{\lambda} + 1 \right) \right)$
- Generate $\beta^{(t)}$ from $\text{Gamma} \left(m + a_3, b_3 + \ln \left(\frac{y_{U(m)}}{\lambda} + 1 \right) \right)$
- Using Metropolis-Hastings (see Metropolis et al., 1953), generate $\lambda^{(t)}$ from (25) with the $N(\lambda^{(t-1)}, \sigma^2)$ proposal distribution, where σ^2 is variance-covariance matrix
- Compute $\lambda^{(t)}$, $\alpha^{(t)}$, and $\beta^{(t)}$. Then compute $R^{(t)} = \frac{\alpha^{(t)}}{\beta^{(t)} + \alpha^{(t)}}$
- Set $t = t + 1$
- Repeat steps 2-5 N times
- We obtain the Bayes MCMC point estimate of R as

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$$E(R | \text{data}) = \frac{1}{N - M} \sum_{i=M+1}^N R^{(i)}, \quad (28)$$

where M is the burn-in period (that is, a number of iterations before the stationary distribution is achieved) and posterior variance of R becomes

$$\hat{V}(R | \text{data}) = \frac{1}{N - M} \sum_{i=M+1}^N \left(R^{(i)} - \hat{E}(R | \text{data}) \right)^2 \quad (29)$$

- To compute the credible intervals of R , usually, take the quintiles of the sample as the end points of the interval. Order $R^{(M+1)}, R^{(M+2)}, \dots, R^{(N)}$ as $R^{(1)}, R^{(2)}, \dots, R^{(N-M)}$. Then the $100(1 - \gamma)\%$ symmetric credible interval is

$$\left(R_{\left(\frac{\gamma}{2}\right)_{(N-M)}}, R_{\left(\left(1 - \frac{\gamma}{2}\right)\right)_{(N-M)}} \right) \quad (30)$$

Estimation of R if λ is Known

Consider the estimation of R and the corresponding highest posterior density (HPD) intervals when λ is known. Assume $x_{U(1)}, x_{U(2)}, \dots, x_{U(n)}$ is the first upper record values observed from $\text{Lomax}(\lambda, \alpha)$, and $y_{U(1)}, y_{U(2)}, \dots, y_{U(m)}$ is the first upper record values observed from $\text{Lomax}(\lambda, \beta)$. Based on these samples, we can estimate R . Recently works on interval estimation of R were discussed in Rezaeia, Tahmasbib, and Mahmoodi (2010), Baklizi (2008a; 2008b), and Shoukri, Chaudhary, and Al-Halees (2005). First, consider the MLE of R and its distributional properties.

MLE of R

The MLE of R , \hat{R} , will be

$$\hat{R} = \frac{\hat{\alpha}}{\hat{\beta} + \hat{\alpha}}, \quad (31)$$

where

$$\hat{\alpha} = n \left[\ln \left(\frac{x_{U(n)}}{\hat{\lambda}} + 1 \right) \right]^{-1}, \quad \hat{\beta} = m \left[\ln \left(\frac{y_{U(m)}}{\hat{\lambda}} + 1 \right) \right]^{-1}. \quad (32)$$

Therefore,

$$\hat{R} = \frac{n \ln \left(\frac{y_{U(m)}}{\hat{\lambda}} + 1 \right)}{m \ln \left(\frac{x_{U(n)}}{\hat{\lambda}} + 1 \right) + n \ln \left(\frac{y_{U(m)}}{\hat{\lambda}} + 1 \right)}. \quad (33)$$

To study the confidence interval of R , consider the distribution of \hat{R} as well as the distributions of $\hat{\alpha}$ and $\hat{\beta}$. Consider first $\hat{\alpha} = n \left[\ln \left(\frac{x_{U(n)}}{\hat{\lambda}} + 1 \right) \right]^{-1}$. Arnold et al. (1998) obtained the pdf of R_n as follows:

$$f_{R_n}(r_n) = f(r_n) \left[-\ln(1 - F(r_n)) \right]^{n-1} / (n-1)! \quad (34)$$

under Lomax(λ, α)

$$f_{R_n}(r_n) = \frac{\alpha \lambda^\alpha}{\Gamma(n)} (r_n + \lambda)^{-(\alpha+1)} \left[\alpha \ln \left(\frac{r_n}{\lambda} + 1 \right) \right]^{n-1}, \quad r_n > 0. \quad (35)$$

Consequently, the pdf of $Z_1 = \hat{\alpha}$ as defined in (32) is given by

$$f_{Z_1}(Z_1) = \frac{(n\alpha)^n}{\Gamma(n) Z_1^{n+1}} \exp\left(-\frac{n\alpha}{Z_1}\right), \quad Z_1 > 0. \quad (36)$$

This is the inverted gamma distribution. Similarly, the pdf of $Z_2 = \hat{\beta}$ as defined in (32) is given by

$$f_{Z_2}(Z_2) = \frac{(m\beta)^m}{\Gamma(m) Z_2^{m+1}} \exp\left(-\frac{m\beta}{Z_2}\right), \quad Z_2 > 0. \quad (37)$$

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Find the pdf of

$$\hat{R} = \frac{\hat{\alpha}}{\hat{\beta} + \hat{\alpha}} = \frac{Z_1}{Z_1 + Z_2} = \frac{1}{1 + Z_2/Z_1} .$$

Consider Z_2/Z_1 . By the properties of the inverted gamma distribution and its relation with the gamma distribution, $n\alpha/Z_1 \sim \text{Gamma}(n, 1)$ and $m\beta/Z_2 \sim \text{Gamma}(m, 1)$. Hence $2n\alpha/Z_1 \sim \chi_{2n}^2$ and $2m\beta/Z_2 \sim \chi_{2m}^2$. By the independence of two random quantities,

$$\frac{2n\alpha/2nZ_1}{2m\beta/2mZ_2} \sim F_{2m, 2n} ,$$

and thus $\frac{Z_2}{Z_1} = \frac{\beta}{\alpha} F_{2m, 2n}$, a scaled F distribution. It follows that the distribution of

\hat{R} is that of $\frac{1}{1 + \frac{\beta}{\alpha} F_{2m, 2n}}$. Because $\frac{1}{\hat{R}} = 1 + \left(\frac{1}{R} - 1\right) F(2m, 2n)$, then

$$R \sim \frac{F(2m, 2n)}{F(2m, 2n) + \left(\frac{1}{\hat{R}} - 1\right)} .$$

The $100(1 - \gamma)\%$ confidence interval of R can be obtained as

$$\left[\frac{F_{\gamma/2}(2m, 2n)}{F_{\gamma/2}(2m, 2n) + \left(\frac{1}{\hat{R}} - 1\right)}, \frac{F_{1-\gamma/2}(2m, 2n)}{F_{1-\gamma/2}(2m, 2n) + \left(\frac{1}{\hat{R}} - 1\right)} \right] . \quad (38)$$

Bayes Estimation of R

Obtain the Bayes estimation of R under assumption that the shape parameters α and β are random variables. It is assumed that α and β have independent gamma priors (22) and (23), respectively, with the parameters $\alpha \sim \text{Gamma}(a_2, b_2)$ and

$\beta \sim \text{Gamma}(a_3, b_3)$. The posterior pdfs of α and β are given by (26) and (27), respectively, because priors α and β are independent. Using standard transformation techniques, and after some manipulations, the posterior pdf of $R = \alpha/\beta + \alpha$ is found to be

$$f_R(r) = C \frac{r^{n+a_2-1} (1-r)^{n+a_3-1}}{\left[r \left(b_2 + \ln \left(\frac{x_{U(n)}}{\lambda} + 1 \right) \right) + (1-r) \left(b_3 + \ln \left(\frac{y_{U(m)}}{\lambda} + 1 \right) \right) \right]^{n+m+a_2+a_3}} \quad (39)$$

if $0 < r < 1$, and 0 otherwise, where

$$C = \frac{\Gamma(n+m+a_2+a_3)}{\Gamma(n+a_2)\Gamma(m+a_3)} \left[b_2 + \ln \left(\frac{x_{U(n)}}{\lambda} + 1 \right) \right]^{n+a_2} \left[b_3 + \ln \left(\frac{y_{U(m)}}{\lambda} + 1 \right) \right]^{m+a_3}. \quad (40)$$

There is no explicit expression for the posterior mean or median of (39). However, the posterior mode can be easily obtained as

$$\frac{d}{dr} f_R(r) = \frac{r^{A_1-1} (1-r)^{A_2-1} \left[2r^2 (B_1 - B_2) + r(2B_2 - 2B_1 - A_2 B_1 + A_1 B_2) + A_1 B_2 \right]}{\left[B_2 (1-r) + B_1 r \right]^{(A_1+A_2+3)}},$$

where $B_1 = b_2 + \ln \left(\frac{x_{U(n)}}{\lambda} + 1 \right)$, $B_2 = b_3 + \ln \left(\frac{y_{U(m)}}{\lambda} + 1 \right)$, $A_1 = n+a_2-1$, and

$A_2 = m+a_3-1$. Note $r \in (0, 1)$, $(d/dr)f_R(r) = 0$ has only two roots. Using the fact that $\lim_{r \rightarrow 0^+} (d/dr)f_R(r) > 0$ and $\lim_{r \rightarrow 1^-} (d/dr)f_R(r) < 0$, it follows that the density function $f_R(r)$ has a unique mode. The posterior mode can be obtained as the unique root of which lies between 0 and 1 of the following quadratic equation:

$$2r^2 (B_2 - B_1) + r(2B_1 - 2B_2 + A_2 B_1 - A_1 B_2) - A_1 B_2 = 0. \quad (41)$$

Consider the following loss function:

$$L(a, b) = \begin{cases} 1, & |a - b| > \varepsilon \\ 0, & |a - b| \leq \varepsilon \end{cases} \quad (42)$$

It is known that the Bayes estimate with respect to the above loss function (42) is the midpoint of the ‘modal interval’ of length 2ε of the posterior distribution. Therefore, the posterior mode is an approximate Bayes estimator of R with respect to the above loss function when the constant ε is small.

The Bayes estimate of R under squared error loss cannot be computed analytically. Alternatively, using the approximate method of Lindley (1980), it can be easily seen that the approximate Bayes estimate of R , say \tilde{R}_{Bayes} , relative to squared error loss function is

$$\tilde{R}_{\text{Bayes}} = \tilde{R} \left[1 + \frac{\tilde{\alpha} \tilde{R}^2}{\tilde{\beta}^2 (n + a_2 - 1)(m + a_3 - 1)} [\tilde{\alpha} (n + a_2 - 1) - \tilde{\beta} (m + a_3 - 1)] \right] \quad (43)$$

where

$$\tilde{\alpha} = \frac{n + a_2 - 1}{b_2 + \ln \left(\frac{x_{U(n)}}{\lambda} + 1 \right)}, \quad \tilde{\beta} = \frac{m + a_3 - 1}{b_3 + \ln \left(\frac{y_{U(m)}}{\lambda} + 1 \right)}, \quad \tilde{R} = \frac{\tilde{\alpha}}{\tilde{\beta} + \tilde{\alpha}}. \quad (44)$$

For comparison purposes, a highest posterior density (HPD) interval of R was computed (Soliman & Al-Aboud, 2008). Due to the unimodality of the posterior distribution (39), the $100(1 - \gamma)\%$ HPD interval $[\omega_L, \omega_U]$ for R is given by the simultaneous solution of the nonlinear equations

$$\int_{\omega_L}^{\omega_U} f_R(r | \text{dat}) dr = 1 - \gamma, \quad f_R(\omega_L | \text{dat}) = f_R(\omega_U | \text{dat}) \quad (45)$$

A Newton-Raphson iteration can be invoked to solve the equations in (45) and thereby the HPD interval is obtained.

Illustrative Example Using Simulated Data

Six upper record values were simulated from Lomax(1, 2.1) and six upper record values from Lomax(1, 2.5), with $R_{\text{Exact}} = 0.4565$. The data were truncated after four

decimal places: the \mathbf{x} upper record values are {1.0638, 1.4488, 7.2166, 7.8652, 11.6919, 34.5528} and the corresponding \mathbf{y} upper record values are {0.2355, 1.0058, 1.5503, 2.0698, 12.8867, 13.0820}.

Case (1)

λ is unknown: Based on the above data, plot the profile log-likelihood function of λ in Figure 2. It is an upside down function and it has a unique maximum. Obtain the MLE of λ using the iterative procedure (16). Using the stopping criterion that the iteration stops whenever two consecutive values are less than 10^{-6} , the iteration stops after 14 steps and it provides the MLE of $\hat{\lambda} = 1.5232$. Using (13) and (14), obtain the MLEs of $\hat{\alpha} = 1.8958$ and $\hat{\beta} = 2.6542$, and hence $\hat{R} = 0.4167$, from (18). The 95% confidence, credible intervals, and corresponding length are reported in Table 1 using exact confidence interval, parametric percentile bootstrap methods, and MCMC technique.

Case (2)

Estimate the parameters assuming λ is known to be 1. Obtain the MLEs of α and β as 1.7079 and 2.4080, respectively. Therefore, the MLE of R becomes $\hat{R} = 0.4150$. The corresponding 95% confidence credible intervals and corresponding length are also reported in Table 1 using MLE, parametric percentile bootstrap methods, and MCMC technique. The posterior probability density function (39) of R for the given data set is plotted in Figure 2. The simulation number of R and Histogram of Rgenerated by MCMC method are plotted in Figure 1.

Table 1. Two-sided 95 % confidence and credible intervals of R when $\lambda = 1$, $\alpha = 2.1$, and $\beta = 2.5$ with prior 0

Methods	λ is unknown			λ is known		
	\hat{R}	95% CI	Length	\hat{R}	95% CI	Length
MLE	0.4167	[0.1789, 0.7007]	0.5217	0.4150	[0.1779, 0.6992]	0.5213
Boot-p	0.4131	[0.2150, 0.6246]	0.4096	0.4189	[0.1796, 0.6921]	0.5125
Boot-t	0.4008	[0.0634, 0.6553]	0.5920	0.4036	[0.0271, 0.7306]	0.7035
Bayes	0.4111	[0.1673, 0.6902]	0.5229	0.4192	[0.1883, 0.7135]	0.5252

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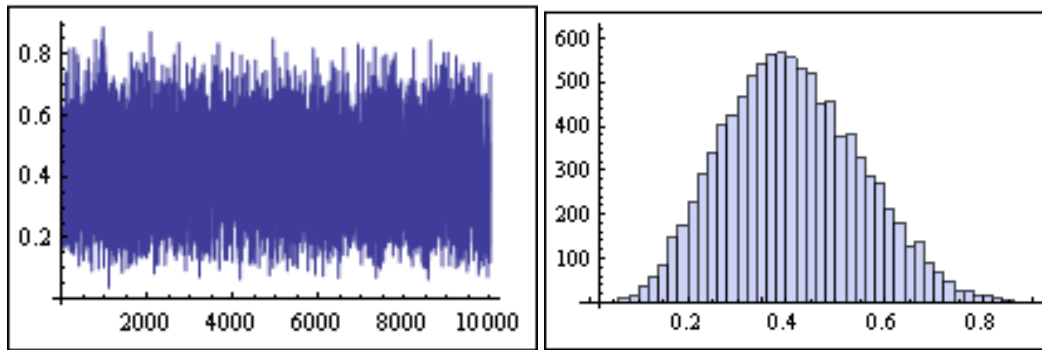


Figure 1. Simulation number (left) and histogram (right) of R generated by MCMC method

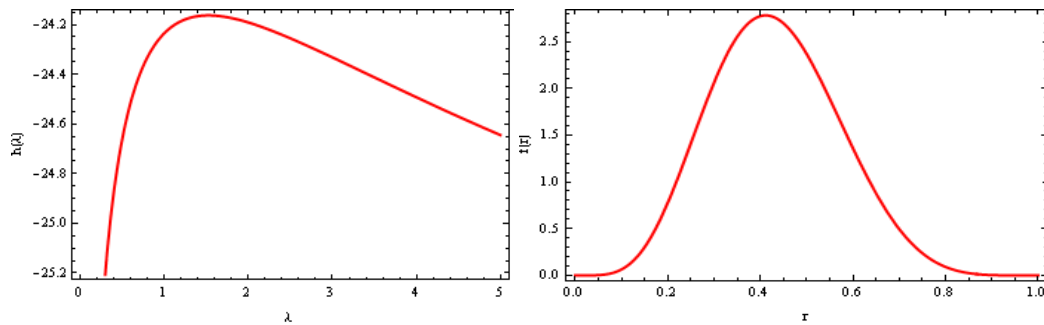


Figure 2. Profile likelihood of λ (left) and posterior pdf of R (right) for given data set presented above

Simulation Results

Some numerical experiments were performed to evaluate the behavior of the proposed methods for different samples, different parameter values, and different hyper parameters. Consider two cases separately to draw inference on R , namely when (i) common scale parameter λ is unknown and (ii) the common scale parameter λ is known. Consider the different sample sizes (n and m) and different hyper parameters ($a_1, b_1, a_2, b_2, a_3, b_3$). In first case, take $\lambda = 2$, $\alpha = 1.5$, and $\beta = 3$. In the second case, take $\lambda = 2$, $\alpha = 3.2$, and $\beta = 2.1$. Without loss of generality, take $\lambda = 2$ in both cases. All the results are based on 1000 replications.

(i) λ is unknown. Compute the estimate of λ using the iterative algorithm (16) with the initial estimate 2. The iterative process stops when the difference between the two consecutive iterates are less than 10^{-7} . Once λ is obtained, then estimate α

and β using (13) and (14), respectively. Finally, obtain the MLE of R using (18). To find the Bayes MCMC estimates, use the non-informative gamma priors for the three parameters (we call it prior 0). Non-informative prior ($a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$) provides prior distributions which are not proper. Also use informative priors, including prior 1, $a_1 = 2, b_1 = 1, a_2 = 3, b_2 = 2, a_3 = 3,$ and $b_3 = 1$, with the values of previous parameters and compute the Bayes estimates and 95% probability intervals based on 10,000 MCMC samples (discard the first 1,000 values as ‘burn-in’). The average Bayes estimates, means squared errors (MSEs), coverage percentages, and average probability interval lengths based on 1000 replications are reported in Table 2.

(ii) λ is known. Obtain the estimates of R by using the ML method and Lindley's approximation approach. Calculate the exact confidence intervals and HPD interval of R , using the same non-informative prior (prior 0) and an informative prior, including (prior 1) to compute the average estimates of R , MSEs, coverage percentages, and average probability interval lengths based on 1,000 replications. The results are reported in Table 3.

Table 2. Simulation results and estimation of the parameters

(n, m)	R_{Exact}	MLE		Bayes using MCMC			
		Mean	MSE	Mean	MSE	Length	Coverage
$\lambda = 2, \alpha = 1.5, \beta = 3$ using prior 0							
(5, 5)	0.3333	0.3184	0.0205	0.3757	0.0115	0.5311	0.9900
(6, 6)		0.3206	0.0203	0.3742	0.0113	0.4863	0.9810
(7, 6)		0.3227	0.0182	0.3696	0.0111	0.4807	0.9750
(7, 7)		0.3459	0.0169	0.3881	0.0110	0.4617	0.9900
(8, 7)		0.3133	0.0154	0.3735	0.0107	0.4431	0.9850
(8, 8)		0.3286	0.0146	0.3775	0.0106	0.4323	0.9950
(9, 8)		0.3326	0.0141	0.3809	0.0104	0.4211	0.9750
(9, 9)		0.3184	0.0140	0.3624	0.0089	0.4004	0.9550
(10, 9)		0.3172	0.0131	0.3682	0.0087	0.3944	0.9550
(10, 10)		0.3224	0.0126	0.3622	0.0082	0.3819	0.9650
$\lambda = 2, \alpha = 1.5, \beta = 3$ using prior 1							
(5, 5)	0.3333	0.3269	0.0202	0.3766	0.0055	0.4399	0.9930
(6, 6)		0.3203	0.0201	0.3698	0.0053	0.4104	0.9950
(7, 6)		0.3246	0.0184	0.3812	0.0052	0.4076	0.9950
(7, 7)		0.3242	0.0151	0.3651	0.0048	0.3895	0.9760
(8, 7)		0.3330	0.0149	0.3780	0.0047	0.3867	0.9950
(8, 8)		0.3378	0.0148	0.3727	0.0045	0.3744	0.9770
(9, 8)		0.3262	0.0146	0.3689	0.0042	0.3653	0.9900
(9, 9)		0.3352	0.0131	0.3663	0.0041	0.3578	0.9660
(10, 9)		0.3335	0.0127	0.3712	0.0039	0.3525	0.9850
(10, 10)		0.3495	0.0124	0.3714	0.0037	0.3443	0.9950

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Table 3. Simulation results and estimation of the parameters

(n, m)	R_{Exact}	MLE				Bayes using Lindely			
		Mean	MSE	Length	Coverage	Mean	MSE	Length	Coverage
$\lambda = 2, \alpha = 3.2, \beta = 2.1$ using prior 0									
(5, 5)	0.6038	0.6407	0.0462	0.4535	0.9550	0.6276	0.0393	0.5851	0.9900
(6, 6)		0.6334	0.0373	0.4361	0.9310	0.6298	0.0286	0.5787	0.9950
(7, 6)		0.5954	0.0369	0.4356	0.9450	0.6356	0.0266	0.5308	0.9700
(7, 7)		0.6288	0.0264	0.4259	0.9600	0.6194	0.0224	0.5294	0.9750
(8, 7)		0.6176	0.0235	0.4162	0.9450	0.6211	0.0219	0.5122	0.9650
(8, 8)		0.6129	0.0213	0.4146	0.9280	0.6241	0.0215	0.4941	0.9800
(9, 8)		0.6099	0.1869	0.4100	0.9330	0.6197	0.0162	0.4677	0.9550
(9, 9)		0.6245	0.0176	0.3930	0.9010	0.6407	0.0149	0.4512	0.9390
(10, 9)		0.6021	0.0170	0.3922	0.8990	0.6378	0.0078	0.4189	0.9400
(10, 10)		0.6059	0.0165	0.3820	0.9090	0.6395	0.0052	0.3972	0.9600
$\lambda = 2, \alpha = 3.2, \beta = 2.1$ using prior 0									
(5, 5)	0.6038	0.5997	0.0454	0.4732	0.9610	0.6084	0.0251	0.5933	0.9450
(6, 6)		0.5964	0.0397	0.4469	0.9800	0.5832	0.0245	0.5532	0.9230
(7, 6)		0.6156	0.0332	0.4348	0.9120	0.5985	0.0214	0.5159	0.9200
(7, 7)		0.6001	0.0327	0.4221	0.9330	0.5979	0.0207	0.5144	0.9450
(8, 7)		0.6077	0.0260	0.4201	0.9350	0.6193	0.0168	0.5083	0.9450
(8, 8)		0.6065	0.0232	0.4131	0.9050	0.6115	0.0162	0.4899	0.9600
(9, 8)		0.6236	0.0191	0.4022	0.9010	0.6261	0.0136	0.4570	0.9850
(9, 9)		0.6201	0.0157	0.3986	0.9200	0.6283	0.0130	0.4047	0.9640
(10, 9)		0.6029	0.0111	0.3930	0.9050	0.6296	0.0089	0.3995	0.9540
(10, 10)		0.6016	0.0093	0.3822	0.9150	0.6194	0.0051	0.3755	0.9600

Conclusion

The problem of estimating $R = P[Y < X]$ for the Lomax distributions was addressed, and classical and MCMC Bayesian analysis for R were developed when both samples on X and Y are in the form of upper record values, observed from the Lomax distribution with different one shape parameter. The general case when all the parameters are unknown was considered, and when the common scale parameter was known. In the first case, the MCMC method provided an alternative method for parameters estimation of the Lomax distribution and, also, for obtaining both point and interval estimators of the stress-strength reliability model R . It is more flexible when compared with the traditional methods, such as MLE, based on the set of upper record values. It is hoped this investigation will be useful for researchers dealing with the kind of data considered.

Observe the following from the results:

- When the common scale parameter λ is unknown, it is observed that the Bayes estimator using MCMC technique works quite well. The MCMC sample were used to construct confidence intervals and that also works quite well. When the common scale parameter λ is known, the maximum likelihood estimator and Bayes estimators were proposed based on the approximate method of Lindley. The confidence interval based on the exact distribution of the MLE works quite very well. Also, a HPD interval was recommended
- Tables 2 and 3 show that, when $m = n$ and m, n increase, then MSEs and average confidence interval lengths, credible interval lengths of the MLEs, and Bayes estimators decrease, and that the coverage percentages are reached to the nominal level in most cases
- From Tables 2 and 3, it is clear that the Bayes estimators based on informative priors (prior 1) perform much better than the Bayes estimators based on non-informative priors (prior 0) or MLEs in terms of biases, MSEs, and lengths of credible intervals.

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